Predictability of Rotating Stratified Turbulence

K. NGAN,* P. BARTELLO, AND D. N. STRAUB

McGill University, Montreal, Quebec, Canada

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ABSTRACT

Although predictability represents one of the fundamental problems in atmospheric science, gaps in our knowledge remain. Theoretical understanding of the inverse error cascade is limited mostly to homogeneous, isotropic turbulence, whereas numerical simulations have focused on highly complex numerical weather prediction models. These results cannot be easily reconciled. This paper describes selected aspects of the predictability behavior of rotating stratified turbulence. The objective is to determine how the predictability varies with scale when the dynamics are more realistic than the idealized models that underlie the classical picture of predictability and yet are free of the parameterizations that complicate interpretation of NWP models. Using a numerical model of the nonhydrostatic Boussinesq equations, it is shown that the predictability decay, as diagnosed by the relative error, is slower for subsynoptic flow. The dependence on the deformation radius, differences between balanced and unbalanced modes, and implications for NWP models are discussed.

1. Introduction

Much of our intuition about atmospheric predictability derives from the pioneering work of Lorenz, Leith, and Kraichnan. In a series of classic papers, they showed that small-scale errors may contaminate the largest scales in a finite time (Lorenz 1969) and that the error spectrum evolves self-similarly (Leith 1971), with characteristic wavenumber decreasing monotonically in time (Leith and Kraichnan 1972, hereafter LK72). These results were obtained using statistical closure models of homogeneous, isotropic turbulence. Although severely idealized, the notion of an inverse error cascade embodied by this classical picture has been of great influence, particularly in the numerical weather prediction (NWP) community.

According to the classical picture, the decay of predictability is determined by turbulent cascades or, more precisely, by the energy spectra and eddy turnover times (LK72). The integral eddy turnover time is given by

\[ \tau(k; k_0) = \left[ \int_{k_0}^{k} k^2E(k')dk' \right]^{-1}. \tag{1} \]

Assuming an isotropic energy spectrum \( E(k) \sim k^{-n} \), \( \tau \) becomes independent of \( k \) (for \( k \to \infty \)) if \( n \geq 3 \)—that is, provided one has steep spectra (e.g., Babiano et al. 1985). This means that there is essentially infinite predictability when the initial error is confined to infinitesimal scales. For \( n < 3 \), however, the integral diverges and an infinitesimal error will contaminate the largest scales in a finite time. Thus, the predictability depends crucially on the energy spectrum. For 3D turbulence with \( n = 5/3 \), LK72 estimate the predictability time to be \( O(10) \) eddy turnover times. For 2D turbulence, with \( n \geq 3 \), the predictability is formally infinite, but with numerical simulations there is a finite, resolution-dependent predictability time (e.g., Boffetta et al. 1997).

In recent years theoretical research examining other measures of predictability has emerged alongside NWP studies that have continued along the path established by Lorenz, Leith, and Kraichnan. Drawing on dynamical systems and information theory, there have been theoretical studies of finite-time Lyapunov exponents (Boffetta et al. 2002), which characterize the sensitivity to perturbation amplitude, the relative entropy (Kleeman and Majda 2005), which quantifies the relaxation of a probability distribution function to its ensemble-averaged or climatological form, and singular

* Current affiliation: Met Office, Exeter, Devon, United Kingdom.

Corresponding author address: K. Ngan, Met Office, Exeter, Devon, EX1 3PB, United Kingdom.
E-mail: keith.ngan@metoffice.gov.uk

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vectors in ensemble forecasts (Palmer 2000). Such approaches have proven especially useful for applications in which a probabilistic framework is preferable (e.g., seasonal predictability and the predictability of climate). By contrast, the relevance of the classical picture, which is concerned with error spectra (see section 3), is essentially limited to initial-value problems and (short-to medium-range) NWP. McWilliams (2008) points out that new diagnostics will be needed to tackle the problems posed by structural instability or sensitivity to model formulation, a component of predictability decay that is distinct from the sensitive dependence on initial conditions highlighted by the classical picture.

Nevertheless, our understanding of predictability decay in the traditional sense (i.e., an inverse error cascade of errors from small to large scales) remains incomplete. Tribbia and Baumhefner (2004) have reviewed the ways in which the predictability behavior of NWP models departs from the classical picture. For example, errors may grow through baroclinic instability, and skill may persist at certain scales, even the smallest (Boer 1994); thus, one may speak of predictability regimes. The primary implication is that in realistic models, one might not have a self-similar inverse error cascade that ineluctably contaminates all scales. Indeed, on certain scales cascade dynamics may play a secondary role as the error grows exponentially through local instabilities.

At first glance this discrepancy should not be especially surprising. NWP models contain representations of physical processes that are not retained in idealized models: Van Tuyl and Errico (1989) and Boer (1994) show that topographic forcing can lead to enhanced predictability in the vicinity of orography. Technical details are another factor: Errico and Baumhefner (1987) and Vukicevic and Errico (1990) show that inappropriate lateral boundary conditions can yield spuriously enhanced mesoscale predictability. Yet even aside from parameterizations, external forcings, and computational issues, one cannot expect the classical picture to be exactly reproduced. Whereas NWP models integrate the primitive equations, the classical picture applies to idealized, “balance” models (e.g., 2D, 3D, and QG turbulence) from which fast inertia–gravity waves are excluded by construction.

The gaps in the classical picture are well illustrated by the problem of atmospheric predictability times. Appeals to the classic work of Lorenz, Leith, and Kraichnan have been made to support the widely quoted figure of 10–14 days, but this analogy is misleading (see, e.g., Basdevant et al. 1981; Tribbia and Baumhefner 2004). This figure is based on the predictability of three-dimensional turbulence. Although atmospheric flow is quasi-two-dimensional on synoptic scales and larger, it is approximately three-dimensional only on the smallest scales, where neither rotation nor stratification plays an important role.

It therefore seems useful to examine the predictability of idealized models (i.e., dynamical cores without parameterizations) that are more realistic than the balance models but less complicated than NWP models. In particular, one would like to elucidate the influence of rotation and stratification or, more specifically, of inertia–gravity waves. Conventional thinking suggests that on large horizontal scales, the dynamics are approximately quasi-geostrophic (QG; i.e., balanced), geostrophic–ageostrophic interactions are weak, and the predictability behavior should be more or less unchanged from that of the balance models (Daley 1981). On smaller scales, however, the behavior could be quite different as the effects of rotation and stratification become weak.

The relative importance of rotation and stratification is measured by the Rossby and Froude numbers,

$$\text{Ro} := \frac{U_0 f L}{N H} \quad \text{and} \quad \text{Fr} := \frac{U_0}{N H},$$

where $f$ is the Coriolis parameter, $N$ the Brunt–Väisälä frequency, and $U_0$, $L$, and $H$ denote characteristic velocity, horizontal, and vertical scales. Many theoretical and numerical studies of rotating stratified turbulence (e.g., Lilly 1983; Bartello 1995; Riley and Lelong 2000) have shown that energy spectra, energy transfers, and geostrophic–ageostrophic interactions are profoundly affected by $\text{Ro}$ and $\text{Fr}$, which demarcate distinct dynamical regimes. For example, the geostrophic (or vortical) energy cascades to larger scales and the ageostrophic (or wave) energy cascades to smaller scales for sufficiently small $\text{Ro}$ and $\text{Fr}$.

The richer cascade phenomenology of rotating stratified turbulence suggests that the classical picture may need modification. Assuming that predictability times are determined by the cascade dynamics, as in homogeneous, isotropic turbulence, the predictability behavior could change qualitatively as $\text{Ro}$ and $\text{Fr}$ vary. To rephrase this in a more physically suggestive way, the predictability could be a function of scale. From the eddy turnover time (1), $\text{Ro}$ and $\text{Fr}$ scale as $k^{(3-n)/2}$ for $n < 3$. An immediate consequence is that $\text{Ro}$ and $\text{Fr}$ will be $O(1)$ at small scales if $n < 3$. This scale dependence has implications for high-resolution NWP models, which incorporate an extremely wide range of scales: on synoptic scales and larger there is quasi-two-dimensional turbulence and $n \approx 3$ (Boer and Shepherd 1983), but on smaller scales the famous $-5/3$ mesoscale energy spectrum emerges (Gage 1979; Koshyk et al. 1999; Skamarock 2004; Takahashi et al. 2006; Hamilton et al.
2 Numerical model and procedure

a. Equations of motion

The nonhydrostatic Boussinesq equations may be written in dimensional form as

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + f \mathbf{z} \times \mathbf{v} &= -\frac{1}{\rho_0} \nabla p + b \mathbf{z} + D(\mathbf{v}) \\
\frac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b &= -N^2 w + D(b) \\
\mathbf{v} \cdot \mathbf{v} &= 0,
\end{align*}
\]

where \( b = \rho' / \rho_0 \) is the (perturbation) buoyancy, \( \theta = \theta_0 + (d\theta/dz)z + \theta' \) is the potential temperature, \( \mathbf{v} \) is the 3D velocity with vertical component \( w \), and \( D \) denotes the dissipation operator. Triply periodic boundary conditions are applied.

Despite the simplifications of constant \( f \) and \( N \), these equations are broadly representative of atmospheric dynamics on a wide range of scales. There are quasi-geostrophic, synoptic-scale dynamics for \( Ro, Fr \ll 1 \), \( Ro/\text{Fr} = O(1) \), and mesoscale dynamics for \( Ro < 1, \text{Fr} \ll \text{Ro} \) (see, e.g., Riley et al. 1981; Lilly 1983). The upshot is that different regimes can be recovered by varying \( Ro \) and \( Fr \).

b. Numerical model

The numerical model is well tested and has been used in previous studies of rotating stratified turbulence (e.g., Bartello 1995; Ngan et al. 2008). Briefly, the model is pseudospectral with leapfrog timestepping and cylindrical truncation; the two-thirds rule is applied to eliminate aliasing (e.g., Durran 1989); a weak Robert filter is used to suppress the computational mode (Asselin 1972); and the grid is isotropic with \( N_h^2 \times N_v \) collocation points. We use an isotropic grid even when \( N_h \neq N_v \) because an anisotropic scaling might not be equally valid on all scales. To confine viscous effects to the smallest scales, we use hyperviscosity, \( D = -\nu W^6 \).

Model parameters for the different runs are specified using dimensional quantities: \( N_h, N_v, N, f, U_0, H_0 \), and \( L_0 \), where \( U_0 \approx 0.5 \) is a characteristic horizontal velocity, \( L_0 = 2\pi \) is the horizontal domain scale, and \( H_0 = 2\pi N_0 / N_h \) is the vertical domain scale. Apart from anisotropic runs (section 3c) and resolution tests (section 3d), \( N \) is varied while \( f \) and the other parameters are fixed. Occasionally we shall refer to the vertical grid spacing, \( \Delta z = 2\pi / N_0 \), \( \nu \) is chosen so that the dissipation at the truncation scale stays constant (\( \nu = \nu_0 / N_0^6 \), where \( \nu_0 = 2.62 \times 10^3 \)). The time step is sufficiently small that buoyancy oscillations are explicitly resolved for all values of \( N \).

Most of our simulations use an isotropic domain with \( N_h = N_v = 180 \); the highest resolution is \( N_h = N_v = 210 \). By contemporary NWP standards, the vertical resolution is excellent but the horizontal resolution is lacking; thus, we cannot resolve the same range of scales as state-of-the-art NWP models. This means that smaller-scale (i.e., mesoscale) dynamics may be inadequately resolved even if synoptic or global dynamics are well represented (see section 2c). Therefore, we choose to examine smaller-scale regimes separately (by employing “mesoscale” parameters), even though in reality, of course, subsynoptic dynamics are embedded within a larger flow. For concreteness the subsynoptic results may be considered products of a limited area model.

c. Procedure

Following previous numerical studies of predictability (e.g., Lilly 1972; Basdevant et al. 1981; Vallis 1983; Tribbia and Baumhefner 2004), the predictability error is calculated from identical-twin experiments. A basic state is generated from random initial conditions and used to define the control and perturbation simulations. The latter differs from the former on account of a small random perturbation; the divergence between the twins is used to diagnose the predictability error.

The initial spinup includes (i) a long quasigeostrophic integration from random initial conditions (see Ngan et al. 2008), (ii) scaling the fields to constant total energy, and (iii) a short integration of the full equations. This is intended to yield a basic state that is more representative of NWP analyses, which typically employ assimilation schemes that penalize or filter out fast waves (e.g., Rawlins et al. 2007). The short integration of the full equations is long enough for (unbalanced) perturbations to grow and decay and for energy spectra
to approach an asymptotic form. Although the basic-state energy does not remain constant as the parameters are changed, the variation is small.

We define the perturbation fields as

\[ \omega^{(p)} = \omega^{(c)} + \alpha_{w} \omega', \quad b^{(p)} = b^{(c)} + \alpha_{b} b', \]  

(4)

where \( \omega \) is the vorticity, \( \alpha_{w} \) and \( \alpha_{b} \) are perturbation amplitudes, and the explicit spatial dependence has been suppressed. In spectral space, we have

\[ \tilde{\omega}_{k} = \tilde{b}_{k}' = \begin{cases} \exp(\pi \eta), & \text{for } |k| \geq k_{\text{cut}}, \\ 0, & \text{for } |k| < k_{\text{cut}}, \end{cases} \]  

(5)

where \(-1 \leq \eta \leq 1\) is a random number, \(|k| = \sqrt{k_{x}^{2} + k_{y}^{2} + k_{z}^{2}}\) is a 3D wavenumber, and \(k_{\text{cut}}\) defines the cutoff wavenumber below which the perturbation is not applied. Unless otherwise stated, \( \alpha_{w} > 0, \alpha_{b} = 0, \) and \( k_{\text{cut}} = 5/6k_{T} \), where the truncation wavenumber \( k_{T} = N_{0}/3 \) on account of dealiasing. Here \( \alpha_{w} \) is chosen so that the (global) perturbation kinetic energy is a specified fraction of the basic state [typically \(|E_{K}^{(p)} - E_{K}^{(c)}|/E_{K}^{(c)} = 10^{-4} = \delta\)].

Variants to the procedure in (4) and (5) have also been considered, such as the following: (i) a random phase perturbation is used in place of an additive one; that is,

\[ \omega^{(p)} = \omega_{0}^{(c)} \exp(\pi \eta) \quad \text{and} \quad b^{(p)} = b_{0}^{(c)} \exp(\pi \eta); \]

(ii) \(k_{\text{cut}}\) is defined by imposing separate conditions on \( k_{h} \) and \( k_{c} \) (e.g., \( k_{h} \leq k_{\text{cut}} \) and \( k_{c} \leq k_{\text{cut}}\)); (iii) the perturbation is restricted to the buoyancy field using \( \alpha_{w} = 0, \alpha_{b} > 0; \) and (iv) the perturbation is projected onto the normal modes (see the appendix). Some of these robustness tests are described in sections 3b and 3d.

To compare results from different simulations, we use a dimensionless, integral time in place of the dimensional time. Specifically,

\[ \tau = \int_{t_{0}}^{t_{f}} dt_{\text{eddy}}, \]

(6)

where \( t_{\text{eddy}} = L_{0}/V \), \( t_{i} \) and \( t_{f} \) are the initial and final times, and \( V \) is a characteristic three-dimensional velocity scale.

We integrate each pair of twins for a minimum of five integral times. Since high-resolution simulations are computationally expensive, complete loss of predictability does not occur in all the runs described below. Nonetheless, the runs are sufficiently long that they are representative of the initial predictability decay. Medium-range weather forecasts are of comparable duration—that is, \( O(10) \) eddy turnover times or roughly 10–20 days. Complete loss of predictability occurs after \( O(10) \) integral times in closure models of 3D turbulence (LK72; Météais and Lesieur 1986).

We classify the runs according to domain-scale Rossby and Froude numbers:

\[ R_{0} := U_{0}/f L_{0}, \quad F_{r} := U_{0}/N H_{0}. \]

(7)

These definitions are convenient because they allow the flow to be classified based on the dynamics of the largest scales; thus, \( R_{0} \) and \( F_{r} \) are resolution independent even for \( n < 3 \). The use of \( H_{0} \) and \( L_{0} \) as characteristic scales is discussed in Ngan et al. (2008). With \( U_{0} \sim 0.5, R_{0} = O(0.1) \). Hereafter we drop the subscripts in \( R_{0} \) and \( F_{r} \).

The Rossby and Froude numbers can be used to define two key parameters. The effects of rotation are characterized by

\[ L_{d} = \frac{N H_{0}}{F_{r}}, \]

(8)

where the deformation radius \( L_{d} = NH_{0}/f \) (Gill 1982) and the domain-scale definitions for \( R_{0} \) and \( F_{r} \) have been used. We say there is synoptic flow if \( L_{d}/L_{0} \sim 1 \); recall that standard quasigeostrophic scaling (Pedlosky 1987) assumes \( L_{d} \sim L_{0} \) and \( R_{0}, F_{r} \ll 1 \). For synoptic flow, \( L_{d}/L_{0} > 1 \) and inertia–gravity waves continue to be important down to the buoyancy scale \( H_{b} \), where the scale-dependent \( F_{r} = O(1) \). The effects of stratification are characterized by

\[ \frac{H_{b}}{H_{0}} = \frac{U_{0}}{N H_{0}}, \]

(9)

which is just the domain-scale \( F_{r} \). We say there is weak stratification for \( H_{b}/H_{0} < 1 \).

The contrasting roles of \( L_{d} \) and \( H_{b} \) are discussed in Ngan et al. (2008). Briefly, \( L_{d}/L_{0} \) determines the structure of the time-dependent base flow, while \( H_{0} \) governs the influence of small-scale turbulence. There is quasi-2D turbulence on scales larger than the deformation radius (i.e., \( L_{d}/L < 1 \) and \( R_{0} \ll 1 \)), mesoscale dynamics for \( L_{d} > L > H_{b} \), and isotropic turbulence for \( H_{b}/H > 1 \).

The simulations to be described below are summarized in Tables 1 and 2. Because the flow is decaying, the values of \( R_{0}, F_{r}, L_{d}/L_{0}, \) and \( H_{b} \) are necessarily approximate.

d. Diagnostics

The predictability errors are calculated directly from the control and perturbation runs. Recalling the definition of the kinetic, potential, and total energies,

\[ E_{K} = \sum_{k \neq k} \frac{\omega^{2}}{k^{2}}, \quad E_{P} = \sum_{|k| > k_{c}} \frac{b^{2}}{N^{2}}, \quad \text{and} \quad E_{T} = E_{K} + E_{P}, \]

(10)
where the error spectra are given by

\[ \Delta_{KE}(k) = \sum_{|k| \leq k} \frac{(\Delta \omega)^2}{k^2} \text{ and } \Delta_{TE}(k) = \sum_{|k| \leq k} \frac{(\Delta \omega)^2}{k^2} + \frac{(\Delta b)^2}{N^2}, \]

(11)

where \( \Delta \omega = \omega^{(p)} - \omega^{(c)} \) and \( \Delta b = b^{(p)} - b^{(c)} \). Similar expressions exist for \( \Delta_{ke}(k_x), \Delta_{Te}(k_x), \Delta_{ke}(k_z), \Delta_{Te}(k_z) \). We shall focus on the relative error (i.e., the normalized error spectra):

\[ r_{KE}(k) = \frac{\Delta_{KE}(k)}{E_k(k)}, \quad r_{TE}(k) = \frac{\Delta_{TE}(k)}{E_T(k)}. \]

(12)

LK72 show that \( r_{KE} \) assumes a self-similar form for 2D and 3D turbulence.

To facilitate representation of the temporal development, we consider the mean relative error:

\[ r_{KE}(\tau) = \frac{\Delta_{KE}}{E_k} \text{ and } r_{TE}(\tau) = \frac{\Delta_{TE}}{E_T}, \]

(13)

where \( \Delta_{KE} = \Sigma_k \Delta_{KE}(k) \) and \( \Delta_{TE} = \Sigma_k \Delta_{TE}(k) \). The time evolution of the error wavenumber \( k_{err} \) has also been studied. It satisfies

\[ r_{KE}(k) > 0.5 \text{ for } k > k_{err}. \]

(14)

LK72 obtained the temporal, power-law scaling of \( k_{err} \) for 2D and 3D turbulence. Note that \( k_{err} \) decreases as the error front propagates to larger scales.

### 3. Error spectra

In this section we examine error spectra for two limiting cases: (i) subsynoptic flow and strong stratification (run 1) and (ii) synoptic flow and weak stratification (run 4). In the synoptic case, \( L_d/L_0 = 0.1 \) and the large-scale flow is dominated by rotation. In the subsynoptic case, \( L_d/L_0 = 10 \) and there is a mesoscale flow in which there is nonnegligible rotation and relatively strong stratification. Other combinations can also be considered (cf. Ngan et al. 2008). For brevity we label the cases as subsynoptic and synoptic, although we shall occasionally refer to the stratification as well.

#### a. Time evolution: Synoptic versus subsynoptic

Figure 1 shows \( r_{KE} \) for subsynoptic (Fig. 1a) and synoptic (Fig. 1b) flow. There is a continual loss of predictability in both cases: just as in classical studies (e.g., LK72), the relative error spectrum spreads from the smallest scales to the largest. By contrast with NWP models (Tribbia and Baumhefner 2004), errors do not grow preferentially at intermediate (baroclinically active) scales. The spectral evolution of \( r_{TE} \) (not shown) is almost identical. The relative error spectra for runs 1 and 4 do not coincide at \( t = 0 \) because the kinetic energy spectra have different slopes. There are some important differences between the cases.

First, the evolution of \( r_{KE} \) appears more self-similar in the synoptic case, with the error front propagating steadily toward larger scales. It is possible that the predictability decay in the subsynoptic case is so much slower that a self-similar decay stage has yet to be reached. Following Kraichnan (1970), Basdevant et al. (1981) distinguish between an error sweeping stage, which is dependent on initial conditions, and a subsequent error generation stage, in which the error grows self-similarly. Second, for subsynoptic flow a complete loss of predictability does not occur: \( r_{KE}(k) = 1 \) even for larger \( \tau \). These differences are underscored by plots of \( k_{err} \) against \( \tau \) (not shown). Whereas for synoptic flow \( k_{err} \sim 1 \) within several integral times, for subsynoptic flow \( k_{err} \) does not decrease from its initial value, even after \( O(10) \) integral times. Note that \( k_{err} \) underrepresents the predictability decay, which is small but nonzero.

We have repeated the simulations described in Fig. 1a with other realizations of the perturbation and basic state. On account of computational limitations, lower resolution is employed \((N_h = 120)\). With an ensemble of basic states (Fig. 2a), there is some dispersion among the ensemble members, but the basic trend persists. With an ensemble of perturbation amplitudes (Fig. 2b), each realization corresponding to a different value of \( \delta \), there is
almost uniform translation of the $\tau_{TE}$ curves, although for $\delta \approx 10^{-3}$ nonlinear effects begin to appear. The latter calculation is analogous to the calculation of finite-size Lyapunov exponents (e.g., Boffetta et al. 2002), in which the sensitivity to perturbation amplitude is studied. The results of Fig. 1 are unlikely to be artifacts of improbable realizations or unrepresentative perturbations.

b. Physical mechanism

The predictability of rotating stratified turbulence departs from the classical picture: the inverse error cascade depends on $L_d$ and the error spectra might not evolve self-similarly. Moreover, the slow predictability decay for subsynoptic flow cannot be directly attributed to energy spectra, or at least to spectral slopes. According to the classical picture, one would expect more rapid predictability decay for shallow energy spectra with $n < 3$. But for both synoptic and subsynoptic flow the spectral slopes for $E_k$ lie between $-3/2$ and $-2$ (not shown).

One can explain this behavior by appealing to the physical properties of waves and turbulence. As has been noted by many authors (e.g., Lilly 1983), the former propagate in physical space whereas the latter induce spectral energy transfers; consequently, errors that are shunted into wave modes might not cascade upscale. Moreover, if there were strong interactions between (vortical) turbulence and (inertia–gravity) waves, deceleration of the inverse error cascade would occur even when initial errors are restricted to the turbulence. For subsynoptic flow these interactions become important at small scales (cf. Ngan et al. 2008), where the Rossby and Froude numbers can be large.

The importance of waves and energy transfers to predictability decay was previously noted by Basdevant et al. (1981) and Vallis (1983). The former reference in particular shows that predictability increases because of the presence of Rossby waves and wave dispersion, which mitigate against nonlinear energy transfers in the inertial range. The effect of inertia–gravity waves would appear to be greater, presumably on account of the increased time scale separation with respect to the turbulence.

More formally, a Lagrangian solution of the linear wave equation $u_{tt} + c^2u_{xx} = 0$ can be written as

$$u = u_0(x_0, t_0), \quad x = x_0 + c(t - t_0),$$

where $x_0$, $t_0$, and $u_0$ respectively denote the initial time, location, and value. It follows immediately that the predictability error in a twin experiment—that is, between solutions

$$u^{(c)} = u_0(x_0 + c(t - t_0)), \quad u^{(l)} = u_0(x_0 + \delta x, t_0),$$

where $x_0 = x_0 + c(t - t_0)$ and $x_0 = x_0 + \delta x + c(t - t_0)$—is bounded for smooth initial conditions $u_0$ and small initial error $\delta x$. The predictability error will also be bounded for small perturbations $\delta u_0$ to the initial conditions;

![Figure 1](http://journals.ametsoc.org/doi/pdf/10.1175/2008JAS2799.1)
for traveling-wave solutions, \( \exp \left( ik(x - ct) + \phi \right) \), the predictability error reduces to a phase error. This argument has been formulated in physical space for the sake of convenience; reformulating it in spectral space does not introduce a cascade.

Obviously this argument is extremely idealized. Nevertheless, it seems plausible that the predictability decay will be driven primarily by the turbulence rather than by the waves. If one thinks of predictability loss as the infinite-dimensional analog of chaos and sensitive dependence to initial conditions in dynamical systems, then the predictability loss due to gravity waves should be weaker. Although wave–wave interactions among gravity waves can yield chaotic dynamics (see, e.g., Ngan and Shepherd 1997 for general background), the chaos should be confined to small scales for weak nonlinearity and small-scale waves [in nonlinear dynamics the phase space geometry is crucial; see, e.g., Wiggins (1990)]. With a separation of length scales and time scales the chaos may average out.

The use of an approximately quasigeostrophic basic state may seem restrictive. However, the physical mechanism described above should still be operative irrespective of the basic state. We investigate this with a modified procedure in which the quasigeostrophic spinup is replaced with the full nonhydrostatic Boussinesq (NHB) equations. Figure 3 compares \( \bar{r}_{TE} \) for both QG and NHB spinup. Changing from the QG to the NHB spinup has an effect on the predictability decay: in the synoptic case, the initial evolution is similar before the curves diverge at later times; in the subsynoptic case, gravity wave activity is significant and the predictability decay is slower, although the decay rates at later times are similar. Nevertheless, the predictability decay remains limited: Fig. 1a actually represents an underestimate of the effect. Again, inertia–gravity waves serve to increase predictability.

### c. Anisotropy

In Fig. 4 we replot the error spectra as functions of \( k_h \) and \( k_z \). For synoptic flow (Figs. 4b,d), the \( k_h \) and \( k_z \) spectra evolve rapidly and on similar time scales. For subsynoptic flow (Figs. 4a,c), however, the predictability decay is more rapid in \( k_h \). This is a consequence of strong anisotropy. With subsynoptic flow and strong stratification, there are qualitatively different dynamics in the horizontal and vertical planes: for \( Ro \ll 1 \) and \( Fr \to 0 \) (Riley et al. 1981; Lilly 1983), the vertical turbulence is layerwise two-dimensional whereas the inertia–gravity waves induced by the vertical stratification are three-dimensional.1 Majda and Embid (1998) show that

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1 It has been shown this scaling can be attained only when the vertical scale is set by something other than the stratification (e.g., viscosity; cf. Waite and Bartello 2006b).
the buoyancy field and fast gravity waves decouple from layerwise 2D turbulence in the limit of low-Fr, finite-Ro dynamics. Thus, one expects more rapid predictability decay in the horizontal.

The influence of anisotropy has also been discussed by Basdevant et al. (1981) in the context of barotropic turbulence. Consistent with our results, they argue that predictability increases as the flow becomes more anisotropic. In our case, the anisotropy arises from vertical stratification rather than zonal jets, but the analogy is clear enough. The emergence of anisotropy indicates that differences between the synoptic and subsynoptic cases cannot be explained solely in terms of decreased nonlinearity.

Note that the initial error spectra in Fig. 4 have nonzero values on intermediate scales (i.e., for \( k_h < k_{\text{cut}} \) and \( k_z < k_{\text{cut}} \)). This is because the initial perturbation is defined using the 3D wavenumber \( k \) [cf. (5)]. When the perturbation is defined using \( k_h \) and \( k_z \) (not shown), so that the error is confined to small scales in both the horizontal and the vertical, the spectral evolution of the error is qualitatively unchanged, though the predictability loss is delayed.

We emphasize that the physical mechanism described above is not fundamentally different from that responsible for the classical picture. Nonlinear dynamics over a wide range of scales leads to an inverse error cascade; the self-similar nature of the turbulent cascade, which is exemplified by the Kolmogorov theory, yields self-similar error spectra. Our point is simply that strong stratification complicates the picture through the introduction of anisotropy and inertia–gravity waves.

d. Resolution dependence

Figure 5 shows snapshots of \( r_{KE}(k) \) for resolutions between \( N_h = N_v = 60 \) and \( N_h = N_v = 210 \). Because the predictability decay occurs at different rates in the two cases, the snapshots are taken at different times. In the synoptic case (Fig. 5b), there is good agreement at small and intermediate scales; in the subsynoptic case (Fig. 5a), the results are resolution dependent. However, the spectral shapes are largely independent of resolution and the contrast with the synoptic case is maintained.

Numerical simulations of stratified turbulence can show great sensitivity to the resolution, in particular to the vertical resolution (e.g., Waite and Bartello 2006a). If the buoyancy scale \( H_b \) is not adequately resolved, then vertical overturning cannot be properly represented and energy spectra assume a bumpy appearance at large scales. Sensitivity to the horizontal resolution is a secondary issue because \( L_d \gg H_b \). A further concern is that geostrophic–ageostrophic interactions might not be correctly represented without adequate resolution (Ngan et al. 2008). For run 4, the buoyancy scale is only marginally resolved (\( H_b / \Delta z \sim 1 \); Table 2).

One can also expect the error spectra to be sensitive to the vertical resolution. Below the buoyancy scale \( k_z > k_h \), where \( k_h \) is the inverse buoyancy scale, there is 3D isotropic turbulence; in this weakly stratified range, errors can grow free from the inhibiting effect of inertia–gravity waves, just as for synoptic flow. These errors may serve as seeds for the inverse error cascade to larger scales; Tribbia and Baumhefner (2004) describe a similar role for baroclinic instability on synoptic scales. The error growth may
occur via an instability involving large-scale geostrophic modes and small-scale ageostrophic modes (see section 5c). For subsynoptic flow and strong stratification, wherein the buoyancy scale is small, a larger number of these small-scale modes will be retained as the resolution is increased. Thus, the inverse error cascade may be sensitive to vertical resolution. Indeed, examination of $r_{KE}$ and $D_{KE}$ confirms that the growth of small-scale errors increases at $N_h = 210$ (not shown). It seems plausible that convergence with an asymptotic value of $r_{KE}(t) < 1$ will eventually be attained, with $H_b$ acting as a barrier to the inverse error cascade, although we do not have the computational resources to verify this. This is, however, a subtle question. Issues such as geostrophic–ageostrophic transfers also come into play.

These considerations also apply to synoptic flow. Even with weak stratification, resolution dependence may set in as subsynoptic scales begin to be resolved, although adequate resolution of the buoyancy scale is unlikely to be an issue. There is limited support for this claim in Fig. 5b.

4. $L_d/L_0$ dependence

In this section we consider the dependence of the predictability decay on the deformation radius. It is useful to map the behavior between the limits described

![Graphs showing anisotropy of error kinetic energy spectra](attachment:fig4.jpg)

Fig. 4. Anisotropy of error kinetic energy spectra depicted in Fig. 1. (a),(c) As in Fig. 1a and (b),(d) as in Fig. 1b. Note that tau values from Figs. 1a,b have been reused. For subsynoptic flow the predictability decay is more rapid in the horizontal plane.
To compare the time evolution for different $L_d/L_0$, we consider the mean relative error $r_{KE}(t)$, as well as the error spectra $r_{KE}(k)$.

Figure 6 examines Froude numbers between 0.1 and 10 in an isotropic domain (the corresponding parameters are listed in Table 1). Only the initial predictability decay, $\tau$, is shown (decaying simulations are computationally expensive because $\tau$ decreases with time). In all cases we see the expected trend: $r_{KE}(t)$ generally increases. The evolution of $r_{TE}(t)$ (not shown) is almost identical. The behavior changes around $L_d/L_0 < 1$, which demarcates the transition from synoptic to subsynoptic flow and corresponds to approximately quasigeostrophic flow. Indeed, while $r_{KE}$ is significantly larger than in the subsynoptic case, the predictability decay remains relatively slow: $r_{KE}(t = 12) \sim 0.3$ (not shown), which is consistent with the results for 2D turbulence (see below).

For $L_d/L_0 = 0.1$ there is a complete loss of predictability by $\tau = O(5)$. For reference, a nonrotating, unstratified case (i.e., 3D turbulence) is also shown.

The trend in Fig. 6a is consistent with the physical mechanism of section 3b. As $L_d/L_0$ increases, the flow changes from synoptic to subsynoptic and there is an increase in gravity wave activity. This change in the nature of the flow makes the flow more predictable. Here the increase in $r_{KE}$ is noticeably smaller for $L_d/L_0 > 1$.

The importance of the deformation radius to predictability was first suggested by Thompson (1957), who showed that errors grow more slowly in a two-layer baroclinic model for $L_d/L > 1$. Because small-scale errors are uncorrelated with the large-scale advecting velocity field, the predictability decay preferentially occurs on synoptic scales. This view takes no account of turbulent cascades, but it can be reconciled with the classical picture by noting that one does not have a pure turbulent cascade on subsynoptic scales (cf. section 3).

One may be tempted to infer that the $r_{KE}$ curves are bracketed by the corresponding curves for 2D and 3D turbulence, the latter representing the maximum growth rate, the former the minimum. However, the analogy breaks down for subsynoptic flow. Using the same resolution employed in Fig. 6a, 2D turbulence becomes essentially unpredictable in $O(20)$ integral times, which contrasts markedly with the results for $L_d/L_0 = 10$. Differences between 2D and strongly stratified turbulence have been emphasized by many authors (see, e.g., Riley and Lelong 2000). The saturation of predictability decay has been discussed by Basdevant et al. (1981).

The trend in Fig. 6a probably cannot be attributed to limited resolution. To show this, we use an anisotropic domain, $\epsilon$: $H_0/L_0 \neq 1$, leaving the stratification and computational grid spacing unchanged (see Table 1); this allows $L_d/L_0$ to be varied independently of the resolution and $N$. Figure 7 shows $r_{KE}$ for $\epsilon$ between 1 and 1/15. As $\epsilon$ and $L_d/L_0$ decrease, synoptic flow returns and predictability decays more rapidly, suggesting

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2 The associated predictability time is shorter than the classical estimate for 3D turbulence, $r = O(10)$. This can be attributed to limited resolution.
that the trend is not an artifact of resolution or strong stratification. Note that $\bar{r}_{KE}$ saturates at smaller values in the anisotropic domain. This is likely a by-product of the missing modes in the vertical.

One may still object to the utility of $r_{KE}(t)$ as a predictability diagnostic. Even in the absence of a scale cascade, an increase in the small-scale error could lead to an increase in $\bar{r}_{KE}$. To investigate this, error spectra for the different $L_d/L_0$ are shown in Fig. 6b (at $t' = 5.8$). The trend as one moves from subsynoptic to synoptic flow is consistent with an inverse error cascade. Nevertheless, there is evidence for in situ error growth: the growth rate is maximized for large $k$ (not shown). Because these errors cannot feed on smaller scales, it seems plausible to ascribe this in situ growth to a small-scale instability (see section 5c).

5. Alternative perturbations

In the preceding sections we restricted attention to vorticity perturbations. In this section we consider more general perturbations. This helps us to assess the claim that the predictability decay is driven by the turbulence and slowed by the waves.

a. Buoyancy perturbations

Assuming naively that the turbulence resides in the vorticity field, one might expect buoyancy perturbations to have a smaller effect on the predictability. Buoyancy perturbations are generated by setting $\alpha_\omega = 0$, $\alpha_b > 1$ [cf. (4)]; $\alpha_b$ is chosen in exactly the same way as $\alpha_\omega$—that is, so that $|E_p^{(\rho)} - E_p^{(\rho)}|^2$, $E_p^{(\rho)} = 10^{-4} = \delta$. Because the buoyancy perturbation affects the potential energy, we consider $r_{TE}(t)$ and $r_{TE}(k)$, although nearly identical results are obtained with the kinetic energy.

Calculations with synoptic and subsynoptic flow indicate that buoyancy perturbations can have a non-trivial effect (Fig. 8). With subsynoptic flow (Fig. 8a), the buoyancy perturbation induces an increase in $r_{TE}$ comparable to that induced by the vorticity perturbation, at least on short time scales ($t \lesssim 2$); on longer time scales the buoyancy perturbation is less predictable, but the growth rates (i.e., slopes) are similar. This suggests that the underlying predictability dynamics are similar. The differences probably arise from increased dissipation of the vorticity relative to the buoyancy. With synoptic flow (Fig. 8b), the buoyancy perturbation trails the vorticity perturbation for all times and differences are more pronounced. Here the buoyancy field partially decouples from the vorticity field. The buoyancy field partially decouples from the vorticity field.

3 This is a slight oversimplification. The buoyancy field decouples from the vorticity field, to $O(Fr^{-2})$, on advective time scales (Lilly 1983). There may be weak coupling on inertial time scales.
The naive expectation about the efficacy of buoyancy perturbations fails because the simplistic identification of the buoyancy field with waves and the vorticity field with turbulence does not hold. It is only for $Fr^2 > 0$ or $Fr \to \infty$ that the turbulence and waves decouple exactly (e.g., Lilly 1983; Majda and Embid 1998). More generally, the buoyancy perturbation couples to the momentum equation (i.e., the turbulence). This means that differences between vorticity and buoyancy perturbations may be elided.

b. Geostrophic and ageostrophic perturbations

Following Daley (1981), the perturbation can be decomposed into geostrophic and ageostrophic components. The normal modes of the nonhydrostatic Boussinesq equations (see the appendix) permit a more satisfactory partitioning into turbulence and waves. We project the vorticity perturbation onto the geostrophic and ageostrophic modes using (A2).

Figure 9 plots $\overline{r_{KE}}$ versus $\tau$ for unfiltered, geostrophic, and ageostrophic perturbations. At all times the geostrophic perturbation leads the ageostrophic perturbation. This is consistent with the idea that the predictability decay is primarily associated with the turbulence. The deviation between the geostrophic and ageostrophic curves is greater for subsynoptic flow (Fig. 9a); with synoptic flow the curves nearly coincide (Fig. 9b).

Although the normal modes can be used for general $Ro$ and $Fr$, strictly speaking the identification of geostrophic and ageostrophic modes is valid only for $Ro$, $Fr \to 0$. But in the synoptic case the stratification is weak and the validity of the normal-mode basis is questionable. This may explain in part why the unfiltered perturbation shows greater predictability decay than both geostrophic and ageostrophic perturbations.
These results agree with those of Daley (1981), who used a primitive equation NWP model to show that geostrophic perturbations generally yield greater predictability decay, although as with our results the differences are not huge. Daley further noted that the rate of decay of geostrophic and ageostrophic perturbations is comparable on sufficiently small vertical scales; because of the increase in Ro and Fr, the time scale separation between waves and turbulence eventually breaks down. This is not observed in the subsynoptic case, presumably because Fr remains small. In the synoptic case, however, differences between the geostrophic and ageostrophic perturbations are small, in accord with Daley’s prediction. There is similar behavior for approximately quasi-geostrophic flow, Fr = 0.1 (run 3; not shown).

c. Geostrophic–ageostrophic interactions

Figure 9 may underrepresent the differences between geostrophic and ageostrophic modes in the same way that buoyancy and vorticity fields couple together, interconversions between geostrophic and ageostrophic modes occur for general Ro and Fr. To circumvent this, we decompose the predictability error into geostrophic and ageostrophic components, whose sum equals the original, unfiltered error. In this way a clearer picture of the effect of waves on the predictability dynamics may be obtained.

Figure 10 shows geostrophic and ageostrophic components,

\[
\frac{\bar{r}_{TE}}{r_{TE}(t)} := \frac{1}{E_T} \sum_k \Delta_g(k), \quad \frac{\bar{r}_{TA}}{r_{TE}(t)} := \frac{1}{E_T} \sum_k \Delta_a(k),
\]

where the geostrophic and ageostrophic errors \(\Delta_g\) and \(\Delta_a\) are defined analogously to (11). In the synoptic case (Fig. 10b), \(\frac{\bar{r}_{TE}}{r_{TE}}\) dominates after an initial adjustment, but in the subsynoptic case (Fig. 10a), \(\frac{\bar{r}_{TE}}{r_{TE}} < \frac{\bar{r}_{TE}}{r_{TE}}\) for all \(\tau\).

The results for subsynoptic flow would appear to contradict our claim that the predictability decay is driven by turbulence and geostrophic modes. However, \(\frac{\bar{r}_{TE}}{r_{TE}}(t)\) can provide a misleading representation of the predictability decay. If there is in situ error growth, \(\frac{\bar{r}_{TE}}{r_{TE}}\) can increase, even in the absence of an inverse error cascade. This is borne out by Fig. 11, which shows snapshots of the unfiltered, geostrophic and ageostrophic components of the error spectra, where \(r_{TE}(k)\) and \(r_{TE}(k)\) are obtained by projection. Here \(r_{TE}\) dominates for large \(k\) while \(r_{TE}(k)\) dominates at small \(k\); in other words, the growth of ageostrophic error is preferentially confined to small scales, whereas the geostrophic error propagates more readily to large scales. The contrast becomes even clearer when the time evolution is examined (not shown).

This behavior is consistent with the notion that linear waves do not induce an inverse scale cascade of error. It is also consistent the presence of an instability (see Ngan et al. 2008 and references therein) in which the
large-scale geostrophic flow drives the growth of ageostrophic perturbations at small vertical scales. The instability, which is most prominent for weakly stratified, synoptic flow, involves small vertical scales but also has a weak effect on larger scales. This is supported by the resolution dependence depicted in Fig. 5: the crossover wavenumber at which $r_{TE,ageo}^5$ shifts to smaller $k$ as $L_d/L_0$ decreases.

6. Summary and discussion

In this work we have shown that the predictability of decaying rotating stratified turbulence differs qualitatively from that of homogeneous isotropic turbulence. By contrast with the classical picture embodied in the pioneering work of Lorenz, Leith, and Kraichnan, the predictability loss, as diagnosed by relative error spectra ($r_{KE}, r_{TE}$) and the mean relative error ($\overline{r}_{KE}, \overline{r}_{TE}$), depends on the Rossby and Froude numbers. Furthermore, the loss is significantly slower for subsynoptic flow, $Ro/Fr < 1$, leading to anisotropy in the inverse error cascade and non-self-similar behavior.

We have argued that departures from the classical picture can be explained by appealing to rotating stratified turbulence phenomenology—in particular, the coexistence of turbulence and waves and the structure of geostrophic–ageostrophic interactions. By analogy with low-order dynamical systems, one may expect the predictability decay to be associated primarily with the turbulence rather than the waves; therefore, as inertia–gravity waves begin to dominate the dynamics and more error energy is transferred into them, larger scales will become more predictable. The inhibiting effect of Rossby waves on predictability decay was previously studied by Basdevant et al. (1981). We speculate that enhanced mesoscale predictability, which has been reported in previous studies (e.g., Anthes et al. 1985), could be related to wave motion.

For finite $Ro$ and $Fr$, one expects the deformation radius $L_d$ to play a role and a close relationship to develop between geostrophic modes and the inverse error cascade. This has been confirmed by comparing the influence of geostrophic and ageostrophic perturbations, as well as by decomposing the error into geostrophic and ageostrophic components. The potential importance of the deformation radius in determining the predictability decay was first suggested by Thompson (1957), although this idea does not seem to have received much attention in recent years.

Many of our results are implicit in the work of Daley (1981), which discussed the connections between predictability decay and geostrophic and ageostrophic perturbations in the context of a hydrostatic NWP model. The distinguishing features of the present work are significantly higher resolution, a model devoid of parameterizations (i.e., a pure dynamical core), a focus on subsynoptic, mesoscale flow, and interpretation in light of rotating stratified turbulence phenomenology.
From a theoretical perspective, the results suggest that the complicated scale dependence exhibited by NWP models (e.g., Boer 1994; Tribbia and Baumhefner 2004) is to a large extent inevitable because the Rossby and Froude numbers, and hence the predictability decay, are also scale dependent. This explains how, even in the absence of physical parameterizations and a realistic basic state, the classical picture may fail to apply. From a practical perspective, the results suggest that model resolution is crucial because the predictability of sub-synoptic flow can be rather sensitive to resolution. This paper has emphasized physical mechanisms governing the predictability of rotating stratified turbulence. Although various tests indicate that the basic trends are robust, we lack the computational resources for a definitive calculation of predictability times. Careful calculation of predictability times (and the associated resolution dependence) would represent a fruitful avenue for future work. Addressing synoptic and subsynoptic predictability within the context of a single high-resolution simulation, thereby shedding further light on the interaction between synoptic and subsynoptic scales, would be desirable. Forced simulations in which error energy siphoned off into wave modes is continually replenished would represent another interesting avenue. Finally, examination of modern statistical diagnostics like the relative entropy and finite-size Lyapunov exponents would provide a valuable complement to the present work, which has focused on error spectra and the inverse error cascade.

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**APPENDIX**

**Normal Modes**

Normal modes have been useful in the study of balance (see, e.g., Daley 1993) and rotating stratified turbulence (Bartello 1995). They consist of geostrophic or vortical modes, $A^{(0)}_k$, and ageostrophic or wave modes, $A^{(±)}_k$.

For the nonhydrostatic Boussinesq equations they take the form (Bartello 1995)

$$A^{(0)}_k = \frac{N\kappa \zeta_k + i\kappa \omega_k T_k}{\alpha_0 \kappa}$$

and

$$A^{(±)}_k = \frac{\alpha_0 \kappa D_k + i\kappa \zeta_k \mp N\kappa T_k}{\sqrt{2\alpha_0 \kappa}}$$

where the scaled variables $\zeta_k = i(k_x u_k - k_y v_k)$, $D_k = k/\kappa(z(i\kappa u_k + ik_x v_k))$, and $T_k = k_h b_k/N$ and the modal

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**A1** Note that the terminology is misleading: both the geostrophic and ageostrophic modes contain a vorticity component.
frequency $\sigma_k = \sqrt{k^2 + N^2 k_z^2}/k$. Here we assume that $k_h, k_z \neq 0$; the special cases $k_h, k_z = 0$ are described in Bartello (1995).

The dynamical fields may be projected onto the geostrophic modes with

$$
(\xi_k, D_k)_G = \left[ \frac{N k_h}{\sigma_k} \overline{A_k^{(0)}} , 0 \right].
$$

(A2)

The ageostrophic projection is obtained by taking differences.

The association of ageostrophic modes with gravity waves is open to debate. For large $\text{Ro}$ and $\text{Fr}$, the time scale separation diminishes and the normal modes do not provide an accurate representation of balanced and unbalanced components, but for strong stratification, $\text{Ro} = O(0.1), \text{Fr} \lesssim O(0.1)$ and this should not be a serious concern.

REFERENCES


Thompson, P. D., 1957: Uncertainty of the initial state as a factor in the predictability of large scale atmospheric flow patterns. Tellus, 9, 275–295.


