Multiscale Methods and Survival Criteria for Diffusive Species in Striped Patterns

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1 Introduction

In this paper, we study the conditions for survival or persistence of species in spatially heterogeneous environments. Our aim is to describe more complex ecological situations in which the difficulties for survival are greater in some parts of the environment than in others. Such heterogeneities are usually ignored in more simple ecological models, in which the coefficients measuring the environment hospitality are constant (Kareiva [6]).

The geometric properties of heterogeneous regions possessing certain type of regularity may be crucial for the success of the colonization of the analyzed environment. For instance, when the observed variations of the environmental quality are spatially well defined (large favourable regions versus large unfavourable regions), the areas and the geographic localizations of the more favourable regions may be very important (Shigesada and Kawasaki [10]).

In this work, we focus on situations when the environments have some specific kind of heterogeneity. Namely, the favourable and unfavourable regions are well defined in small space scales with respect to the parameters of the problem.

Patterns like the one described above are commonly found in various types of natural environments such as some forests (Kareiva [6]). If distances between the trees

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are neglected, the forest may be viewed as a continuum in a macroscopic scale. The resulting models are then described by differential equations with constant coefficients as in the seminal work of Ludwig, Aronson, and Weinberger [8].

A motivation for this work comes from the fact that many common patterns are characterized by a specific type of spatial heterogeneity, namely, heterogeneity only in one spatial dimension. Typical examples of this pattern are found in large plantations where almost continuously occupied rows alternate with empty strips used for harvesting and other management purposes. Of course such a pattern results in a sharp mesoscopic variation of biological conditions for dispersal and growth of some organisms in the direction across the lines.

The following model, the main subject of this paper, aims to describe dispersal process in a stripped pattern environment.

The environment under consideration is modelled by a bounded plane domain $\Omega$. From a practical point of view, it can be considered as a rectangle:

$$\Omega := (0, L) \times (0, M) \subset \mathbb{R}^2. \quad (1.1)$$

We suppose that a certain species population inhabits this region and we denote its density by $u(x, y, t), (x, y) \in \Omega, t > 0$.

Various authors have proposed models in terms of evolution of semilinear partial differential equations (Kareiva [6], Cantrell and Cosner [4], Shigesada and Kawasaki [10]). Such models take the following form generalizing the traditional logistics models:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(x, y) \nabla u) + r_1 \left( \gamma(x, y) - \frac{u}{K} \right) u. \quad (1.2)$$

The parameters $D$ and $r_1$, respectively, describe the population dispersal in the environment and how its individuals reproduce and/or die. Note that, in contrast to usual models, here the parameters may vary depending on the position indicating the environmental influence on the locomotion and reproduction of the considered organisms. It is permitted, in particular, that the intrinsic growth rate $r_1$ is adjusted by the function $\gamma$, $|\gamma| \leq 1$, which assumes negative values reflecting the existence of unfavourable regions for the species $u$.

We intend to study situations in which the function $\gamma$ in (1.2), which models the intrinsic growth of $u$, possesses fast oscillations between the highly favourable and less
favourable regions. For this purpose, we will consider a fast variable

\[ \xi = \frac{x}{\epsilon}, \]  

(1.3)

where \( \epsilon := L/n \) and \( n \) is the number of environmental quality oscillations observed in \((0, L)\). We assume that such a constant \( \epsilon \) is small, that is, we assume that differences of the environment appear only at a very small scale compared to the spatial dimensions of the whole region (in this case \( L \)).

In order to observe how the sudden variations of the environmental quality influence its colonization, we introduce a function \( r \) defined for any \( \xi \in \mathbb{R} \) by

\[ r(\xi) := r_1 \gamma(x, y) = \begin{cases} r_1, & 0 \leq \xi < \beta, \\ -r_2, & \beta \leq \xi < 1, \end{cases} \]  

(1.4)

and periodically extended \((r(\xi + 1) = r(\xi))\) where \( r_1, r_2 \) are positive constants. The most favourable regions for the development of \( u \) correspond to the set of all points \((x, y) \in \Omega \) such that \( r = r_1 \), while the less favourable regions correspond to the set of points \((x, y) \in \Omega \) where \( r = -r_2 \). This negative value indicates the inhospitable regions. The constant \( \beta \), \( 0 < \beta < 1 \), determines the small scale variation between the favourable and unfavourable subregions. The smaller the value of \( \beta \), the more favourable the environment is, because of the shortage of less favourable regions. The homogeneous and entirely favourable regions correspond to \( \beta = 1 \). Clearly, the most unfavourable case is when the whole region \( \Omega \) is inhospitable: \( \beta = 0 \).

Note that in this model the intrinsic growth depends only on the variable \( x \). This represents the situation in a plantation pattern formed by \( y \)-strips of contiguous plants. The less favourable regions model the absence of hospitable plants.

We will also suppose that the spatial heterogeneity of the environment may influence the locomotion of the organisms. For this purpose, we define a function \( \hat{D} \) in the following way:

\[ \hat{D}(\xi) = \begin{cases} D_1, & 0 \leq \xi < \beta, \\ D_2, & \beta \leq \xi < 1. \end{cases} \]  

(1.5)
Further, the fast variations are modelled into the diffusion coefficient by

$$D(x, y) := \hat{D}(\xi).$$

(1.6)

We point out that the diffusion coefficient assumes only positive values. In order to cover a greater number of ecological situations and following Shigesada and Kawasaki [10], we do not specify the magnitudes of the values assumed by \( \hat{D} \) in the environment \( \Omega \). For instance, the less favourable regions for the development of the studied species may allow easier locomotion, which means \( D_1 < D_2 \).

Aiming to make the paper useful for wider groups of readers, we organize it as follows. In the first sections, we present the results and some comments on the proposed models. The titles of the corresponding sections end with “I.” Then, in the next sections, we give more mathematical details and proofs. In this second part, we use the same section titles as in the first one, but ending with “II.” In this way, the reader who is not interested in the details may read only the first part.

2 The one-dimensional model, I

In this section, we concentrate on the one-dimensional model. Biologically that means that uniformity occurs along each strip and variations show up only in the transversal direction. This model has the advantage that it can illustrate the main ideas of the homogenization method in a simpler way than in the two-dimensional one where more technical difficulties occur.

To begin with, we will consider, following the seminal work of Ludwig, Aronson, and Weinberger [8], the following equivalent one-dimensional model:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \hat{D}(\xi) \frac{\partial u}{\partial x} \right) + \left( r(\xi) - \frac{r_1}{K} u \right) u,$$

(2.1)

which describes the population dynamics of a species inhabiting a one-dimensional environment: the interval \( \Omega = (0, L) \).

It is supposed that the exterior of \( \Omega \) is totally inhospitable or lethal, mathematically represented by the Dirichlet boundary condition:

$$u(0, t) = u(L, t) = 0, \quad t > 0,$$

(2.2)

together with \( u(x, 0) = u_0(x) \), which gives the initial distribution of individuals in \( \Omega \).
Since we look for conditions for the success of the colonization of the region $\Omega$, we must analyze first the stability of the trivial solution (in this case the identically zero solution) under "small" perturbations:

$$u = 0 + \delta(x, t), \quad (2.3)$$

where $\delta$ is such that $\delta^2 \ll \delta$, that is, the square of the perturbation $\delta$ can be neglected in comparison with its own value. Hence, we obtain the following linear differential equation for $\delta$:

$$\frac{\partial \delta}{\partial t} = \frac{\partial}{\partial x} \left( \hat{D}(\xi) \frac{\partial \delta}{\partial x} \right) + r(\xi) \delta \quad \text{in } \Omega \times (0, \infty), \quad (2.4)$$

satisfying initial and boundary conditions like $u$. The linearity of the latter equation comes from the use of the Fourier method, searching the solution $\delta$ in the form

$$\delta(x, t) = e^{\sigma t} w_\sigma(x). \quad (2.5)$$

Thus we are led to analyze the signs of the constants $\sigma$ in the spectral problem:

$$\frac{\partial}{\partial x} \left( \hat{D}(\xi) \frac{\partial w}{\partial x} \right) + r(\xi) w = \sigma w, \quad (2.6)$$

$$w(0) = w(L) = 0. \quad (2.7)$$

For instance, if all eigenvalues $\sigma$ are negative, then the studied species will be extinct. On the other hand, some positive value $\sigma$ indicates species' persistence.

The major technical difficulty to analyze the spectral problem (2.6) is the presence of nonconstant coefficients, which have fast variation. This reveals the explicit dependence on the fast variable $\xi = x/\varepsilon$. For instance, the computation of numerical solutions for (2.6) is extremely expensive due to the fact that the numerical approximation error is, in general, proportional to $1/\varepsilon$ (Bakhvalov and Panasenko [1]). This approach requires very small steps and hence very large time for execution of the algorithms in order to obtain reliable results. Moreover, a "macroscopic" model is necessary since that is exactly the appropriate scale to control and to influence the system.

To obtain a macroscopic model, we will use the so-called homogenization method (Keller [7], Bakhvalov and Panasenko [1]), which consists in substituting the
problem (2.6) by a simpler equation obtained from the first one by asymptotic analysis of multiple scales which results in an approximation to the original problem. This method is widely used in engineering, when variations of the materials properties are present in very small scales with respect to the dimensions of interest of the problem (Bakhvalov and Panasenko [1]).

Applying the homogenization method to model (2.6) and (2.7) results in an effective equation associated to the following spectral problem:

\[
\frac{\partial^2 w_0}{\partial x^2} + \tau w_0 = \sigma w_0,
\]

\[w_0(0) = w_0(L) = 0,\]  \hspace{1cm} (2.8)

where \(w_0\) will represent the approximate solution of the problem (2.6) and the constant coefficients \(\overline{D}\) and \(\tau\) are given by

\[
\overline{D} := \left(\int_0^1 \frac{1}{\hat{D}(z)} \, dz\right)^{-1},
\]

\[
\tau := \int_0^1 r(z) \, dz.
\]  \hspace{1cm} (2.9)

With constant coefficients, the analysis of the proposed problem becomes similar to that one performed by Ludwig, Aronson, and Weinberger [8]. Their result states that the critical size of the region (i.e., the value under which extinction occurs) is given by

\[
L^* := \pi \sqrt{\frac{\overline{D}}{\tau}},
\]  \hspace{1cm} (2.10)

which, in terms of the parameters of the original problem, can be written as

\[
L^* = \pi \sqrt{\frac{D_1 D_2}{[\beta D_2 + (1 - \beta) D_1][\beta r_1 - (1 - \beta) r_2]}}.
\]  \hspace{1cm} (2.11)

Note that not all values of \(\beta\) are permitted for the survival of the species. Indeed, suppose that the unfavourable regions for the development of this species were much larger than the favourable regions in microscale. This observation can be translated mathematically as

\[
\beta < \beta^* := \frac{r_2}{r_1 + r_2}.
\]  \hspace{1cm} (2.12)

In such case, we have biological extinction independent of the size L of the studied region, which is compatible with the expectations.
Note that by the definition (2.12), \( \beta^* \) is an increasing function of \( r_2 \). Therefore, for the population to persist in the environment exhibiting large quality differences, the favourable subregions must greatly predominate the unfavourable ones. Another fact that can be observed by the definition of \( L^* \) is the existence of a vertical asymptote at \( x = \beta^* \). This parameter measures the difficulty for the population survival in a region where unfavourable subregions predominate in the small scale. In the limit \( \beta \to 1 \), one can see that \( L^* = \pi \sqrt{D_1/r_1} \), which is exactly the situation corresponding to a totally favourable environment.

We also observe that the predominance of unfavourable regions in the small length scale is not the unique factor which could lead to extinction. Indeed, if the constant \( \beta \) is fixed and the quality differences of the environment are sufficiently large, then the survival of the species will be compromised. More precisely, if

\[
r_2 > r_2^* := \frac{\beta}{1 - \beta} r_1,
\]

(2.13)

then survival is impossible in such environment independently of its measure. Analogously, the environment must have a sufficiently large favourable subregion in order to allow a successful colonization. In other words, for persistence, the intrinsic growth coefficient \( r_1 \) for the favourable regions must satisfy the inequality

\[
r_1 > r_1^* := \left( \frac{1}{\beta} - 1 \right) r_2.
\]

(2.14)

In conclusion, we would like to note that the one-dimensional model provides useful insights for the higher-dimensional models.

3 The two-dimensional model, I

In this section, we develop the model for a two-dimensional space. At this stage, we are interested in the consequences of a fast spatial variation of the intrinsic growth and diffusion across one of the spatial dimensions. Therefore, we will analyze the following spectral problem:

\[
\nabla \cdot (\bar{D}(\xi) \nabla w) + r(\xi)w = \sigma w \quad \text{in } \Omega,
\]

(3.1)

\[
w = 0 \quad \text{on } \partial \Omega,
\]

(3.2)

with respect to the sign of the eigenvalues \( \sigma \), where \( w \) is the eigenfunction associated by
the separation of variables of the solution, \( \Omega \) is a rectangular region of the form \((0, L) \times (0, M)\), and \( r \) is given by (1.4). If all eigenvalues are negative, then the species will be extinct; otherwise, it will survive in \( \Omega \).

We proceed in a similar way as in the previous section by using the homogenization method with just a little adaptation due to the fact that the coefficients of (3.1) depend only on the variable \( x \). Through analogous calculations, we find the approximate solution \( w_0 \) by solving the following spectral problem with constant coefficients:

\[
\mathcal{D} \frac{\partial^2 w_0}{\partial x^2} + \mathcal{D} \frac{\partial^2 w_0}{\partial y^2} + rw_0 = \sigma w_0 \quad \text{in } \Omega,
\]

\[
w_0 = 0 \quad \text{on } \partial \Omega.
\]

(3.3)

The coefficients \( \mathcal{D} \) and \( \sigma \) are defined by a harmonic mean as in the previous section (see (2.9)). However, the effective (homogenized) diffusion coefficient \( \mathcal{D} \) is defined by an arithmetic mean:

\[
\mathcal{D} := \int_0^1 \hat{D}(z)dz,
\]

(3.4)

which results in an anisotropic diffusion model.

The Fourier method, with some modifications, gives us the following estimate for the minimal value of the area \( |\Omega| \) of \( \Omega \) in order that the species survive:

\[
|\Omega| < \Lambda^* := \frac{2\pi^2}{\mathcal{D} D} \left[ \mathcal{D}_1 D_2 (\beta D_1 + (1 - \beta) D_2) \right]^{1/2}.
\]

(3.5)

Further, we may express the critical value \( \Lambda^* \) of the area as function of the problem parameters by the following expression:

\[
\Lambda^* = \frac{2\pi^2}{\beta r_1 - (1 - \beta) r_2} \left[ \mathcal{D}_1 D_2 (\beta D_1 + (1 - \beta) D_2) \right]^{1/2}.
\]

(3.6)

Comparing the critical values of the minimal length and minimal area, respectively, obtained in (2.11) and (3.6), we conclude that both analyses can be carried out in a similar way. Below, we will briefly comment on the results obtained for the critical area.
The expression (3.6) reveals the strong influence of the small scale on the survival capabilities of a species with vital dynamics satisfying (2.1) or its two-dimensional analog (3.1). The spatial pattern of favourable or unfavourable regions, as well as their qualitative differences, has decisive importance and shows up in the expression for the minimal survival area. If there are extremely unfavourable subregions, or if they are abundant, the region as a whole may become too hostile and extinction occurs. Moderate degrees of inhospitability related to abundance of favourable regions in the small length scale could result, in the large length scale of observation, in a favourable region for the studied species.

We will conclude the discussion with some results on the critical measure of more general two-dimensional environments than the rectangular one considered above. In this case, we cannot consider the diffusion coefficient and the intrinsic growth rate as functions of the rapid variable \( \xi \) since the average operator with respect to periods \( \langle \cdot \rangle \) is not well defined. However, this model can also be generalized a little by introducing a first-order term, or convection, which describes the movement tendencies of organisms in a preestablished direction. This direction is determined by the constant vector \( \mathbf{b} := (b_1, b_2) \in \mathbb{R}^2 \). The following results are inspired by the paper of Murray and Sperb [9] where the estimates of the minimal area of the refuge candidates were not completely derived.

We consider a spectral problem similar to the homogenized problem (3.3), namely,

\[
D_1 w_{xx} + D_2 w_{yy} + rw + b \cdot \nabla w = \sigma w \quad \text{in } \Omega,
\]

\[w = 0 \quad \text{on } \partial \Omega,\]

where the diffusion coefficients \( D_1 \) and \( D_2 \) are positive constants, possibly distinct in case of anisotropic diffusion, and

\[
\nabla := \left( \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{array} \right).\]

(3.8)

The intrinsic growth rate \( r \) is supposed to be a positive constant since we already know that the survival chances for this type of problem exist only in such cases.

The Faber-Krahn inequality (Garabedian [5]) and a technical result contained in Murray and Sperb [9] imply the following estimate on the critical value of area:

\[|\Omega| < A^* := \frac{4 \pi \delta^2}{4r - (d \cdot b)^2} \sqrt{D_1 D_2},\]

(3.9)
where \( j_0 \) is the first positive zero of the Bessel function of first kind and order zero, and 
\[
d := (d_1, d_2)
\]
with
\[
d_1 = \frac{1}{\sqrt{D_1}},
\]
\[
d_2 = \frac{1}{\sqrt{D_2}},
\]
(3.10)

We note that the expression in (2.11) makes sense only if \(|b|\) is small. This expression implies that in the case of lethal boundary the loss of individuals as a result of the convection process must not be too large, should the environment in question be a refuge for this species. We then obtain a critical value for the module of the convection vector \( b \) as a function of the intrinsic growth rate and the diffusion coefficients:
\[
|b| < \frac{2\sqrt{r}}{|d|} = 2\sqrt{\frac{D_1 D_2}{D_1 + D_2}}.
\]
(3.11)

Therefore, the loss of organisms through the lethal boundary as a result of the transportation along the vector \( b \) could be compensated for by great values of the intrinsic growth rates or random movements, which may serve as a kind of “resistance” to convection movements.

The estimate for the minimal area \( A^* \) obtained in (3.9) does not take into account the other geometrical properties of the refuge, for example, the boundary form. Let us suppose that the organisms are randomly moving in \( \Omega \) with equal chances of movement in any direction. If \( \Omega \) has a more complicated form, then the individuals have more chances to meet the lethal boundary if located close to it. It is natural then to expect that the principal eigenvalue increases. And actually this is the case, in accordance with Bhattacharya [3]. In that paper, an estimate analogous to the Faber-Krahn inequality is obtained taking into account a quantity \( \alpha(\Omega) \) called asymmetry of the region \( \Omega \). This function measures the similarity between \( \Omega \) and a round region. Using this result, we can obtain the following estimate for \( A^* \) taking into account the asymmetry of the domain \( \Omega \):
\[
A^* = \frac{4\pi j_0^2}{4r - (d \cdot b)^2} \sqrt{D_1 D_2} \left( 1 + \frac{61}{200} \alpha(\Omega)^3 \right).
\]
(3.12)

Observe that the larger the value of asymmetry \( \alpha \) is, the larger the critical value \( A^* \), which makes it more difficult for the species survival in the refuge region.
4 The mathematical details

4.1 The one-dimensional model

We recall that in Section 2 the following one-dimensional model was discussed:

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \hat{D} \left( \frac{x}{\varepsilon} \right) \frac{\partial u}{\partial x} \right) + r_{1} \left( \gamma \left( \frac{x}{\varepsilon} \right) - \frac{u}{K} \right) u \quad \text{in} \ (0, L),
\]

\[
u(0, t) = u(L, t) = 0,
\]

\[
u(x, 0) = u_0(x), \quad 0 \leq x \leq L,
\]

where

\[
r(\xi) := r_{1} \gamma(x, y) = \begin{cases} r_{1}, & 0 \leq \xi < \beta, \\ -r_{2}, & \beta \leq \xi < 1, \end{cases}
\]

\[
r(\xi + 1) = r(\xi), \quad \xi \in \mathbb{R}, \quad r_{1} > 0, \quad r_{2} > 0,
\]

\[
\hat{D}(\xi) = \begin{cases} D_{1} > 0, & 0 \leq \xi < \beta, \\ D_{2} > 0, & \beta \leq \xi < 1, \end{cases}
\]

with \(\hat{D}(\xi + 1) = \hat{D}(\xi), \xi \in \mathbb{R}\).

Associated to (4.1), we have the following linearized problem:

\[
\frac{\partial \delta}{\partial t} = \frac{\partial}{\partial x} \left( \hat{D} \left( \frac{x}{\varepsilon} \right) \frac{\partial \delta}{\partial x} \right) \frac{\gamma}{\varepsilon} \delta \quad \text{in} \ (0, L),
\]

\[
\delta(0, t) = \delta(L, t) = 0,
\]

\[
\delta(x, 0) = \delta_{0}(x), \quad 0 \leq x \leq L.
\]

Using the Fourier method, the solution of the latter problem is represented in the form

\[
\delta(x, t) = e^{\sigma t} w_{\sigma}(x).
\]

Hence, we arrive at the spectral problem

\[
\frac{\partial}{\partial x} \left( \hat{D} \left( \frac{x}{\varepsilon} \right) \frac{\partial w}{\partial x} \right) + r \left( \frac{x}{\varepsilon} \right) w = \sigma w,
\]

\[
w(0) = w(L) = 0.
\]
We will develop an approximate model for the spectral problem (4.7) by the multiple scales method (Keller [7]). Actually, the solution to this approximate model solves another eigenvalue problem related to the original one in (4.7). However, in contrast to (4.7), which has coefficients depending on the fast variable \( x/\varepsilon \), the coefficients of the homogenized problem are constant, a fact which enormously simplifies the analysis of the problem in question.

The main characteristic of the method consists of uncoupling the so-called slow variable (the variable \( x \)) from the fast variable \( x/\varepsilon \), assuming that \( w \) has the following form (Keller [7], Bakhvalov and Panasenko [1]):

\[
w = w\left(\frac{x}{\varepsilon}, \varepsilon\right). \tag{4.9}\]

It is supposed that the unknown function \( w \) has distinct behaviours in the small and large length scales. It is also assumed that \( w \) is 1-periodic function of the fast variable \( \xi \) and that the diffusion coefficient is a differentiable function of \( \xi \).

Using the notation

\[
\begin{align*}
u_{x_1 x_2 \cdots x_n} & := \frac{\partial^n u}{\partial x_1 \partial x_2 \cdots \partial x_n}, \\
ung_{x_1 x_2 \cdots x_n} & := \frac{\partial^n u}{\partial x_1 \partial x_2 \cdots \partial x_n},
\end{align*} \tag{4.10}\]

we have

\[
\frac{1}{\varepsilon} \tilde{D}_{\xi} \left( w_x + \frac{1}{\varepsilon} w_{\xi} \right) + \tilde{D}(\xi) \left( w_{xx} + \frac{2}{\varepsilon} w_{x\xi} + \frac{1}{\varepsilon^2} w_{\xi\xi} \right) + r(\xi) w = \sigma w. \tag{4.11}\]

By grouping the increasing powers of \( \varepsilon \), the last equation becomes

\[
\frac{1}{\varepsilon^2} \tilde{D}(\xi) w_{\xi\xi} + \frac{1}{\varepsilon} \left( \tilde{D}(\xi) w_x \right)_{\xi} + \tilde{D}(\xi) w_{xx} + r(\xi) w = \sigma w. \tag{4.12}\]

At this stage, we suppose that the two length scales are independent and that the singularity is fully "modelled" by \( \xi = x/\varepsilon \) so that the solution \( w(x, \xi, \varepsilon) \) depends regularly on the parameter \( \varepsilon \). This enables us to write the following asymptotic expansion:

\[
w(x, \xi, \varepsilon) = w_0(x, \xi) + \varepsilon w_1(x, \xi) + \varepsilon^2 w_2(x, \xi) + \cdots + \varepsilon^n w_n(x, \xi) + o(\varepsilon^n), \tag{4.13}\]
where the functions $w_n(x, \xi)$ are bounded, independent of the parameter $\varepsilon$, and periodic in the fast variable $\xi$, with period 1. Substituting (4.13) into (4.12) and regrouping the increasing powers of the parameter $\varepsilon$, we obtain

\[
\varepsilon^{-2} (\hat{D}(\xi)w_{0,\xi}) + \varepsilon^{-1} \left[(\hat{D}(\xi)w_{1,\xi}) + (\hat{D}(\xi)w_{0,x}) + \hat{D}(\xi)w_{0,xx}\right] \\
+ \left[(\hat{D}(\xi)w_{2,\xi}) + (\hat{D}(\xi)w_{1,x}) + \hat{D}(\xi)w_{1,xx} + r(\xi)w_{0}\right] + \cdots \\
+ \varepsilon^n \left[(\hat{D}(\xi)w_{(n+2),\xi}) + (\hat{D}(\xi)w_{(n+1),x}) + \hat{D}(\xi)w_{(n+1),xx} + r(\xi)w_{n}\right] + o(\varepsilon^n) = \sigma w_0 + \varepsilon (\sigma w_1) + \varepsilon^2 (\sigma w_2) + \cdots + \varepsilon^n (\sigma w_n) + o(\varepsilon^n),
\]

(4.14)

where the comma denotes the respective partial derivatives. The boundary conditions are

\[
w_0(0, \xi) + \varepsilon w_1(0, \xi) + \varepsilon^2 w_2(0, \xi) + \cdots + \varepsilon^n w_n(0, \xi) + o(\varepsilon^n) = 0, \\
w_0(L, \xi) + \varepsilon w_1(L, \xi) + \varepsilon^2 w_2(L, \xi) + \cdots + \varepsilon^n w_n(L, \xi) + o(\varepsilon^n) = 0.
\]

(4.15)

Now, as usual, we equate the coefficients of the corresponding powers of $\varepsilon$ in (4.14) and obtain the following recursive sequence of linear differential equations:

\[
(\hat{D}(\xi)w_{0,\xi}) = 0, \\
(\hat{D}(\xi)w_{1,\xi}) + (\hat{D}(\xi)w_{0,x}) + \hat{D}(\xi)w_{0,xx} = 0, \\
(\hat{D}(\xi)w_{2,\xi}) + (\hat{D}(\xi)w_{1,x}) + \hat{D}(\xi)w_{1,xx} + r(\xi)w_{0} = \sigma w_0, \\
(\hat{D}(\xi)w_{(n+2),\xi}) + (\hat{D}(\xi)w_{(n+1),x}) + \hat{D}(\xi)w_{(n+1),xx} + r(\xi)w_{n} = \sigma w_n,
\]

(4.16) (4.17) (4.18) (4.19)

where $n = 1, 2, \ldots$. If the initial functions $w_0$ and $w_1$ are known, then one can obtain the next functions in the sequence by substitution and resolution of the followed equations.

The purpose of this work is to characterize the function $w_0$ as a solution of an equation simpler than (4.7) satisfied by $w$. By integration it follows from (4.16) that

\[
\hat{D}(\xi)w_{0,\xi}(x, \xi) = C(x),
\]

(4.20)
and, since $\hat{D}$ does not vanish identically,

$$w_{0,\xi}(x, \xi) = \frac{C(x)}{D(\xi)}. \quad (4.21)$$

Now we introduce the averaging operator $\langle \cdot \rangle$ defined as follows:

$$\langle f \rangle(x) := \int_0^1 f(x, \xi) d\xi. \quad (4.22)$$

We emphasize that $x$ and $\xi$ are considered as independent variables in this integral. Applying the operator $\langle \cdot \rangle$ to (4.21) and taking into account the periodicity of the function $w_n, n = 0, 1, 2, \ldots$, in the rapid variable $\xi$, we have

$$\langle w_{0,\xi}(x, \xi) \rangle = \int_0^1 w_{0,\xi}(x, \xi) d\xi = w_0(x, \xi) \bigg|_{\xi=0}^{\xi=1} = 0, \quad (4.23)$$

which, by (4.21), implies that the function $w_0$ is independent of the rapid variable $\xi$, that is,

$$w_0 = w_0(x). \quad (4.24)$$

Then we can write (4.17) as

$$[\hat{D}(\xi)(w_{1,\xi} + w_{0,x})]_\xi = 0. \quad (4.25)$$

Hence,

$$\hat{D}(\xi)(w_{1,\xi} + w_{0,x}) = C_1(x),$$

$$w_{1,\xi} + w_{0,x} = C_1(x)\hat{D}^{-1}(\xi). \quad (4.26)$$

Applying again the averaging operator to the latter align, we obtain

$$w_{0,x} = C_1(x)\langle \hat{D}^{-1}(\xi) \rangle,$$

that is,

$$C_1(x) = D w_{0,x}, \quad (4.28)$$
where $\mathcal{D}$ (which will be called the \textit{coefficient of effective diffusion}) is calculated by the harmonic mean of the function $\hat{D}$, namely,

$$\mathcal{D} := \left( \int_0^1 \frac{1}{\hat{D}(\xi)} \, d\xi \right)^{-1}. \quad (4.29)$$

Therefore, by (4.29) and (4.26), we have

$$w_{1,\xi} = \left( \frac{\mathcal{D}}{\hat{D}(\xi)} - 1 \right) w_{0,x}, \quad (4.30)$$

that is,

$$w_1(x, \xi) = N_1(\xi)w_{0,x} + F_1(x), \quad (4.31)$$

where

$$N_1(\xi) := \int_0^\xi \left( \frac{\mathcal{D}}{\hat{D}(\eta)} - 1 \right) \, d\eta \quad (4.32)$$

and $F_1$ is an arbitrary function of the slow variable $x$.

Applying one more time the mean operator to (4.18) gives

$$\hat{D}(\xi)w_{1,x}\bigg|_{\xi=0}^{\xi=1} + \int_0^1 \hat{D}(\xi)w_{1,\xi}\, d\xi + \left( \int_0^1 \hat{D}(\xi)\, d\xi \right)w_{0,xx} + \left( \int_0^1 r(\xi)\, d\xi \right)w_0 = \sigma w_0. \quad (4.33)$$

Since

$$\int_0^1 \hat{D}(\xi)w_{1,\xi}\, d\xi = \int_0^1 \left[ \hat{D}(\xi) \left( \frac{\mathcal{D}}{\hat{D}(\xi)} - 1 \right) w_{0,xx} \right] \, d\xi = \mathcal{D}w_{0,xx} - \left( \int_0^1 \hat{D}(\xi)\, d\xi \right)w_{0,xx} \quad (4.34)$$

and, by the definition of $\hat{D}$,

$$w_{1,x}\bigg|_{\xi=0}^{\xi=1} = (F_1'(x) + N_1(\xi)w_{0,xx})\bigg|_{\xi=0}^{\xi=1} = \left[ \int_0^1 \left( \frac{\mathcal{D}}{\hat{D}(\xi)} - 1 \right) \, d\xi \right] w_{0,xx} = 0, \quad (4.35)$$

where $' := d/dx$, the equation (4.33) is then simplified resulting in the following stationary diffusion equation:

$$\mathcal{D}w_{0,xx} + rw_0 = \sigma w_0, \quad (4.36)$$
where $D$ is the (effective) diffusion coefficient defined by (4.29) and the effective coefficient of intrinsic growth $\bar{r}$ is defined by

$$\bar{r} := \int_0^1 r(\xi) d\xi = (r_1 + r_2)\beta - r_2.$$  \hspace{1cm} (4.37)

The partial differential equation with constant coefficients (4.36) is called homogenized since it uses only “macroscopic” information from the mathematical model.

The boundary conditions (4.15) imply that $w_0$ satisfies

$$w_0(0) = w_0(L) = 0.$$  \hspace{1cm} (4.38)

We remark that the above calculation concerns classical solutions. If the involved parameters are weakly differentiable functions, one should consider generalized solutions of the above sequence of problems (Bensoussan, Lions, and Papanicolaou [2]).

In the case $r \equiv 0$, there exists an estimate on the admitted error when the homogenized problem (4.36) is considered as an approximation of the original one. Let $\sigma_i$ be a sequence of eigenvalues of the original problem and let $\bar{\sigma}_i$ be a sequence of eigenvalues of the homogenized problem. Then the following estimate holds:

$$\left| \frac{1}{\sigma_i} - \frac{1}{\bar{\sigma}_i} \right| \leq c(D_1, D_2) \varepsilon$$  \hspace{1cm} (4.39)

(see Bakhvalov and Panasenko [1]).

Now we will obtain an estimate on the minimal measure of the domain necessary for the survival. The calculations simplify if the problem is adimensionalized by the following scaling:

$$\bar{x} := x \sqrt{\frac{r_1}{D}}.$$  \hspace{1cm} (4.40)

After substituting and removing the bar of the variable $x$ (for the sake of notational economy), (4.36) reads as

$$-w_{0,xx} = \lambda w_0,$$  \hspace{1cm} (4.41)

where

$$\lambda := \frac{\bar{r} - \sigma}{r_1}.$$  \hspace{1cm} (4.42)
The boundary conditions become

\[ w_0(0) = w_0(\ell) = 0, \quad (4.43) \]

where

\[ \ell := L \sqrt{\frac{r_1}{D}} \quad (4.44) \]

The solution of this spectral problem is a pair \((\lambda_n, v_n)\), where the eigenvalues \(\lambda_n\) are given by

\[ \lambda_n := \frac{n^2 \pi^2}{\ell}, \quad (4.45) \]

and the eigenfunctions by

\[ v_n(x) := \sin\left(\frac{n\pi x}{\ell}\right). \quad (4.46) \]

By \((4.42)\) the eigenvalues of the problem \((4.36), (4.38)\) are given by

\[ \sigma_n := r - r_1 \frac{n^2 \pi^2}{\ell^2} = r - \frac{n^2 \pi^2 D}{L^2}, \quad n = 1, 2, \ldots. \quad (4.47) \]

Since

\[ \sigma_1 > \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq \cdots, \quad (4.48) \]

if

\[ \sigma_1 = r - \frac{\pi^2 D}{L^2} < 0, \quad (4.49) \]

all the eigenvalues of the problem \((4.36), (4.38)\) are negative, resulting in the linear stability of the trivial solution, thus implying eventual extinction. Since there is no information \textit{a priori} for the sign of the effective intrinsic growth coefficient \(\bar{r}\), we need to consider both cases separately.

(1) \(\bar{r} \leq 0\). Clearly the negativity of the effective intrinsic growth and the positivity of the effective diffusion (by definition of \(D\)) imply by \((4.47)\) the negativity of the eigenvalue of the problem \((4.36), (4.38)\) for any value of the constant \(L\). Therefore, the less favourable subregion for the development of the studied species predominates in the small length scale resulting in an extinction scenario \textit{independent of the dimensions of}
the region to be colonized. This condition is satisfied if, for instance, we keep \( r_1 \) and \( r_2 \) fixed and decrease the value of the constant \( \beta \).

(2) \( r > 0 \). In this case the extinction happens only for sufficiently large values of the constant \( L \). This indicates that there exists a critical value of the measure of the one-dimensional region \( \Omega \) below which the species cannot survive. This threshold value is obtained by resolving (4.49) with respect to \( L \):

\[
L < L^* := \pi \sqrt{\frac{D}{r}}.
\]  

(4.50)

Using the definitions of the effective coefficients of diffusion and intrinsic growth given in (4.29) and (4.37), respectively, we can write the condition (4.50) for extinction in the following way:

\[
L < L^* = \pi \sqrt{\frac{D_1 D_2}{[\beta D_2 + (1 - \beta) D_1] [\beta r_1 - (1 - \beta) r_2]}}.
\]  

(4.51)

For an examination of this formula for \( L^* \), it is clear that if the other parameters are kept fixed, \( L^* \) as a function of \( \beta \) is not defined for all values of \( \beta \in [0, 1] \). At the point

\[
\beta^* := \frac{r_2}{r_1 + r_2},
\]

(4.52)

there is a vertical asymptote to the graph of \( L^* \) which shows how extremely difficult the environmental colonization can be when unfavourable subregions prevail in that sense.

4.2 The two-dimensional model, II

The two-dimensional analog of the one-dimensional model (4.7), (4.8) is represented by the following problem:

\[
\nabla \cdot \left( D \left( \frac{x}{v} \right) \nabla w \right) + r \left( \frac{x}{v} \right) w = \sigma w \quad \text{in} \ \Omega,
\]

(4.53)

\[
w = 0 \quad \text{on} \ \partial \Omega,
\]

(4.54)

\( \Omega \) being a rectangular region (see Section 3).
As in the preceding subsection, we assume that

\[ w = w\left(x, y, \frac{x}{\epsilon}, \epsilon\right). \quad (4.55) \]

Since the application of the homogenization method is quite similar to the one-dimensional case, we will only briefly present the corresponding calculations.

Substituting (4.55) in (4.53) and grouping the terms in increasing powers of \( \epsilon \) (keeping the notation \( \xi = x/\epsilon \) for the fast variable), we obtain

\[
\frac{1}{\epsilon x}(\hat{D}(\xi) w_x)_{\xi} + \frac{1}{\epsilon} \left[(\hat{D}(\xi) w_x)_{\xi} + \hat{D}(\xi) w_{x,\xi} + \hat{D}(\xi)(w_{xx} + w_{yy}) + r(\xi)w = \sigma w. \right. \quad (4.56)
\]

Henceforth, we suppose that the slow variable \( x \) and the fast variable \( x/\epsilon \) are independent as well as that the function \( w \) can be asymptotically expanded with respect to the parameter \( \epsilon \) as follows:

\[
w(x, y, \xi) = w_0(x, y, \xi) + \epsilon w_1(x, y, \xi) + \epsilon^2 w_2(x, y, \xi) + \cdots + \epsilon^n w_n(x, y, \xi) + o(\epsilon^n), \quad (4.57)\]

where we assume that the functions \( w_n \) are bounded and 1-periodic functions of the fast variable \( \xi \). Substituting (4.57) in (4.56), we get

\[
\epsilon^{-2}(\hat{D}(\xi) w_{0,\xi})_{\xi} + \epsilon^{-1} \left[(\hat{D}(\xi) w_{1,\xi})_{\xi} + (\hat{D}(\xi) w_{0,x})_{\xi} + \hat{D}(\xi) w_{0,xx} + w_{0,yy} + r(\xi)w_0\right] + \cdots + \epsilon^n \left[(\hat{D}(\xi) w_{n,\xi})_{\xi} + (\hat{D}(\xi) w_{0,x})_{\xi} + \hat{D}(\xi)(w_{0,xx} + w_{0,yy}) + r(\xi)w_n\right] \\
= \sigma w_0 + \epsilon (\sigma w_1) + \epsilon^2 (\sigma w_2) + \cdots + \epsilon^n (\sigma w_n) + O(\epsilon^n). \quad (4.58)
\]

Equating the coefficients of the corresponding powers of \( \epsilon \), we obtain the sequence of linear partial differential equations

\[
(\hat{D}(\xi) w_{0,\xi})_{\xi} = 0, \quad (4.59)
\]

\[
(\hat{D}(\xi) w_{1,\xi})_{\xi} + (\hat{D}(\xi) w_{0,x})_{\xi} + \hat{D}(\xi) w_{0,xx} = 0, \quad (4.60)
\]

\[
(\hat{D}(\xi) w_{2,\xi})_{\xi} + (\hat{D}(\xi) w_{1,x})_{\xi} + \hat{D}(\xi)(w_{0,xx} + w_{0,yy}) + r(\xi)w_0 = \sigma w_0, \quad (4.61)
\]

\[
(\hat{D}(\xi) w_{(n+2),\xi})_{\xi} + (\hat{D}(\xi) w_{(n+1),x})_{\xi} + \hat{D}(\xi)(w_{n,xx} + w_{n,yy}) + r(\xi)w_n = \sigma w_n, \quad (4.62)
\]
where \( n = 1, 2, \ldots \). Now we define the averaging operator \( \langle \cdot \rangle \) as follows:

\[
\langle f \rangle(x, y) := \int_0^1 f(x, y, \xi) d\xi.
\]  

(4.63)

By integrating (4.59) and applying the operator \( \langle \cdot \rangle \), we conclude that the function \( w_0 \) is independent of the fast variable \( \xi \):

\[
w_0 = w_0(x, y).
\]  

(4.64)

By integrating and applying the operator \( \langle \cdot \rangle \) to (4.60), we get that the function \( w_1 \) is expressed as a function of \( w_0 \):

\[
w_1(x, y, \xi) = F_1(x, y) + N_1(\xi)w_{0,x},
\]  

(4.65)

where \( F_1 \) is an arbitrary function smooth enough and

\[
N_1(\xi) := \int_0^\xi \left( \frac{D}{\bar{D}(\eta)} - 1 \right) d\eta.
\]  

(4.66)

Recall that \( \bar{D} \) is defined in (4.29). Using (4.34) and (4.35), after some tedious work, we arrive at the following linear partial differential equation with constant coefficients:

\[
\bar{D}w_{0,xx} + \bar{D}w_{0,yy} + \bar{r}w_0 = \sigma w_0,
\]  

(4.67)

where \( \bar{D} \) and \( \bar{r} \) are defined in (4.29) and (4.37), respectively, and the second effective diffusion coefficient \( \bar{D} \) is defined by

\[
\bar{D} := \int_0^1 \hat{D}(\xi)d\xi.
\]  

(4.68)

The boundary conditions which \( w_0 \) satisfies are

\[
w_0 = 0 \quad \text{on } \partial \Omega,
\]  

(4.69)
which in the specific case of rectangle \( \Omega = (0, L) \times (0, M) \) assume the form

\[
\begin{align*}
  w_0(x, 0) &= w_0(x, M) = 0, \quad 0 \leq x \leq L, \\
  w_0(0, y) &= w_0(L, y) = 0, \quad 0 \leq y \leq M. 
\end{align*}
\]  \tag{4.70}

Our purpose is to obtain an estimate for the critical value \( A^* \) of the area such that for smaller values it is not possible for the population to colonize the environment \( \Omega \). We assume here similar estimates of the error as in (4.40).

We rescale the independent variables by the following adimensionalization:

\[
\begin{align*}
  \bar{x} := x \sqrt{\frac{r_1}{D'}}, \\
  \bar{y} := y \sqrt{\frac{r_1}{D'}}. 
\end{align*}
\]  \tag{4.71}

Then, omitting the bars, (4.67) reduces to

\[-(w_{0,xx} + w_{0,yy}) = \lambda w_0, \tag{4.72}\]

where

\[
\lambda := \frac{r - \sigma}{r_1}. \tag{4.73}\]

After the rescaling, the boundary conditions become

\[
\begin{align*}
  w_0(x, 0) &= w_0(x, m) = 0, \quad 0 \leq x \leq \ell, \\
  w_0(0, y) &= w_0(\ell, y) = 0, \quad 0 \leq y \leq m, 
\end{align*}
\]  \tag{4.74}

where

\[
\begin{align*}
  \ell := L \sqrt{\frac{r_1}{D'}}, \\
  m := M \sqrt{\frac{r_1}{D'}}. 
\end{align*}
\]  \tag{4.75}

The pair \((\lambda_{np}, v_{np})\) is a solution of the above spectral problem, where the eigenvalues are given by

\[
\lambda_{np} = \pi^2 \left( \frac{n^2}{\ell^2} + \frac{p^2}{m^2} \right). \tag{4.76}\]
and the eigenfunctions by

\[ v_{np}(x, y) = \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{p\pi y}{m}\right), \]  

for \( m, n = 1, 2, \ldots \). By (4.76) it follows that the eigenvalues \( \sigma_{np} \) of the spectral problem (4.67), (4.70) are

\[ \sigma_{np} = \tau - r_1 \pi^2 \left( \frac{n^2}{\ell^2} + \frac{p^2}{m^2} \right) = \tau - \pi^2 \left( \frac{n^2 D}{L^2} + \frac{p^2 D}{M^2} \right). \]  

Hence, for any \( m, n \geq 1, 2, \ldots \),

\[ \sigma_{np} \leq \sigma_{11}. \]  

Thus the following inequality is a sufficient condition for extinction of the species:

\[ \sigma_{11} = \tau - \pi^2 \left( \frac{D}{L^2} + \frac{D}{M^2} \right) < 0. \]  

Obviously if \( \tau \leq 0 \), this condition is satisfied independently of \( L \) and \( M \).

Now suppose that \( \tau > 0 \). Then the elementary inequality

\[ 2ab \leq a^2 + b^2 \]  

implies the following condition for extinction:

\[ \tau - \pi^2 \left( \frac{2\sqrt{D D}}{LM} \right) < 0. \]  

Since \( |\Omega| = LM \), we obtain the desired estimate for \( A^* \):

\[ |\Omega| < A^* := \frac{2\pi^2}{\tau} \sqrt{D D}. \]
The latter inequality can be written as

\[ |\Omega| < A^* = \frac{2\pi^2}{\beta r_1 - (1 - \beta) r_2} \left[ \frac{D_1 D_2 (\beta D_1 + (1 - \beta) D_2)}{\beta D_2 + (1 - \beta) D_1} \right]^{1/2}, \tag{4.84} \]

in terms of original parameters.

As in the one-dimensional case, the graph of \( A^* \) as function of the parameter \( \beta^* \) has an asymptote indicating again the colonization hardships.

### 4.3 On the more general two-dimensional models

In Section 3, we considered the following spectral problem:

\[
\begin{align*}
D_1 w_{xx} + D_2 w_{yy} + r w + b \cdot \nabla w &= \sigma w \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\tag{4.85}
\]

where \( \Omega \) is a bounded plane region not necessarily rectangular, \( D_1, D_2 \) (diffusion coefficients), and \( r \) (intrinsic growth rate) are positive constants, and \( b \) — the advection velocity. Analogous calculations lead to the following conclusions.

Suppose that

\[ \sigma_1 < 0, \tag{4.86} \]

or

\[ \lambda_1(c) > 1, \tag{4.87} \]

where

\[ \lambda_1(c) = 1 - \frac{\sigma_1}{r} \tag{4.88} \]

is the fundamental eigenvalue of the spectral problem

\[
\begin{align*}
- (w_{xx} + w_{yy}) + c \cdot \nabla w &= \lambda w \quad \text{in } U, \\
w &= 0 \quad \text{on } \partial U,
\end{align*}
\tag{4.89}
\]

\( U \) being the image of \( \Omega \) through the scaling

\[
\begin{align*}
\bar{x} &:= x \sqrt{\frac{r}{D_1}}, \\
\bar{y} &:= y \sqrt{\frac{r}{D_2}},
\end{align*}
\tag{4.90}
\]
and $c := (c_1, c_2)$ is the scaled advection with components

\[
\begin{align*}
    c_1 &= \frac{b_1}{\sqrt{rD_1}}, \\
    c_2 &= \frac{b_2}{\sqrt{rD_2}}.
\end{align*}
\]  

(4.91)

In that case the species $u$ will go extinct in $\Omega$.

There exist some lower estimates on the fundamental eigenvalue of the problem (4.89) for general regions. In this regard, the theorem of Faber and Krahn (Garabedian [5]) states that among the regions with fixed area, the circle is the one that gives the least value of the fundamental eigenvalue of (4.89) with $c \equiv 0$. The related Faber-Krahn inequality is given by

\[
\lambda_1 \geq \frac{\pi j_0^2}{|U|},
\]

(4.92)

where $j_0^2 \approx 2.40483$ is the first positive zero of the Bessel function of first kind $J_0(x)$.

Further, we need a result proved in Murray and Sperb [9], which relates $\lambda_1(c)$ and $\lambda_1$, namely,

\[
\lambda_1(c) = \lambda_1(\|c\|) = \lambda_1 + \frac{\|c\|^2}{4}.
\]

(4.93)

This result, (4.87), and the Faber-Krahn inequality (4.92) lead to another sufficient condition for unsuccessful colonization of $\Omega$:

\[
\frac{\pi j_0^2}{|U|} + \frac{\|c\|^2}{4} > 1
\]

(4.94)

or

\[
|U| < \frac{4}{4 - \|c\|^2 \pi j_0^2}.
\]

(4.95)

Since

\[
|U| = \frac{r}{\sqrt{D_1D_2}}|\Omega|,
\]

(4.96)
going back to the original parameters, we write the following condition for extinction:

\[ |\Omega| \leq A^* := \frac{4\pi j_0^2}{4r - (d \cdot b)^2} \sqrt{D_1 D_2}, \tag{4.97} \]

where \( d := (d_1, d_2) \) is the vector with coordinates

\[ \begin{align*}
  d_1 &= \frac{1}{\sqrt{D_1}}, \\
  d_2 &= \frac{1}{\sqrt{D_2}}.
\end{align*} \tag{4.98} \]

Note that the expression in (4.97) makes sense only if

\[ 4r - (d \cdot b)^2 > 0. \tag{4.99} \]

Hence, by the Cauchy-Schwarz inequality, we obtain

\[ \|b\| < \frac{2\sqrt{r}}{\|d\|} = 2\sqrt{\frac{D_1 D_2}{D_1 + D_2}}. \tag{4.100} \]

The Faber-Krahn inequality (4.92) relates the fundamental eigenvalue of the problem (4.89) and the area of the considered region. However, it does not take into account other topological characteristics as, for instance, the form of the boundary of the region \( U \). The recent paper Bhattacharya \[3\] establishes an inequality similar to (4.92) involving a constant which measures how much the boundary is similar to a circle. This constant, called *asymmetry* of the region, is defined by

\[ \alpha(U) := \inf_x \frac{|U \setminus B(x, R)|}{|U|}, \tag{4.101} \]

where \( B(x, R) \) is the disk centered in \( x \) and of radius \( R \) having the same area as \( U \). Roughly, this constant is proportional to the sum of areas of regions belonging to \( U \) but not belonging to the disk centered in the point for which that sum is minimal.
The inequality obtained in Bhattacharya [3] reads as

\[ \lambda_1 \geq \frac{\pi j_0^2}{|U|} \left( 1 + \frac{61}{200} \alpha(U)^3 \right). \tag{4.102} \]

Note that the scale (4.90) does not alter the form of circles. Thus

\[ \alpha(U) = \alpha(\Omega). \tag{4.103} \]

Hence, and by (4.95) and (4.96), we obtain a better estimate for the minimal area \( A^* \), namely,

\[ |\Omega| < A^* = \frac{4\pi j_0^2}{4r - (d \cdot b)^2} \sqrt{D_1D_2} \left( 1 + \frac{61}{200} \alpha(\Omega)^3 \right). \tag{4.104} \]

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