On the Tests for Gravitational Theories in Terms of an Artificial Satellite

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We examined whether or not Singer's test and others, if any, would be useful to discriminate between general relativity and other theories of gravitation. Emphasis was laid upon whether curvedness of a space-time could be detectable or not by means of these tests.

§ 1. Introduction

The general theory of relativity is ordinarily regarded as the firmly established classical field theory of gravitation.* According to this theory the metric tensor of a curved space-time is nothing but the gravitational potential in such a way that the motion of a test particle is described by a geodesic of the space-time. Contrary to this idea of Einstein, Gupta† proposed another view that Einstein's field equations should be reinterpreted as the non-linear field equations in the flat space-time. Then, has the general theory of relativity been verified by the literally well established experimental or observational facts? Especially is it the case for Einstein's idea of geometrization of gravitation? As is well known, the observational basis of this theory consists simply in the so-called three tests in the solar gravitational field, but phenomena like the gravitational wave have not yet been discovered, although the theory offered a reasonable interpretation of the equivalence** between inertia and gravitation. In addition, the tests in question are not necessarily crucial, apart from the priority of Einstein for the explanation*** and prediction of phenomena. It is because these phenomena can be equally explained in terms not only of the above stated interpretation due to Gupta, but also of different gravitational theories†† based on the flat space-time. It is very desirable, therefore, that another test is found so that we can predict some result which is different from the one due to general relativity, by using other theories of gravitation permitting us to explain the three tests.

* Except for the standpoint on which gravitation is regarded as a secondary effect of another field‡‡, say, neutrino or electromagnetic field, we may say that most of the recent attempts to quantize the gravitational field stand on this viewpoint‡‡.

†† There is a different opinion due to Dicke†§ concerning the limit of the validity of the principle of equivalence.

*** of the anomalous advance amounting to 43″ per century of the perihelion of Mercury, for which Newtonian theory was incompetent.
Singer\textsuperscript{5}) has recently proposed a method of measuring the ratio \( \mathcal{J} = (dt / dt_0 - 1) \) between the rate of a clock on an artificial satellite and that of a similar clock on the earth, and he has pointed out that this gives the fourth test of the general theory of relativity. According to him, the above \( \mathcal{J} \) is given by the following expression:

\[
\mathcal{J} = (Gm_{\text{at}} / c^2 R_0) \left\{ (0.5 - h) / (1 + h) \right\}
\]

\[
\cong 6.96 \times 10^{-10} (0.5 - h) (1 + h)^{-1}, \tag{1.1}
\]

where \( h \) is the height of the artificial satellite in the unit of the radius of the earth \( R_0 \), the other symbols being as usual. Though Singer’s test is like that of the red-shift of the spectral lines in the gravitational field of the sun or white dwarfs, there is a considerable difference; i.e., contrary to the latter which is strongly influenced by the Doppler effect due to thermal and turbulent motions of stellar atmospheres, such a complexity of the situation may not arise in the former case.\textsuperscript{*}

Singer’s formula (1.1) was, however, derived under the assumption that (1) the gravitational field of the earth can be represented by Schwarzschild’s space-time and (2) the orbit of a satellite is circular. Since not only the earth is rotating, but has ellipticity, it is necessary to investigate whether the first of the above assumptions is correct or not. But as Hoffmann\textsuperscript{6}) and Das\textsuperscript{7}) showed, the effects due to the rotation and ellipticity of the earth are revealed by small quantities of higher order, and in addition the effects due to them have tendency to cancel each other.

In this paper, therefore, we shall investigate at first the influence of the eccentricity and the phase (specified by the eccentric anomaly), after extending Singer’s formula (1.1) to the case of an elliptic orbit. Next we shall examine whether the formula thus obtained is characteristic of the general theory of relativity in the view-point of Einstein himself. For this purpose, we shall attack the same problem from the standpoint of different gravitational theories due to, say, Whitehead and Birkhoff.

Moreover we shall inspect another conceivable test in terms of geodesic deviation instead of geodesic motion. (In the case of an artificial satellite, the equation of geodesic deviation can be interpreted as representing the relative motion of the cap to the main body.) Because this equation includes the curvature tensor \( R_{\alpha \beta \gamma \delta} \), explicitly, the value of which tells us directly the curvedness of a space-time.

§ 2. Geodesic equation in Schwarzschild’s space-time

As is well known, the line-element of Schwarzschild’s space-time is given by:

\[
ds^2 = c^2 (1 - 2Gm / c^2 r) dt^2 - dr^2 / (1 - 2Gm / c^2 r) - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{2.1}
\]

If we choose the coordinate system so that a test-particle moves initially in the plane \( \theta = \pi / 2 \), the geodesic equations in this space-time are

\textsuperscript{*} See, for instance, p. 12 of Singer’s paper cited as reference 5) in this paper.
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\[ d\tau = \frac{d\varphi}{hu^2(\varphi)}, \]  

\[ (du/d\varphi)^2 + u^2 = c^2(\frac{1}{h^2} - \frac{1}{b^2}) + \frac{(2Gm/b^2)u + (2Gm/c^2)u^3}{h^2}, \]  

\[ dt/d\tau = \beta (1 - 2Gmu/c^2)^{-1}, \]

where \( u = 1/r \), \( d\tau = ds/c \), \( h \) and \( \beta \) are integration constants. The third term on the right-hand side of the orbital equation (2·3) is the well known term in terms of which the advance of the perihelion of Mercury is explained.

In Singer's problem, however, it is sufficient to take Newtonian approximation for the calculation of the orbit of the artificial satellite, i.e., we can adopt

\[ (du/d\varphi)^2 + u^2 = c^2(\beta^2 - 1)/b^2 + \frac{(2Gm/b^2)u}{h^2}, \]  

in place of (2·3). Then from (2·2) and (2·3') we can obtain for an elliptic orbit,

\[ u = \frac{(1 + e \cos \varphi)}{a(1 - e^2)} = 1/a(1 - e \cos E), \]  

\[ n\tau = E - e \sin E, \quad \tan (\varphi/2) = \sqrt{(1 + e)/(1 - e)} \tan (E/2), \]  

\[ \beta^2 = 1 - Gm/c^2a \]

with usual notations; here we assume \( \varphi = 0 \) at \( \tau = 0 \) for convenience.

Though it is sufficient to take the above approximation in the calculation of the orbital motion of the artificial satellite, the essence of Singer's problem lies in considering the deviation of \( dt/d\tau \) from unity. Accordingly, inserting (2·5) and (2·7) into the right-hand side of (2·4), and neglecting the terms higher than \( (Gm/c^2a)^3 \) compared with unity, we obtain

\[ dt/d\tau = 1 - \frac{Gm}{c^2a} \left[ \frac{2}{1 - e \cos E} - \frac{1}{2} \right]. \]

Considering (2·6), we can integrate the above equation as follows:

\[ t = \tau - (Gm/c^2a) \left( 1/2n \right) (3E + e \sin E). \]

§ 3. Generalization of Singer's formula for \( \mathcal{J} \)

As was shown by Hoffmann, effects due to the rotation and ellipticity of the earth can be ignored. So we shall assume that the gravitational field of the earth can be expressed by Schwarzschild's space-time. Then we can take the value of \( m_{\oplus} \) as the value of \( m \) appearing in § 2. Accordingly we obtain from (2·8):

\[ \frac{dt}{d\tau} = 1 + \frac{Gm_{\oplus}}{c^3R_{\oplus}} \left( \frac{1 + h}{2} \right)^{-1} \left( \frac{3 + e \cos E}{1 - e \cos E} \right), \]

where \( a = (1 + h)R_{\oplus} \), \( R_{\oplus} \) = the radius of the earth, \( t \) is the reading of the clock on the artificial satellite. On the other hand the expression of the reading \( t_{\oplus} \) of the clock which is at rest on the earth can be given, from (2·1) with \( m = m_{\oplus} \) by:
From the above two expressions, we calculate $J$ corresponding to (1·1) as:

$$
J = (6.96 \times 10^{-10}) \left[ (\psi(e; E) - h) / (1 + h) \right]
$$

(3·3)

with

$$
\psi(e; E) = (1/2) \left( 1 + 3e \cos E \right) / (1 - e \cos E).
$$

(3·4)

Especially when $e=0$ (circular orbit) $\psi(0; E)=0.5$, so that (3·3) reduces to Singer's formula (1·1).

Now for brevity's sake, we assume that an observer on the earth is situated on the orbital plane of the artificial satellite, i.e. on the equatorial plane of the earth. Then, the conditions securing him to observe the artificial satellite are reduced to the one that the perigee distance $a(1-e)$ is larger than the radius of the earth $R_e$, i.e. $h$ must satisfy the following inequality:

$$
h \geq e / (1-e).
$$

(3·5)

Next we shall study how $J$ depends on the eccentricity $e$ and the eccentric anomaly $E$ of the orbit of the artificial satellite. As is easily seen from (3·3), the sign of $J$ depends on the magnitude of $\psi(e; E)$ compared with $h$. $\psi(e; E)$ is, however, such a periodic function of $E$ that it decreases monotonously in the phase $(0, \pi)$ and increases in $(\pi, 2\pi)$, if the value of $e$ is fixed. Thus we obtain from (3·4):

$$
\begin{align*}
\psi_{\text{max}} &= \psi(e; 0) = \frac{1}{2} \left( 1 + 3e \right), \\
\psi_{\text{int}} &= \psi(e; \pi/2) = 0.5, \quad \text{irrespective of the value of } e, \\
\psi_{\text{min}} &= \psi(e; \pi) = \frac{1}{2} \left( 1 - 3e \right).
\end{align*}
$$

(3·6)

As is easily seen, when $e \neq 0$, $\psi_{\text{int}}=0.5$ is splitted up and down into $\psi_{\text{max}}$ and $\psi_{\text{min}}$ respectively. And the sign of $\psi_{\text{int}}$ depends on the value of $e$, i.e. $\psi_{\text{int}} \geq 0$ when $e \leq 1/3$.

From the above, signs of $J$ can be specified as follows:

$$
\begin{align*}
J &> 0 \quad \text{(red-shift), when } h < \psi_{\text{min}} \quad \text{(if } e < 1/3), \\
J &< 0 \quad \text{(blue-shift), when } h > \psi_{\text{max}}.
\end{align*}
$$

(3·7)
And when \( \psi_{\text{max}} > h > \psi_{\text{min}} \), the sign of \( J \) varies as the phase of \( E \). As an example we illustrate such a situation for the case \( e < 1/3 \) in Fig. 1.

§ 4. Gravitational theories based on the flat space-time due to Whitehead and Birkhoff

In this section we shall deal with Whitehead's theory* and Birkhoff's one* as representative theories which are based on the flat space-time and at the same time by which we can interpret the so-called three tests in the solar gravitational field similarly to the general theory of relativity.

(W-theory). This is an action-at-a-distance theory obtained by reforming Newton's inverse-square law of gravitation in a Lorentz covariant fashion. Therefore this must be automatically rejected when the quantization of the gravitational field is concerned. However, no theory would be better than this, if we could test the appropriateness of the theory without considering the problem of quantization. Meanwhile, Schild\(^{10}\) showed that we could construct various kinds of action-at-a-distance theories of Whitehead's type by starting from Schwarzschild's line-element. The essential point of his procedure consists in transforming the geodesic equation by the use of quasi-Newtonian coordinate systems and making each term in the equation thus transformed correspond to the Lorentz covariant quantity in the flat space-time. In addition, he asserted that, contrary to general relativity, these theories predict in general the presence of a secular acceleration\(^{**}\) of the center of mass in two-body problem.

(B-theory). Contrary to the W-theory, this is a kind of tensorial gravitational field theory. Of this theory, however, there exists Weyl's\(^{7}\) critique, and further Gupta\(^{10}\) remarked that difficulty appears in the quantization problem, i.e. the energy of the gravitational field does not take a positive definite value. But it is to be noticed that these critiques are concerned not with pointing out the discrepancy of the theoretical consequences from the observational evidences, but with formalistic points and quantization problem. The original interpretation of the red-shift of spectral lines is, however, \textit{ad hoc} (due to the energy loss of photon). But this was explained by Moshinsky\(^{11}\) more naturally by taking account of the gravitational effect on the Maxwell and Dirac equations. Moreover, it is to be remarked that the secular acceleration of the center of mass of two bodies takes place also in this theory (just the same form as the one derived erroneously by Levi Civita in general relativity).

In the following we shall study the corresponding formulae of these theories to (3·3) with (3·4) derived by the general theory of relativity.

§ 5. One-body problem in W-theory

Now we shall consider one-body problem which corresponds to Schwarzschild's solution in general relativity. Taking the coordinate system with origin at the gravitating

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* In the following we shall designate them as W- & B-theories, respectively.
** It is very difficult, however, to judge whether or not such a secular acceleration exists from the observation of binary stars.
mass, we can describe the Lagrangian of a test particle in the gravitational field from which the equation of motion can be derived:

$$2L = \left(1 - \frac{2Gm}{c^2 r}\right)^2 + \frac{4Gm}{c^2 r} \frac{\dot{r}}{r} - \frac{1}{c^2} \left[\left(1 + \frac{2Gm}{c^2 r}\right)^2 + r^2 \left(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2\right)\right], \quad (5.1)$$

where a dot denotes differentiation with respect to parameter $\tau$, and the limitation to $\tau$ is specified by the condition $2L=1$.

If we take $\theta = \pi/2$ as in § 2, the Eulerian equation of (5.1) becomes

$$d\tau = d\varphi / hu^2 (\varphi), \quad (5.2)$$

$$(du/d\varphi)^2 + u^2 = c^2 (\beta^2 - 1) / h^2 + (2Gm/h^2) u + (2Gm/c^2) u^3, \quad (5.3)$$

$$dt/d\tau = \left[\beta + \frac{2Gmb}{c^2} \frac{du}{d\varphi}\left(1 + \frac{2Gm}{c^5} u\right)^{-1}\right], \quad (5.4)$$

where $u = 1/r$, $h$ and $\beta$ are integration constants the meanings of which are the same as in § 2.

As is easily seen, if we ignore the difference in the physical meaning of $\tau$, (5.2) and (5.3) are of the same form as (2.2) and (2.3), and (5.4) is obtained from (2.4) by means of putting $\{ \}$ in place of $\beta$. Moreover, as in § 2, it is allowed in the Newtonian approximation to put $t = T$ in the above equation. Then inserting (2.5)-(2.7) in § 2 into (5.4) and neglecting the terms higher than $(Gm/c^2 a)^2$, we obtain:

$$dt/d\tau = 1 + \frac{Gm}{c^2 a} \left[\frac{2}{1 - e \cos E} - \frac{1}{2}\right] + (\pm) \left(\frac{Gm}{c^2 a}\right)^{3/2} \frac{2e \sin E}{(1 - e \cos E)^2}, \quad (5.5)$$

where $(\pm)$ denotes the signs of $h$. On the other hand, for the test particle which is at rest in the gravitational field, we get from (5.1)

$$dt/d\tau = 1 + (Gm/c^2 R), \quad (5.6)$$

where $r = R = \text{const}$.

Applying the above two equations to the problem of the artificial satellite, we obtain the expression of $J$ with the same procedure as in § 3 as follows:

$$J = 6.96 \times 10^{-10} \frac{\psi(e; E) - h}{1 + h} - (\pm) \frac{1.54 \times 10^{-14} (1 + h)^{-3/2} 2e \sin E}{(1 - e \cos E)^2}, \quad (5.7)$$

where $\psi(e; E)$ is given by (3.4). Comparing the above formula with (3.3), we find that the second term on the right-hand side is a correction term. But we can neglect this term, because it is very small compared with the first term and, furthermore, it vanishes when $e = 0$. Therefore (5.7) and (3.3) are practically equivalent.

Now we shall show that such a similarity of (5.7) to (3.3) is not accidental. Putting $d\tau = ds/c$, we obtain from (5.1)

$$ds^2 = c^2 \left(1 - \frac{2Gm}{c^2 r}\right)dt^2 + \frac{4Gm}{c r} dr dt - \left(1 + \frac{2Gm}{c^2 r}\right)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (5.8)$$

Reinterpreting parameter $\tau$ as proper time in general relativity, we can regard the above
expression as the line-element of a spherically symmetric space-time. Then, after the following coordinate transformation

$$i = t + \frac{2Gm}{c^2} \log \left( \frac{c^2 r}{2Gm} - 1 \right),$$

(5.9)

(5.8) reduces to

$$ds^2 = c^2 \left( 1 - \frac{2Gm}{c^2 r} \right) dt^2 - \frac{dr^2}{1 - \frac{2Gm}{c^2 r}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(5.10)

which is nothing but Schwarzschild's line-element. (cf. (2.1)). In short the above procedure is reciprocal to Schild's. And, if we interpret (5.7) from the standpoint of general relativity, the appearance of the second term on the right-hand side is due to the fact that the coordinate time $t$ specified by (5.9) is used in place of $i$.

§ 6. One-body problem in B-theory

While the one-body problem in W-theory has an intimate connection with Schwarzschild's space-time, the situation is different with the corresponding problem in B-theory:

1. The trajectories of test particles are not geodesics in any curved space-time, i.e. the specification of the equations of motion of a test particle is made under another proposal.*

2. Contrary to W-theory, the following relation is an integral of the equation of motion:

$$1 = \left( \frac{dt}{d\tau} \right)^2 - \left( \frac{dr}{d\tau} \right)^2 - r^2 \left( \frac{d\theta}{d\tau} \right)^2 + r^2 \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 e^{-u},$$

(6.1)

specifying the flat space-time.

Now, the explicit form of the equations of motion is as follows when $\theta = \pi/2$:

$$dt = \frac{d\phi}{hw^2(\phi)} \exp \left( \frac{2Gm}{c^2} u \right),$$

(6.2)

$$(du/d\phi)^2 + u^2 = (c/h)^2 \exp \left[ 2Gmu/c^2 \right] \left[ \exp \left[ 2Gmu/c^2 \right] - \beta^{-2} \right],$$

(6.3)

d$\tau/d\tau = \beta \exp \left[ Gmu/c^2 \right]$ (6.4)

where $u = 1/r$, $h$ and $\beta$ are integration constants.

Comparing (6.2)-(6.4) with (2.2)-(2.4), we can easily understand their differences. Nevertheless, it is to be remarked that the advance of the perihelion of Mercury and the deflection of light ray at solar limb can be derived without ad hoc hypothesis.

Then, what is the matter with Singer's test? Inserting the Newtonian approximate

* See the article of Barajas 12. When we confine ourselves to the one-body problem the equations of motion can be derived from the following variational principle:

$$\delta \left[ \exp \left( -\frac{2Gm}{c^2 r} \right) (c^2 + v^2) \right] dt = 0,$$

which is the same as that proposed by A. G. Walker 13 from the standpoint of Milne's kinematic relativity.
solutions \((2\cdot5)-(2\cdot7)\) of \((6\cdot2)\) and \((6\cdot3)\) into \((6\cdot4)\) and neglecting higher order terms, we obtain

\[
\frac{dt}{d\tau} = 1 + \frac{Gm}{c^2a} \left[ \frac{1}{1 - e \cos E} - \frac{1}{2} \right]. \tag{6\cdot5}
\]

On the other hand, considering that \((6\cdot1)\) is an integral of the equation of motion, we obtain the following expression concerning the particle at rest in gravitational field:

\[
\frac{dt}{d\tau} = 1. \tag{6\cdot6}
\]

Comparing the above expressions with \((3\cdot1)\) and \((3\cdot2)\) or \((5\cdot5)\) and \((5\cdot6)\), we can find a remarkable difference. In fact, if we derive the corresponding formula to \((3\cdot3)\) from the above, it follows:

\[
\mathcal{A} = 6.96 \times 10^{-10} \left[ \frac{1 + e \cos E}{1 - e \cos E} \right] \left( 1 + \frac{\mu^2}{2} \right), \tag{6\cdot7}
\]

which means that \(\mathcal{A}\) is always positive for any \(e\) and \(E\). Thus, at first sight Singer’s test would be useful to make discrimination between B-theory and general relativity.

It is evidently the case so far as we take the standpoint on which the red-shift of spectral lines is interpreted as caused by the energy loss of photon. Because Singer’s test is to directly read the number (counted by a scaler) of ticks of the clock, contrary to the usual measurement of the shift of spectral lines. But it is more appropriate to stand on the viewpoint due to Moshinsky as for the problem of red-shift (cf. § 5). On this viewpoint, however, the formula of red-shift can be represented as follows:

\[
\frac{dt'}{dt} = 1 + \frac{Gm}{c^2}, \tag{6\cdot8}
\]

where \(t'\) and \(t\) are times based on the frequency of spectral line in the presence of and in the absence of the gravitational field, respectively.

The atomic clock is, however, necessary to carry out Singer’s test actually. But the time by the atomic clock is based on the unit defined by the reciprocal of frequencies of atomic or molecular lines. Therefore, the relation \((6\cdot8)\) must be taken into account in Singer’s problem. Then the following relations must be used instead of \((6\cdot5)\) and \((6\cdot6)\):

\[
\frac{dt'}{d\tau} = \frac{dt'}{dt} \cdot \frac{dt}{d\tau} = 1 + \frac{Gm}{c^2a} \left[ \frac{2}{1 - e \cos E} - \frac{1}{2} \right], \tag{6\cdot9}
\]

\[
\frac{dt'}{d\tau} = 1 + \frac{Gm}{c^2R}, \tag{6\cdot10}
\]

where \(R\) is constant. Comparing these equations with \((3\cdot1)\) and \((3\cdot2)\), we can find that they are completely the same. Consequently the corresponding formula to \((3\cdot3)\) is identical with \((3\cdot3)\) itself.

Lastly it must be pointed out that the above procedure specified by \((6\cdot8)\) is also applied to Gupta’s formalism mentioned before (cf. § 1).
§ 7. Geodesic deviation in Schwarzschild’s space-time and its corresponding one in B-theory

As is well known, whether a space-time is flat or not depends not on the values of metric tensor $g_{\mu\nu}$ and its first derivative but only on the fact whether curvature tensor $R_{\alpha\beta\gamma\delta}=0$ or $\neq 0$. For this reason any prediction made by the geodesic equation only does not tell us directly whether the space-time is curved or not, because in this equation $g_{\mu\nu}$ and at most its first derivative are contained. In this respect, Pirani emphasized in his theory of gravitational radiation an importance of the following equation of geodesic deviation:

$$\frac{\partial^2 \gamma^\mu}{\partial \tau^2} + R^\mu_{\alpha\beta\gamma} \lambda^\alpha \gamma^\beta \lambda^\gamma = 0,$$  \hspace{1cm} (7·1)

where $\partial$ denotes covariant derivative, and $\lambda^\alpha$ is a tangential vector along one geodesic such that $\lambda_a \lambda^a=1$ and $\gamma^a$ is a vector perpendicular to $\lambda^a (\lambda_a \gamma^a=0)$.

Then does the equation (7·1) play a significant role also in a static gravitational field such as Schwarzschild’s? Unfortunately it would no be the case, since there exist the following situations: (i) In contrast with the gravitational radiation whose essence consists in the appearance of the discontinuity of $R_{\alpha\beta\gamma\delta}$, there is no such discontinuity in Schwarzschild’s space-time, (ii) We can only estimate the mixture of the second term on the left-hand side of (7·1) and $\sum (\partial^2 \gamma^\mu / \partial \tau^2 - d^2 \gamma^\mu / ds^2)$, which is of the same order as the former, i.e. we cannot separate the former term from the latter in practice.

Moreover, almost the same equation as (7·1) would be derived also from, say, B-theory, if we take account of the physical meaning of (7·1). For this equation can be interpreted as expressing the relative motion of one test particle on a geodesic line to another one on the neighbouring geodesic line (such a relative motion can be treated even from the standpoint of Newtonian dynamics).

In the following we shall clarify that such a conjecture is valid in the spherically symmetric and static gravitational field, assuming one of geodesic curves as a circular orbit (for the sake of mathematical simplicity).

(General relativity). The solution of (2·2)-(2·4) for a circular orbit is as follows:

$$\frac{d\varphi}{dt} = \left( \frac{Gm}{r_\circ^3} \right)^{1/2}, \quad \frac{dt}{d\tau} = \left( 1 - \frac{3Gm}{c^2 r_\circ} \right)^{-1},$$  \hspace{1cm} (7·2)

where $r_\circ=\text{const}$. Considering (2·1) and (7·2), we obtain the following after calculating (7·1):

$$X + n_\circ (1 - \frac{6Gm}{c^2 r_\circ}) \dot{X} = 0,$$

$$\ddot{X} + 2n_\circ \left( 1 - \frac{2Gm}{c^2 r_\circ} \right)^{1/2} \dot{X} = 0,$$  \hspace{1cm} (7·3)

$$\dot{Y} + n_\circ^2 Y = 0.$$
where \((X, Y, Z)\) are physical components of \((\gamma^1, \gamma^2, \gamma^3)\), a dot denotes differentiation with respect to \(t\) and \(n_0 = (Gm/r_0^3)^{1/2}\). As is easily seen, if we put parentheses appeared in the first and second equations of \((7 \cdot 3)\) as unity, then these equations are Newtonian counterparts. And the number 6 in the factor of parentheses of the first equation is familiar in Einstein's formula for the advance of the perihelion of Mercury, i.e. \(\frac{6\pi Gm}{c^2 a (1 - e^2)}\) per revolution.

\((B\text{-theory})\). The solution of \((6 \cdot 2)-(6 \cdot 4)\) for a circular orbit is:

\[
\frac{d\varphi}{dt} = \left(\frac{Gm}{r_0^3}\right)^{1/2}\left(1 - \frac{Gm}{c^2 r_0}\right)^{-1/2},
\]

\[
\frac{dt}{d\tau} = \left(1 - \frac{Gm}{c^2 r_0}\right)^{1/2}\left(1 - \frac{2Gm}{c^2 r_0}\right)^{1/2}.
\]

Now we shall calculate the motion of the second test particle \(Q\) relative to the first one \(P\) referring to the rotating coordinate system \((X', Y', Z')\) as illustrated in Fig. 2. Then, in terms of the method of small oscillation, we obtain

\[
\ddot{X} + n^2\left(1 - \frac{2Gm}{c^2 r_0}\right)\left(1 - \frac{4Gm}{c^2 r_0}\right)\dot{X} = 0,
\]

\[
\ddot{Z} + 2n\left(1 - \frac{Gm}{c^2 r_0}\right)\dot{Z} = 0,
\]

\[
\ddot{Y} + n^2Y = 0,
\]

where \((X, Y, Z)\) are the same quantities as those in \((7 \cdot 3)\) and \(n = \dot{\varphi}\).

Taking into account that \(n\) in \((7 \cdot 5)\) corresponds to \(n_0\) in \((7 \cdot 3)\), we can see that these two systems of equations coincide with each other, if we ignore the terms higher than \((Gm/c^2 r_0)^2\) compared with 1. Thus it may be said that there appears neither difference between \((7 \cdot 1)\) and the corresponding equation in \(B\text{-theory}\), nor additional prediction from \((7 \cdot 1)\) to those due to the geodesic equation; the effect due to the factor \(1 - \frac{6Gm}{c^2 r_0}\) in the first equation of \((7 \cdot 3)\) is equivalent to the advance of perihelion in geodesic motions.

§ 8. Conclusions

From the above consideration we may conclude that Singer's test is also useless to.
discriminate between (1) the orthodox interpretation of gravitation in general relativity and another due to Gupta, and (2) general relativity and W- & B-theories, because the same formula \(3 \cdot 3\) can be derived with some proper, but not \textit{ad hoc} procedure in terms of either theory. Moreover, it turns out to be clear that the test due to geodesic deviation is also useless so far as the static gravitational field is concerned.

Then do these circumstances originate from the very nature of the gravitational field? Perhaps they do not. The question whether a space-time is flat or not must have real significance just in the same way that the surface of the earth is closed (concept in-the-large). In this respect, the facts clarified in this paper are simply that it is next to impossible to decide whether a space-time is curved or not so long as the static gravitational field is concerned. Thus the problem is led to finding other objects or methods, if any, suitable to verify whether a space-time is curved or not.

Eventually the investigation of the gravitational field of the celestial bodies may not be suitable for this purpose, because it is not clear whether the properties can be observed or not, if the difference of the properties in-the-large between Schwarzschild's and flat space-time become known.

On the other hand, in cosmology it is evidently not so and the consideration in-the-large would play an important role. It is not desirable, however, to use cosmology as the test of a gravitational theory\(^{16}\). If so, except the problem of quantization of the gravitational field, there is little probability of finding out the object by which the appropriateness of the theory is discriminated.

Nevertheless, W-theory is based on an action-at-a-distance and B-theory encounters a difficulty in quantization procedure so long as Gupta's critique is accepted. So there is left only the general theory of relativity. Thus, the only way we could take is either to follow the line of thought due to Gupta et al.\(^{17}\) or to attack the problem with Wheeler's idea\(^{18}\) which is, though difficult, more faithful to Einstein's thought. But so long as the applicability of general relativity to cosmology is accepted, it seems to us that the latter way is more plausible. For if we take such a view-point, the world model based on the flat space-time is incompetent to explain the red-shift-magnitude relation of galaxies without \textit{ad hoc} hypothesis other than the concept of cosmic expansion.

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