Lateral Variations of Density in the Mantle

Jafar Arkani-Hamed*

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Summary

The lateral variations of the Earth's gravitational field, deduced from orbital data of artificial satellites, indicate the existence of lateral density variations within the Earth. A density model is computed for the mantle with the following constraints: (1) the model presents perturbations to Gutenberg's earth model which are specified by spherical harmonics through the sixth order and degree; (2) the density anomalies are confined to the mantle and the crust; (3) the anomalies of the crust are determined for \( n = 2, \ldots, 6 \) and \( m = 0, \ldots, n \) from crustal thickness, crustal \( P \) wave velocity, and \( P_n \) velocity, and those of the upper mantle for \( n = 2 \) and \( 3 \) and \( m = 0, \ldots, n \) are related to the lateral variations of seismic travel-time residuals; (4) the unknown density anomalies of the mantle are determined such that the total shear strain energy of the Earth is a minimum; (5) the gravitational potential of the deformed Earth (subject to the density anomalies) on its surface equals the first six orders of the spherical harmonic representation of the measured geopotential; and (6) an isotropic, elastic, and cold mantle, and a liquid core are assumed in the stress analysis.

The density anomalies thus obtained exhibit a decreasing feature with depth. In the crust they are of the order of \( 0.3 \, \text{g cm}^{-3} \), in the upper mantle \( 0.1 \, \text{g cm}^{-3} \), and in the lower mantle \( 0.04 \, \text{g cm}^{-3} \), which are within the values deduced from independent seismic measurements.

The lateral variations of the associated stress differences at shallow depths correlate with the surface feature of the Earth. This correlation disappears in the deep mantle.

1. Introduction

Lateral undulations of geopotential, deduced from orbital data of close satellites, indicate the existence of lateral variations of density within the Earth. In the present study we are concerned with the density anomalies confined to the crust and mantle. Kaula (1963) made a start on this problem, but we wish to extend his work by making use of seismic data on crustal thickness and crustal seismic velocities to fix the crustal density variations. This extension is made possible by the close relationship between density and seismic velocity in igneous rocks (Birch 1961). Similar determinations of upper mantle density variations are made by using seismic travel-time residuals. We compute the density variations of the mantle by using these seismically inferred

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loads as inputs, and by minimizing the total shear strain energy in the Earth, while satisfying the satellite gravity data.

The selected density perturbations are then compared with the density variations which can be deduced from independent seismic measurements.

2. Density variations of crust

To determine crustal loads, we must consider both the topographic features and the lateral variations of the density in the crust. For topography we utilize spherical harmonic coefficients of equivalent rock topography given by Lee & Kaula (1967). Then we consider a surface layer with 50 km thickness as representing the crustal region. The bottom of this layer lies almost everywhere below the Mohorovicic discontinuity (Arkani-Hamed & Toksoz 1968). Thus, the lateral variations of the density in this layer contain both the density variations of the actual crust and the density variations due to the lateral undulations of the depth of the Mohorovicic discontinuity. Using the seismic P wave velocities at 297 stations (tabulated by Arkani-Hamed & Toksoz 1968) and Birch's (solution 8, 1961) experimental relationship between density ($\rho$) and P wave velocity ($V_p$) of rocks,

$$\rho = 0.41 + 0.3597 \times V_p$$

(2.1)

the density of each sub-layer is calculated. Then, the average density of the surface

![Graph](https://academic.oup.com/gji/article-abstract/20/5/431/760309)

**Fig. 1.** Average density of the surface layer versus the actual crustal thickness.
Lateral variations of density in the mantle

Fig. 2. Lateral variations of the average density of the surface layer (g cm⁻³).

Layer beneath the above-mentioned stations is computed through the following equation:

\[ \bar{\rho}_j = \sum_{i=1}^{N_j} \rho_{ij} \cdot H_{ij} / \sum_{i=1}^{N_j} H_{ij} \]  

(2.2)

where \( \rho_{ij} \) and \( H_{ij} \) are, respectively, the density and the thickness of the \( i \)-th sub-layer under the \( j \)-th station. \( N_j \) and \( \bar{\rho}_j \) are the number of the sub-layers (the surface layer is divided into \( N_j \) sub-layers) and the average density of the surface layer beneath the \( j \)-th station, respectively.

Fig. 1 displays \( \bar{\rho}_j \) versus the actual crustal thickness which shows a strong correlation between these quantities (their linear correlation coefficient is \(-0.83\)). Since the stations utilized are distributed somewhat evenly over the Earth, we assume that such a correlation exists between the spherical harmonic representations of \( \bar{\rho} \) and the actual crustal thickness. Therefore, an empirical relationship

\[ \bar{\rho} = 3.311 - 0.0239 \times \text{CRTH} + 0.000247 \times \text{CRTH}^2 \]  

(2.3)

Table 1

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\( n \) = Order of the harmonics.

\( m \) = Degree of the harmonics.
Table 2

Spherical harmonic coefficients of the gravitational potential of the crust and the observed geopotential ($10^7$ ergs)

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G1 = Equivalent rock topography.
G2 = Surface layer.
G3 = Observed geopotential.

Table 2 is the list of the spherical harmonic coefficients of the gravitational potentials of the equivalent rock topography, computed through

$$\phi_n = 4\pi G a_{\sigma_m} / (2n+1),$$

and those of the surface layer, determined from

$$\phi_n = 4\pi G (a^{n+3} - R^{n+3}) \bar{\rho}_n / [a^{n+1} \cdot (2n+1)(n+3)].$$

Here $a$, $G$, $n$, $\sigma_m$, and $R$ are, respectively, the mean radius of the Earth, the gravitational constant, the order of the spherical harmonic, the coefficient of the $n$-th order harmonic of the surface mass density corresponding to the equivalent rock topography (the equivalent rock topography is reduced to the surface mass by using $2.7 \text{ g cm}^{-3}$ for the average density of its materials, Lee & Kaula 1967), and the radial distance to the bottom of the surface layer. The spherical harmonic coefficients of the observed geopotential, listed in the same table for easy comparison, are an order of magnitude smaller than those of the foregoing potentials. This table together with a small correlation coefficient between the observed geopotential and the equivalent rock topography (Arkani Hamed & Toksöz 1968) indicates that some compensating density anomalies, quite independent from the surface.
feature of the Earth, exist in the mantle. The existence of such anomalies has also been concluded by Munk & MacDonald (1960).

3. Density variations of the upper mantle (A)

The lateral variations of the seismic structure of the upper mantle (down to 400 km depth) are manifested in those of the seismic travel-time residuals (expressed in spherical harmonics through the third order by Arkani-Hamed & Toksoz 1968) which are assumed to be due to the lateral variations of \(P\) wave velocity in the surface layer and the upper mantle.

The spherical harmonic coefficients of the travel-time residuals of the surface layer, the crustal residuals (CSTTR), are determined through the following equation:

\[
C_{\text{STTR}} = -2.78 \times 50 \frac{\Delta \rho}{V_p^2} \quad (3.1)
\]

where \(V_p\) is the average velocity (in \(\text{km} \cdot \text{s}^{-1}\)) and \(\Delta \rho\) is the appropriate spherical harmonic coefficient of the density variations (in \(\text{g} \cdot \text{cm}^{-3}\)) of the surface layer. Equation (3.1) is derived from

\[
t = -H \frac{\Delta V_p}{V_p^2} \quad (3.2)
\]

and equation (2.1).

The travel-time residuals of the upper mantle, the mantle residuals (MSTTR), are computed by

\[
M_{\text{STTR}} = STTR - CSTTR. \quad (3.3)
\]

### Table 3

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</tr>
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<td>-0.041</td>
</tr>
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<td>20– 25</td>
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**Long.** = Longitude in degrees.

**Co-lat.** = Co-latitude in degrees.

**STTR** = Observed residuals in seconds.

**CSTTR** = Crustal residuals in seconds.

* = Reference point.
Here STTR are the observed travel-time residuals. Mantle residuals thus computed are greater than crustal ones and have, in general, opposite signs from those of the crustal's. This situation is also observed at several stations where, using the average crustal velocities and $P_n$ velocities (tabulated by Arkani-Hamed & Toksöz 1968), the travel-time residuals of the surface layer with respect to a reference point are calculated and listed in Table 3. The residuals of the selected reference point are almost equal to zero. Fig. 3 illustrates the residuals thus computed versus the observed residuals, included in the same table. Their linear correlation coefficient ($= -0.47$) indicates that areas with positive crustal residuals (low $P$ wave velocity of the surface layer) are associated with negative observed residuals and vice versa. This result is geophysically important. It implies that wherever the surface layer velocities are low (the continents) the upper mantle velocities are high. Hence the surface layer compensates for the large lateral variations of the upper mantle velocity.

From the upper mantle residuals we determine the upper mantle density variations through Birch's (1964) formula,

$$\Delta \rho = 0.3788 \Delta V_p.$$  \hfill (3.4)

In the absence of any realistic equation of state for the upper mantle materials, the use of this formula is in order. Equations (3.2) and (3.3) yield the following relationship:

$$\Delta \rho = -0.3788 \times M \text{STTR} \sum_i \left( \frac{H_i}{V_i^2} \right)$$  \hfill (3.5)
where $H_i$ and $V_i$ are, respectively, the thickness and the velocity of the $i$-th layer in the mantle. In the numerical calculations, Gutenberg’s earth model is used and it is assumed that $\Delta \rho$ does not change with depth. If we use Jeffreys’s model instead, there will be no significant difference in the results. Both of these models are listed by Alterman, Jarosh & Pekeris (1961). Table 4 gives the spherical harmonic coefficients of $\Delta \rho$ as computed through equation (3.5).

### Table 4

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### 4. Density variations in the upper mantle (B) and the lower mantle

In the present section we compute a special model of density variations of the upper mantle (for harmonics between the third and the seventh orders), and of the lower mantle (for low harmonics through the third order). This model takes into account the surface equivalent rock topography, the density variations of the surface layer, and the observed geopotential. It also minimizes the total shear strain energy resulting from the density variations in the Earth.

The procedure is based on the study of the elastic deformation of a gravitating earth model, subjected to a laterally inhomogeneous density distribution. The following assumptions are made throughout the treatment:

1. The Earth is considered to be a cold spherical body with a linear elastic and isotropic mantle, a liquid core, and with no rotational motion;
2. The elastic moduli of the Earth are considered to vary only radially; and
3. The lateral variations of density are confined to the crust and the mantle.

These assumptions do not represent the conditions existing inside the real Earth. The real Earth is hot, with lateral and radial variations in temperature. It is nearly a viscoelastic body for the loading at the low spherical harmonics considered in the present study. The probable convection currents in the core and the mantle contribute to the measured geopotential. Despite these shortcomings, the mathematical simplifications introduced by the assumptions make the calculations feasible. Furthermore, it is shown that the equilibrium equations of a thermo-visco-elastic earth model are analogous to those of an elastic model. Therefore, the solution of the elastic model will yield a clue to the deformation of the real Earth.

The techniques employed for this study are the same as those utilized by Kaula (1963). We start with the final equations governing the deformation of the mantle (Kaula 1963; Arkani-Hamed 1969). These equations are:

$$\frac{d\mathbf{Y}}{dr} = \mathbf{M}. \mathbf{Y} + \mathbf{D}.$$  \hspace{1cm} (4.1)

The components of $\mathbf{Y}$ are the radial component of displacement, radial stress, radial dependence of tangential displacement, radial dependence of tangential stress, perturbations of the gravitational potential, and gradient of the gravitational potential.
minus the contribution of the radial displacements thereto. Zero indices denote the unperturbed state.

\[
M = \begin{pmatrix}
M_{11} & M_{12} & M_{13} & 0 & 0 & 0 \\
M_{21} & M_{22} & M_{23} & M_{24} & 0 & M_{26} \\
M_{31} & 0 & M_{33} & M_{34} & 0 & 0 \\
M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & 0 \\
M_{51} & 0 & 0 & 0 & 0 & M_{56} \\
0 & 0 & M_{63} & 0 & M_{65} & M_{66}
\end{pmatrix}
\]  

(4.2)

with

\[M_{11} = -2\lambda/[r(\lambda+2\mu)],\]
\[M_{12} = 1/(\lambda+2\mu),\]
\[M_{13} = n(n+1)\lambda/[r(\lambda+2\mu)],\]
\[M_{21} = -4\rho g/r + 4\mu(3\lambda+2\mu)/[r^2(\lambda+2\mu)],\]
\[M_{22} = -4\mu/[r(\lambda+2\mu)],\]
\[M_{23} = n(n+1)[\rho g/r - 2\mu(3\lambda+2\mu)/(r^2(\lambda+2\mu))],\]
\[M_{24} = n(n+1)/r,\]
\[M_{26} = -\rho,\]
\[M_{31} = -1/r,\]
\[M_{33} = 1/r,\]
\[M_{34} = 1/\mu,\]
\[M_{41} = \rho g/r - 2\mu(3\lambda+2\mu)/[r^2(\lambda+2\mu)],\]
\[M_{42} = -\lambda/[r(\lambda+2\mu)],\]
\[M_{43} = 2\mu[\lambda(2n^2+2n-1)+2\mu(n^2+n-1)]/[r^2(\lambda+2\mu)],\]
\[M_{44} = -3/r,\]
\[M_{45} = -\rho/r,\]
\[M_{51} = 4\pi G \rho,\]
\[M_{56} = 1,\]
\[M_{63} = -4\pi G \rho (n+1)/r,\]
\[M_{66} = -2/r,\]

and

\[\mathbf{D} = (0, g\Delta \rho, 0, 0, 0, -4\pi G \Delta \rho).\]  

(4.3)

Here \(\lambda\) = Lame constant, \(\mu\) = rigidity, \(g\) = gravitational acceleration, \(G\) = gravitational constant, \(\rho\) = density, and \(\Delta \rho\) = the density perturbations in which we are interested. In the case of the liquid core (\(\mu = 0\) and \(\Delta \rho = 0\)), Longman (1963) showed that equation (4.1) can be reduced to the following equations:

\[\rho g y_1 - y_2 - \rho y_5 = 0\]  

(4.4)

and

\[
d/dr \begin{pmatrix} y_5 \\ y_7 \end{pmatrix} = \begin{pmatrix}
0 & 1 \\
(n(n+1)/r^2 - 4\pi G \rho^2/\lambda - 2/r) & -2/r
\end{pmatrix} \begin{pmatrix} y_5 \\ y_7 \end{pmatrix}
\]  

(4.5)
The appropriate boundary conditions for the equilibrium state of the Earth are:

I—On the deformed surface of the Earth:

(i) Stress-free boundary condition,
\[ y_2 - g_0 \sigma = 0 \]  
(4.6)
and  
\[ y_4 = 0 \]  
(4.7)
where it is assumed that the present state of the Earth is its equilibrium state. In other words, \( \sigma \) includes the topography before deformation and the radial displacement of the Earth's surface after deformation.

(ii) Dirichlet boundary condition for the gravitational potential:
\[ y_5 = \phi' \]  
(4.8)
where \( \phi' \) is the corresponding spherical harmonic coefficient of the observed geopotential.

(iii) Neumann boundary condition for the gravitational potential (Pekeris & Jarosh 1958):
\[ y_i + (n+1)y_3 = 4\pi G\sigma. \]  
(4.9)

II—At the centre of the Earth (Kaula 1963):
\[ y_i = 0, \quad (i = 1, \ldots, 6) \]  
(4.10)
and  
\[ d/dr(y_i) = 0, \quad (i = 1, 3, 5) \]  
(4.11)

III—At the core–mantle boundary:

All the \( y \)'s are continuous except \( y_3 \) which is arbitrary because of the liquidity of the core.

Once \( y_1, y_3, \) and \( y_5 \) are known at the core–mantle boundary, other components of \( \mathbf{Y} \) can be obtained there, and equation (4.1) can be integrated if \( \Delta \rho \) is also known throughout the mantle. In such a case, the problem is overdetermined since there are four boundary conditions at the Earth's surface from which we should obtain the three parameters of the core–mantle boundary \( (y_1, y_3, \) and \( y_5 \)). However, \( \Delta \rho \) is unknown and we are, in fact, dealing with an underdetermined system. A geophysically interesting solution is the one which yields minimum stress differences in the mantle. Thus, if the mantle is subjected to the surface topography and the internal density anomalies, it will be in a static state. For the sake of simplicity, however, we minimize the total shear strain energy of the mantle. Although this minimization does not guarantee a local minimum stress difference, it reduces the magnitude of the stress differences within the whole Earth.

The total shear strain energy of the mantle is (Sokolnikoff 1956):
\[ E_{sh} = \int \mu(e_{ij} e_{ij} - \frac{1}{3} v^2) dv \]
\[ = 4\pi \int_b^a r^2 \mu dr \mathbf{Y}^T \cdot \mathbf{Y} \]  
(4.12)
where \( e_{ij} \) is the \((i, j)\)-th element of the strain tensor, \( b \) is the radius of the core–mantle
boundary, $Y^T$ is the transpose of $Y$, $v$ is the dilation, and

$$
\mathbf{P} = \begin{pmatrix}
  P_{11} & P_{12} & P_{13} \\
  P_{12} & P_{22} & P_{23} \\
  P_{13} & P_{23} & P_{33} \\
  0 & 0 & P_{44}
\end{pmatrix}
$$

(4.13)

with

$$
P_{11} = 2(3\lambda + 2\mu)^2/[3r^2(\lambda + 2\mu)^2],$$
$$P_{12} = -2(3\lambda + 2\mu)/[3r(\lambda + 2\mu)^2],$$
$$P_{13} = -n(n+1)P_{11}/2,$$
$$P_{22} = 2/[3(\lambda + 2\mu)^2],$$
$$P_{23} = -n(n+1)P_{12}/2,$$
$$P_{33} = 2n^2(n+1)^2(3\lambda^2 + 4\mu^2 + 6\mu\lambda)/[r^2(\lambda + 2\mu)^2] - n(n+1)/r^2,$$

and

$$P_{44} = n(n+1)/2^2.$$

In the numerical computations Longman’s (1963) procedure is employed. It is assumed that the Earth is a spherically layered body where the density perturbation inside each layer does not change with depth. The detailed formulation of the procedure is presented in the Appendix. As a test of our calculations, we solved Takeuchi, Saito & Kobayashi’s (1962) problem for the perturbations specified by the third-order harmonic, and obtained similar results.

The unknown density anomalies in which we are interested are determined for Gutenberg’s Earth model by adopting the following boundary conditions and constraints:

1. The gravitational potential at the surface of the deformed earth is assumed to be equal to the observed geopotential (Kaula 1967).
2. Lee & Kaula’s (1967) equivalent rock topography is used as the topography on the deformed earth.
3. The surface density model, computed in Section 2, is used for all the harmonics considered ($n = 2, \ldots, 6; m = 0, \ldots, n$).
4. For the low-order harmonics through the third order, the density variations of the upper mantle are taken to be those determined in Section 3, while those of the lower mantle are calculated through the procedure explained in this section. For the higher harmonics, however, we assume that the lower mantle is spherically symmetric, and the corresponding density variations are confined to the upper mantle and the surface layer. In this case, the density perturbations of the upper mantle are determined by the foregoing technique.
5. Both the total shear strain energy and the amplitudes of the density perturbations of the mantle, associated with a weighting factor, are minimized. If we minimize only the shear strain energy, the density anomalies would oscillate along the radius. Such a condition has also been observed by Kaula (1963, personal communication). The inclusion of the amplitude term in the minimization process, however, tends to eliminate the oscillatory feature.

The final model is selected by a trial and error variation of the weighting factor.
6. The region with unknown density perturbations is divided into $L$ layers with radially independent density anomalies.

Table 5 is the list of the density models calculated for different values of $L$ and $w_d$, where $w_d$ is the weighting factor of the amplitudes of the density perturbations. The variations of the total shear strain energy with $L$ and $w_d$ are illustrated in Figs 4 and 5. Included in the figures are the perturbations specified by zonal harmonics. The energy values are normalized to the energy associated with $L = 3$ and $w_d = 0$, respectively.

Models (2) and (8) are selected as the final density models of the mantle because: (1) they produce a nearly minimum total shear strain energy; (2) they just cease to

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{shear_strain_energy.png}
\caption{Total shear strain energy of the Earth versus $L$. Normalized for the energy corresponding to the case when $L = 3$.}
\end{figure}
show oscillatory feature with depth; and (3) they happen to divide the Earth into regions in agreement with other geophysical investigations (Chinnery & Toksöz 1967; Toksöz, Chinnery & Anderson 1967; Press 1968). The spherical harmonic coefficients of the models are tabulated in Table 6. Using these coefficients, the lateral variations of the final density anomalies of the upper mantle and the lower mantle are contoured in Figs 6 and 7, respectively. Fig. 8 shows their radial variations under shield ($\theta = 40^\circ$ and $\phi = 200^\circ$), tectonic ($\theta = 100^\circ$ and $\phi = 180^\circ$), and oceanic ($\theta = 50^\circ$ and $\phi = 200^\circ$) areas. The upper mantle is characterized by positive

**Table 5**

<table>
<thead>
<tr>
<th>Model</th>
<th>Density models</th>
<th>WD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-3 3</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>1-3 3</td>
<td>0.001*</td>
</tr>
<tr>
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<td>0.01</td>
</tr>
<tr>
<td>7</td>
<td>4-6 3</td>
<td>0.0</td>
</tr>
<tr>
<td>8</td>
<td>4-6 3</td>
<td>$10^{-5}$*</td>
</tr>
<tr>
<td>9</td>
<td>4-6 3</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>4-6 3</td>
<td>$5 \times 10^{-3}$</td>
</tr>
<tr>
<td>11</td>
<td>4-6 3</td>
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<td>14</td>
<td>4-6 5</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>15</td>
<td>4-6 7</td>
<td>$10^{-5}$</td>
</tr>
</tbody>
</table>

* = Selected models.
Lateral variations of density in the mantle

Table 6
Spherical harmonic coefficients of the density anomalies in the mantle (10^{-2} g cm^{-3})

<table>
<thead>
<tr>
<th>Depth (km)</th>
<th>Upper Mantle</th>
<th>Depth (km)</th>
<th>Lower mantle</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50-125</td>
<td>125-225</td>
<td>225-400</td>
</tr>
<tr>
<td>n m</td>
<td>Even Odd</td>
<td>Even Odd</td>
<td>Even Odd</td>
</tr>
<tr>
<td>2 0</td>
<td>2.175 0.0</td>
<td>2.175 0.0</td>
<td>2.175 0.0</td>
</tr>
<tr>
<td>2 1</td>
<td>0.377 1.483</td>
<td>0.377 1.483</td>
<td>0.377 1.483</td>
</tr>
<tr>
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<td>0.522 -1.132</td>
<td>0.522 -1.132</td>
<td>0.522 -1.132</td>
</tr>
<tr>
<td>3 0</td>
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<td>0.084 0.0</td>
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</tr>
<tr>
<td>3 1</td>
<td>0.326 -0.599</td>
<td>0.326 -0.599</td>
<td>0.326 -0.599</td>
</tr>
<tr>
<td>3 2</td>
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<td>-1.654 0.774</td>
<td>-1.654 0.774</td>
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<tr>
<td>3 3</td>
<td>0.146 1.007</td>
<td>0.146 1.007</td>
<td>0.146 1.007</td>
</tr>
<tr>
<td>4 0</td>
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<td>1.027 0.0</td>
<td>1.027 0.0</td>
</tr>
<tr>
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<td>0.925 -0.786</td>
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<tr>
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<td>-0.503 0.276</td>
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<td>-1.391 0.0</td>
<td>-1.391 0.0</td>
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<td>-0.060 0.066</td>
<td>-0.060 0.066</td>
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<tr>
<td>5 2</td>
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<td>-0.024 0.361</td>
<td>-0.024 0.361</td>
</tr>
<tr>
<td>5 3</td>
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<td>-0.672 0.878</td>
<td>-0.672 0.878</td>
</tr>
<tr>
<td>5 4</td>
<td>-0.612 -0.689</td>
<td>-0.612 -0.689</td>
<td>-0.612 -0.689</td>
</tr>
<tr>
<td>5 5</td>
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<td>1.231 0.598</td>
<td>1.231 0.598</td>
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<td>5 6</td>
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</tr>
<tr>
<td>5 7</td>
<td>-0.593 -0.212</td>
<td>-0.593 -0.212</td>
<td>-0.593 -0.212</td>
</tr>
<tr>
<td>5 8</td>
<td>-0.230 -0.883</td>
<td>-0.230 -0.883</td>
<td>-0.230 -0.883</td>
</tr>
<tr>
<td>5 9</td>
<td>0.247 0.959</td>
<td>0.247 0.959</td>
<td>0.247 0.959</td>
</tr>
<tr>
<td>6 0</td>
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<td>-0.054 0.017</td>
<td>-0.054 0.017</td>
</tr>
<tr>
<td>6 1</td>
<td>-0.034 0.0</td>
<td>-0.034 0.0</td>
<td>-0.034 0.0</td>
</tr>
<tr>
<td>6 2</td>
<td>-0.012 0.0</td>
<td>-0.012 0.0</td>
<td>-0.012 0.0</td>
</tr>
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<td>6 3</td>
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<td>-0.004 -0.003</td>
<td>-0.004 -0.003</td>
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<td>6 4</td>
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<td>-0.026 0.120</td>
<td>-0.026 0.120</td>
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<td>6 5</td>
<td>-0.011 0.0</td>
<td>-0.011 0.0</td>
<td>-0.011 0.0</td>
</tr>
<tr>
<td>6 6</td>
<td>-0.03 -0.162</td>
<td>-0.03 -0.162</td>
<td>-0.03 -0.162</td>
</tr>
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</table>

density anomalies under the shields and negative density anomalies under the oceans. The large density anomalies beneath Southwestern Africa may not be realistic. This is probably due to the unrealistic harmonic representation of the seismic travel-time residuals which are used as the inputs in the present analysis. The large change at 400 km depth is most likely due to the modelling effect. In general, the density perturbation decrease with depth. In the crust they are of the order of 0.3 g cm^{-3}, in the upper mantle 0.1 g cm^{-3}, and in the lower mantle 0.04 g cm^{-3}.

Figs 9 and 10 display the lateral variations of the radial displacements and the perturbations of the gravitational potential at 800 km depth. Their radial variations at specific latitudes and longitudes are illustrated in Fig. 11. It is evident from the figure that the shield areas are lifted up and the oceanic areas are depressed. But the maximum displacements, which occur at 800 km depth, indicate that, down to 800 km, the shield areas are contracted while the oceanic areas are expanded, and
from 800 km to the core–mantle boundary, the materials beneath the shields are expanded while those under the oceans are contracted.

The stress differences associated with the final density model are determined. Fig. 12 shows their lateral variations at 30 km depth, and it illustrates the strong correlation between the surface feature of the Earth and the maximum stress differences at shallow depths. In general, shield areas and oceanic basins are characterized by small stress differences, while the tectonic areas appear to have large stress differences. In the deep mantle, however, the effects of the surface features on the stress differences disappear.
Lateral variations of density in the mantle

Fig. 8. Radial variations of the density perturbations under shield, tectonic, and oceanic areas (g cm$^{-3}$).

Table 7

Correlation of Kaula's (1963) crustal model with the density of the surface layer

<table>
<thead>
<tr>
<th>$n$</th>
<th>Degree power g cm$^{-2}$</th>
<th>Correlation coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Surface layer</td>
<td>Kula's crust</td>
</tr>
<tr>
<td>2</td>
<td>0.0040</td>
<td>0.0006</td>
</tr>
<tr>
<td>3</td>
<td>0.0053</td>
<td>0.0015</td>
</tr>
<tr>
<td>4</td>
<td>0.0038</td>
<td>0.0013</td>
</tr>
</tbody>
</table>
It is worth mentioning that the real earth can not support such stress differences, and they will diminish through the anelastic deformations of the real earth unless there is a dynamic process that creates and/or maintains the density anomalies within the Earth. This subject will be discussed in a separate paper.

To compare our density model with Model 1 of Kaula (1963), the degree powers of Kaula's crustal density and the density of the surface layer presented in Section 2 are computed and listed in Table 7. It is evident from the table that the density anomalies of our surface layer are about twice as large as Kaula's crustal density variations. This table also includes the degree correlation coefficients, and displays a negative correlation for the second-order harmonics and positive ones for the higher harmonics. In the case of the mantle, Kaula has only listed the maximum values of the density perturbations which are about two orders of magnitude smaller than the ones listed in Table 6. This difference is due to: (1) the large upper mantle density anomalies deduced from p wave travel-time residuals; (2) the difference between his
Lateral variations of density in the mantle

Fig. 11. Radial variations of the radial displacements and the perturbations of the gravitational potential.

Fig. 12. Lateral variations of the stress differences at 30 km depth (bars).
polynomial model and the layered model adopted in the present study; and (3) the additional harmonics (fifth and sixth orders) included in this work.

Most recently Kaula (1969) concluded that the density variations of the order of $0.1 \text{ g cm}^{-3}$ exist in the upper mantle. This value agrees quite well with the density anomalies obtained by the present work.

5. Other geophysical data supporting our density model

In this section, we compare our selected density model with what can be deduced from seismic data.

Analysis of surface wave dispersion measurements has shown that the oceanic upper mantle shear wave velocity is about $0.3 \text{ km s}^{-1}$ less than the continental one (Dorman, Ewing & Oliver 1960; Takeuchi, Saito & Kobayashi 1962; and Toksoz, et al. 1967). Assuming that the variations of $p$ and $S$ wave velocities of the upper mantle are related by

$$\Delta V_p = \left[\frac{(2e-2)}{(2e-1)}\right]^\frac{1}{2} \cdot \Delta V_s,$$

(5.1)

where $e = 0.28$ is Poisson’s ratio, the corresponding lateral variation of $p$ wave velocity is found to be about $0.5 \text{ km s}^{-1}$. On the other hand, Hayles & Doyle’s (1967) empirical relationship for the upper mantle beneath the United States, 

$$\Delta V_p = 0.8 \Delta V_s$$

(5.2)

yields a value of about $0.2 \text{ km s}^{-1}$ for the variation of $V_p$. Therefore, lateral variations of approximately $0.3 \text{ km s}^{-1}$ seem to be plausible for a $p$ wave velocity in the upper mantle. Using Birch’s (1964) equation, these variations correspond to the density perturbations of about $0.1 \text{ g cm}^{-3}$ which agrees with the average density difference between the shield and the oceanic upper mantle obtained in our study.

Using seismic data Iyer, Pakiser & Stuart (1969) computed the average density distribution under the United States, down to 530 km through Birch’s (1964) relations between $p$ only wave velocity and density of the upper mantle. The results indicate that density in the upper mantle is less than that in Birch’s (1964) model, which is adopted as the standard density model in our studies. Our results, on the other hand, show that density in the upper mantle beneath the United States is greater than that of the standard model. The following points may serve to explain this discrepancy.

1. The United States is nearly as large as the smallest anomaly (sixth-order spherical harmonic) considered in our studies. We have confined such anomalies to the crust and to the upper mantle. Thus, in order to meet the observed gravitational field and to take into account the large negative gravitational potential of the crustal layer, it is necessary to have a positive density anomaly in the upper mantle (beneath the United States). This positive density anomaly can also be explained by the hypothesis of the crustal formation through the differentiation of the upper mantle material (Clark & Ringwood 1964).

2. Having a negative density anomaly in the upper mantle requires the existence of large positive density anomaly in the lower mantle in order to yield the observed geopotential. At the present stage of our understanding, it is difficult to accept such large magnitudes for the higher order (sixth-order spherical harmonics) variation of density in the lower mantle.

Very little study has been devoted to lateral variations in the lower mantle. From his studies of $dt/\Delta A$ curves of seismic arrivals at LASA, Montana, Fairborn (1968) deduced a value of about $0.1 \text{ km s}^{-1}$ for the lateral variations of the $p$ wave velocity at about 1900 km depth. Assuming that this velocity variation is due to the density variation, we determine the latter by Anderson’s (1967) equation of state,

$$\rho = 0.048 \bar{m} \phi^{0.323}$$

(5.3)
where
\[ \phi = V_p^2 - 4V_s^2/3, \] (5.4)
and \( \bar{m} \) is the mean atomic weight. Differentiating equation (5.3) with respect to \( V_p \) and at constant radius, we obtain
\[ \left( \frac{\partial \rho}{\partial V_p} \right)_r = 0.0155 \bar{m} \phi^{-0.677} \left( \frac{\partial \phi}{\partial V_p} \right)_r + 0.048\phi^{-3.23} \left( \frac{\partial \bar{m}}{\partial V_p} \right)_r. \] (5.5)

Setting \( \epsilon = 0.29 \) (Birch 1952), \( V_p = 12.8 \text{ km/s}^{-1}, \) \( V_s = 6.9 \text{ km/s}^{-1} \) (from the model used by Fairborn), and \( \bar{m} = 22, \) equation (5.5) is then reduced to
\[ \left( \frac{\partial \rho}{\partial V_p} \right)_r = 0.24 + 0.21 \left( \frac{\partial \bar{m}}{\partial V_p} \right)_r. \]

If we assume, furthermore, that there are no lateral variations of the chemical composition at 1900 km depth, \( (\partial \bar{m}/\partial V_p), \) would equal zero. In that case equation (5.6) yields a lateral density variation of \( 0.024 \text{ g cm}^{-3} \) at 1900 km depth. This variation is larger than the maximum value \( (= 0.01 \text{ g cm}^{-3}) \) obtained in our study. Thus, in both the upper and the lower mantles, our selected density anomalies are comparable to, or less than, those implied by the seismic observations.

6. Conclusion

The main interest in this study has been the determination of the lateral density variations in the mantle which give rise to a gravitational potential similar to the one deduced from artificial satellite data, and which also takes into account the lateral variations of crustal thickness and \( P \) wave travel-time residuals. We have been concerned with broad features specified by spherical harmonics through the sixth order. From this study the following conclusions are drawn:

1. Radial variations of the density models, computed by minimizing only the shear strain energy of the mantle, have oscillatory features. These oscillations disappear when we minimize both the shear strain energy and the amplitudes of the density perturbations.

2. The selected density model is characterized by a decrease with depth. A maximum value of about \( 0.3 \text{ g cm}^{-3} \) is found in the crustal region. The upper mantle density variations are within \( 0.1 \text{ g cm}^{-3} \), and the lower mantle ones are within \( 0.04 \text{ g cm}^{-3} \). These density variations are of the order of the values indicated by independent seismic studies.

Acknowledgment

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References


This appendix is devoted to the explanation of the procedure which we employ in order to compute the density perturbations of the mantle.

Before integrating equations (4.1) and (4.5), we normalize the variables by the following transformations (Longman 1963):

\[
\begin{align*}
\lambda_1 &= \lambda/\lambda^*, \\
d &= \Delta \rho/\rho^*, \\
z_5 &= y_5/\gamma_0^*, \\
\mu_1 &= \mu/\mu^*, \\
z_1 &= y_1/\gamma, \\
z_6 &= y_6/\gamma_0^*, \\
\rho_1 &= \rho_0/\rho_0^*, \\
z_2 &= y_2/\lambda^*, \\
z_7 &= y_7/\gamma_0^*, \\
g_1 &= g_0/\gamma_0^*, \\
z_3 &= y_3/\gamma, \\
C &= a \rho_0^* \gamma_0^*/\lambda^*,
\end{align*}
\]  

where the asterisk denotes the values at the Earth's surface. These transformations change equations (4.1-5) into the following forms, respectively:

\[
\frac{dZ}{dr} = M' Z + \Delta \rho_1 \cdot D_1, 
\]  

with

\[
M' = \begin{pmatrix}
M'_{11} & M'_{12} & M'_{13} & 0 & 0 & 0 \\
M'_{21} & M'_{22} & M'_{23} & M'_{24} & 0 & M'_{26} \\
M'_{31} & 0 & M'_{33} & M'_{34} & 0 & 0 \\
M'_{41} & M'_{42} & M'_{43} & M'_{44} & M'_{45} & 0 \\
M'_{51} & 0 & 0 & 0 & 0 & M'_{56} \\
0 & 0 & M'_{63} & 0 & M'_{65} & M'_{66}
\end{pmatrix},
\]

and

\[
M'_{11} = -2 \lambda_1/[r_1(\lambda_1 + 2 \mu_1)],
\]

\[
M'_{12} = 1/(\lambda_1 + 2 \mu_1),
\]

\[
M'_{13} = n(n+1) \lambda_1/[r_1(\lambda_1 + 2 \mu_1)],
\]

\[
M'_{21} = -4C \rho_1 g_1/r_1 + 4 \mu_1(3 \lambda_1 + 2 \mu_1)/[r_1^2(\lambda_1 + 2 \mu_1)],
\]

\[
M'_{22} = -4 \mu_1/r_1(\lambda_1 + 2 \mu_1),
\]

\[
M'_{23} = n(n+1)[C \rho_1 g_1/r_1 - 2 \mu_1(3 \lambda_1 + 2 \mu_1)/(r_1^2(\lambda_1 + 2 \mu_1))],
\]

\[
M'_{24} = n(n+1)/r_1,
\]

\[
M'_{26} = -C \rho_1,
\]

\[
M'_{31} = -1/r_1,
\]

\[
M'_{33} = 1/r_1,
\]

\[
M'_{34} = 1/\mu_1,
\]

\[
M'_{41} = C \rho_1 g_1/r_1 - 2 \mu_1(3 \lambda_1 + 2 \mu_1)/[r_1^2(\lambda_1 + 2 \mu_1)],
\]

\[
M'_{42} = -\lambda_1/[r_1(\lambda_1 + 2 \mu_1)],
\]
The boundary conditions take the following form:

1. At the surface of the Earth, $r_1 = 1$,

$$S = \begin{pmatrix} z_2 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} -g_0 \sigma / \lambda^* \\ 0 \\ 4\pi G \sigma / g_0^* \\ -(n+1)z_5 \end{pmatrix}, \quad (A.7)$$

2. At the centre of the Earth, $r_1 = 0$,

$$z_i = 0, \quad (i = 1, \ldots, 6) \quad (A.8)$$

$$dz_i / dr_1 = 0, \quad (i = 1, 3, 5) \quad (A.9)$$

3. At the core–mantle boundary, $r_1 = b/a$,

$$\begin{array}{c|c}
\text{core} & \text{mantle} \\
\hline
z_1 & z_1 = \alpha \\
z_2 & z_2 \\
\hline
\text{arbitrary} & z_3 = \beta \\
0 & 0 \\
z_5 & z_5 \\
z_7 - B\rho_1 z_1 & z_6 \\
\end{array} \quad (A.10)$$

The total shear strain energy is now expressed by

$$E_{sb} = 4\pi \int_{b/a}^{1} r_1^2 dr_1 \mu_1 \mathbf{Z}^T \mathbf{P}_1 \mathbf{Z} \quad (A.11)$$

where

$$\mathbf{Z} = (z_1, z_2, z_3, z_4, z_5, z_6) \quad (A.12)$$

and $\mathbf{P}_1$ has the same expression as $\mathbf{P}$ with all the parameters having subscripts 1.
Integration in the core: At the Earth's centre $z_5$ and $z_7$ vanish while some of the coefficients appearing in equation (A.6) are infinite. To start the computation with non-zero values of $z_5$ and $z_7$, and with finite values of the coefficients, we introduce a homogeneous sphere of radius $\varepsilon$ (= 200 km) at the centre of the Earth, the central sphere. On the surface of this sphere $z_5$ and $z_7$ are expressed in terms of the power series of $\varepsilon_1$ (= $\varepsilon/a$). The coefficients appearing in the series are determined within an unknown factor ($q_0$) through equations (A.5) and (A.6) (Longman 1963). We then obtain the values of $z_5$ and $z_7$ at the core–mantle boundary by integrating equation (A.5), using the matrizant method:

$$Z'_{r_1=b/a} = \Omega^{b/a}_{z_5} \cdot Z'_{r_1=\varepsilon_1}$$  \hspace{1cm} (A.13)

where $\Omega^{b/a}_{z_5}$ is the matrizant of the core (between $r_1 = \varepsilon_1$ and $r_1 = b/a$), and

$$Z' = \begin{pmatrix} \frac{z_5}{z_7} \end{pmatrix} / q_0.$$  \hspace{1cm} (A.14)

From the boundary conditions at the core–mantle boundary, and equation (A.5), the value of $Z$ is calculated at that boundary and inside the mantle by

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ C\rho_1 g_1 & 0 & -C\rho_1 \frac{z_5'}{q_0} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{z_5'}{q_0} \\ -B\rho_1 & 0 & \frac{z_7'}{q_0} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ q_0 \end{pmatrix} = J \cdot X'$$  \hspace{1cm} (A.15)

where the elements of vector $X'$ (= $\alpha$, $\beta$, and $q_0$) are unknown.

Integration in the mantle: We divide the mantle into $K$ layers, Fig. 13, thin enough so that in each layer $M'$ and $D_1$ can be considered as constants. Then we compute the value of $Z$ at the top of the $i$-th layer, $Z_i$, in terms of the matrizant of the medium located between the core–mantle boundary and the top of the $i$-th layer, $\Omega_i^{j, i}$, the value of $Z$ at the core–mantle boundary, $Z_0$, and the effect of the source vector of the $j$-th layer, located in this medium, on the $Z_i$, $r_j'$, from the following equation (Gantmacher 1960):

$$Z_i = \Omega_0^{i, i} Z_0 + \sum_{j=1}^{i} \Gamma_j^{i, j},$$  \hspace{1cm} (A.16)

where $\Gamma_j^{i, j}$ is determined through

$$\Gamma_j^{i, j} = \frac{i}{r_i (j+1)} \cdot \int_{r_i (j-1)}^{r_i (j+1)} \Omega_t^{j, j} \Delta \rho_1 (\tau) D_1 (\tau) d\tau.$$

We make the following assumptions in order to obtain a general form of the solution in the mantle:

1. All the layers from $i = 1$ to $i = n_1$ have no density anomalies;
2. The layer from $i = n_1 + 1$ to $i = n_2$ have some unknown density anomalies; and
3. The rest of the layers, $i = n_2 + 1$ to $i = K$, have some known density anomalies.

The known density anomalies are the crustal and the upper mantle density perturbations obtained in Sections 2 and 3.
Let

\[ \eta = (\Delta \rho_1(n_1 + 1), \ldots, \Delta \rho_1(n_2)), \]  
(A.17)

\[ D = (\Delta \rho_1(n_2 + 1), \ldots, \Delta \rho_1(K)), \]  
(A.18)

\[ X = (X', \eta) \]  
(A.19)

\[ U_j = [\Omega_0^i \cdot J: \Gamma_{n_1 + 1}^i : \ldots : \Gamma_j^i], \quad j \leq n_2 \]  
(A.20)

and

\[ U' = [\Gamma_{n_1 + 1}^i : \ldots : \Gamma_f^i], \]  
(A.21)

where \( j > n_2 \). Using these definitions together with equation (A.15), equation (A.16) is reduced to the following form:

\[ Z_i = \begin{cases} U(i) \cdot X, & i \leq n_2 \\ U(i) \cdot X + U'(i) \cdot D, & i > n_2 \end{cases} \]  
(A.22)

(A.23)

and the boundary conditions at the Earth’s surface are:

\[ S = F \cdot Z + F' \cdot D. \]  
(A.24)

Here \( S, F, \) and \( F' \) are found by omitting the first and the third rows of \( Z, U, \) and \( U' \), respectively.
Minimization of the total shear strain energy and the amplitudes of the density perturbations: To parameterize the problem, we apply a weighting factor to the amplitude term. Thus, the function to be minimized is:

\[ I = E_{sh} + w_d \sum_{j=n_1+4}^{j=n_2} x_j^2 \]

\[ = (X^T W_1 X + X^T W_2 D + D^T W_2^T X + D^T W_3 D) + \]

\[ + (X^T X - X'X').w_d. \quad (A.25) \]

Here

\[
\begin{pmatrix}
W_1 \\
W_2 \\
W_3 
\end{pmatrix} = \sum_{i=1}^{K} \left( r_i^3(i) - r_i^3(i-1) \right) / 3.
\]

\[
(U^T Q U) \quad (U^T Q U')
\]

\[
(U^T Q U')
\]

and

\[ Q = 2\pi \mu_1 (i)(P_1 (i) + P_1 (i-1)) \quad (A.27) \]

minimization of \( I \), subjected to the constraints given by equation (A.24), is a generalized least square problem which can be put into the following simultaneous first-order linear equations by adopting the method of Lagrangian multipliers:

\[
\begin{pmatrix}
W_1' \\
F\\
0
\end{pmatrix} \cdot \begin{pmatrix}
X \\
L
\end{pmatrix} = \begin{pmatrix}
-W_2 D \\
(P-F') D
\end{pmatrix} = \begin{pmatrix}
V_1 \\
V_2
\end{pmatrix},
\]

where

\[ W_1'(i, j) = W_1(i, j) + \begin{cases} 
2, & n_2 \geq (i = j) \geq n_1 + 3 \\
0, & \text{otherwise}
\end{cases} \]

and \( L \) is the Lagrangian multiplier. The solution vector \( X \), is given by (Arley & Buck 1950)

\[
X = W_1''^{-1} V_1 - W_1''^{-1} F (F W_1''^{-1} F)^{-1} F W_1''^{-1} V_1
\]

\[ + W_1''^{-1} F (F W_1''^{-1} F)^{-1} V_2. \quad (A.29) \]