Estimating Bolus Velocities from Data—How Large Must They Be?

PETER D. KILLWORTH *

National Oceanography Centre, Southampton, Southampton, United Kingdom

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ABSTRACT

This paper examines the representation of eddy fluxes by bolus velocities. In particular, it asks the following: 1) Can an arbitrary eddy flux divergence of density be represented accurately by a nondivergent bolus flux that satisfies the condition of no normal flow at boundaries? 2) If not, how close can such a representation come? 3) If such a representation can exist in some circumstances, what is the size of the smallest bolus velocity that fits the data?

The author finds, in agreement with earlier authors, that the answer to the first question is no, although under certain conditions, which include a modification to the eddy flux divergence, a bolus representation becomes possible. One such condition is when the eddy flux divergence is required to balance the time-mean flux divergence. The smallest bolus flow is easily found by solving a thickness-weighted Poisson equation on each density level. This problem is solved for the North Pacific using time-mean data from an eddy-permitting model. The minimum bolus flow is found to be very small at depth but larger than is usually assumed near the surface. The magnitude of this minimum flow is of order one-tenth of the mean flow. Similar but larger results are found for a coarse-resolution model.

1. Introduction

Eddy parameterizations are in almost universal use in the ocean component of climate models, which are too coarse to permit eddies directly. Although there remains debate about diapycnic aspects of parameterizations, most extant versions represent eddy effects by an additional small “bolus” velocity whose role is to advect temperature and salinity (and thus density) to bring these fields closer to observations. McDougall and McIntosh (2001) give an approximate formula for the bolus velocity in terms of averaged eddy quantities (though we shall argue that for general eddy buoyancy fluxes, there may be no completely accurate way to represent them with bolus velocities). The size of the bolus velocity is typically $10^{-4}$ to $10^{-5}$ m s$^{-1}$, save possibly near the surface and in mixed layers.

There have been several approaches in the past to estimate bolus velocities. Rix and Willebrand (1996), Treguier et al. (1997), Gille and Davis (1999), Lee and Coward (2003), and Eden (2006) have analyzed eddy-permitting models and diagnosed the bolus velocities or streamfunctions. Ferreira et al. (2005) have followed McDougall and McIntosh’s (2001) suggestion and recast the equations of motion using the total (mean plus eddy) velocity as the main variable, which yields a frictionlike term in the momentum equations provided that the bolus velocity is indeed small compared to the Eulerian mean velocity. By using an adjoint, they were able to get a best fit to the frictional term (proportional to the eddy streamfunction). Values for the eddy streamfunction were rather large, reaching values of 2 m$^2$ s$^{-1}$, which implies large bolus velocities, of order up to 4 cm s$^{-1}$ in some regions. Equivalently, the best-fit eddy diffusivity was found to be of order 5000 m$^2$ s$^{-1}$, much larger than usually employed [also found by Eden (2006) in the Southern Ocean]. Note that Marshall et al.’s (2006) estimates were smaller, as they were unaffected by rotational flux components. While closer to observations than traditional Gent and McWilliams’s (1990, hereafter GM) calculations, their runs still yielded tracer fields some way from observations. There are indications, at least within a GM formulation, that the thickness diffusivity may be a strong function of depth, with high values near the surface and...
small values at depth; many values were negative (Eden et al. 2007b).

Indeed, the evidence from several different attempts at eddy parameterization, including our own nonlocal formulation (Killworth and Nurser 2008, unpublished manuscript), is that while parameterizations do improve the model results, measures of the distance between a model and data (e.g., the temperature field) are usually larger than the distances between the various small corrections produced by different bolus formulations. Given Ferreira et al.’s (2005) results, there may be two reasons for this. Either our choice of eddy parameterization is in error or the choice of numerical values entering the parameterization is at fault. Because the traditional choices for parameterization do show improvement over runs without parameterization, it seems sensible to examine the second possibility.

Thus, the main question in this paper is whether the parameterized bolus velocities—assuming these to be a good representation of eddy effects—need to be larger in at least some locations than hitherto believed.

However, before this question can be answered, we must first answer a set of preliminary questions:

1) Can an arbitrary eddy flux divergence of density be represented accurately by a nondivergent bolus flux that satisfies the condition of no normal flow at boundaries?
2) If not, how close can such a representation come?
3) If such a representation can exist in some circumstances, what is the size of the smallest bolus velocity that fits the data?

The answer to the first question is known, although still debated. It must, however, be answered afresh to create the formalism for the other questions. There is a skew flux definition of the bolus velocity based on the eddy fluxes themselves, although this does not satisfy all the requirements of no normal flow at boundaries (e.g., Treguier et al. 1997; Held and Schneider 1999; Plumb and Ferrari 2005). Any parameterization of the bolus flow that has normal flow at boundaries plays havoc with conservation (Killworth 2001), and so this condition cannot be neglected. Nurser and Lee (2004) and Eden et al. (2007a) give an elegant discussion of ways to represent eddy fluxes as well as a discussion of the boundary conditions.

All these papers show that there is a diapycnal component to the eddy terms, yet this is seldom, if ever, included in eddy parameterizations. This is largely due to McDougall and McIntosh (2001), who showed the existence of an approximate bolus velocity, but their formulation assumed that third-order terms were negligible, and estimates suggest this is not the case (Canuto and Dubovikov 2006). The question, therefore, remains how well a bolus flow can represent eddy fluxes.

We shall show that a general flux divergence of density cannot be represented completely accurately by a bolus term, although if the flux divergence is adjusted by a function of density, and the divergence satisfies certain integral constraints over surfaces of constant density, then a bolus term is adequate. (This does not hold for the mixed layer or any other area where the density does not vary in the vertical.)

One set of circumstances for which we will show the bolus term to be adequate is statistically steady flow. In this case the mean density does not vary temporally, and the implied divergence of the density flux $F = \nabla \cdot (\mathbf{u} \rho') = -\nabla \cdot (\mathbf{u} \rho'')$. Here, a bar is some time average, and primes are temporal fluctuations about that mean. We shall assume a steady state so the two expressions for $F$ are equal. If the mean flow lies along density contours, no bolus terms would be necessary to balance the flow. Note that only the divergence is relevant, so that rotational eddy fluxes are irrelevant. We then seek to create a three-dimensional, divergence-free bolus field $\mathbf{u}^*$ such that

1) $\mathbf{u}^* \cdot \mathbf{n} = 0$ at boundaries, where $\mathbf{n}$ is the outward normal at the boundary;
2) $\nabla \cdot (\mathbf{u}^* \rho') = -F$ everywhere; and
3) $|\mathbf{u}^*|^2$, integrated over the domain, is as small as possible.

Requirements 1 and 2 merely define a physical bolus velocity. Requirement 3 is added for two reasons. First, it removes the trivial solution $\mathbf{u}^* = -\bar{\mathbf{u}}$. This would imply—apart from removing all advection from the tracers—that the bolus terms were as large as the mean flow everywhere. Condition 3 avoids this, by requiring the bolus flow to be as small as possible. The question then becomes: How small can it be?

As a source of time-mean data, one could use an inverse method, deriving the baroclinic horizontal velocity from thermal wind, and using least squares to deduce the barotropic components and the vertical velocity offset. Alternatively, a time mean of an eddy-permitting calculation could be used. (This has the advantage of being able to cross the equator and thus in principle be solved globally.) The latter approach will be used here.

The use of a time mean implies that at any mean surface density outcrop, there will be some diabatic effects due to the averaged movement of the outcrop; such effects are included in the term $F$. It is at such

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1 It is not clear how unsteady the residual ocean, after the time average, can be. At the very least, $|\overline{\mathbf{p}}| \ll |\mathbf{u} \cdot \nabla \mathbf{p}|$ must hold. The role played in bolus velocities by, for example, seasonal variations, remains unclear.
locations that bolus terms may need to be large, and this hypothesis will be tested.

In this paper, therefore, we first show that a general flux divergence of density cannot be represented accurately by a bolus term (section 2). In conditions where this can be done, we pose the problem of generating a bolus velocity that will remove the mean flux divergence while being as small as possible (section 3). An extension to this method to ensure purely baroclinic bolus flow is given in appendix A. The velocity will turn out to be of order one-tenth of the mean flow on any given density surface. In particular, the bolus flow must be quite large near the surface and can be very small at depth for data from a realistic numerical model (section 6). We shall also show that a bolus velocity cannot be an adequate representation of eddy terms in the mixed layer (appendix B).

2. Representing eddy flux divergences by a bolus term

In this section we shall use the term $E$ to represent the eddy flux divergence $-\nabla \cdot (u^* \rho^\prime)$ and retain the term $F$ for some possibly modified flux divergence to be used later. We ask whether there can exist a bolus velocity $u^*$ such that $\nabla \cdot (u^* \rho^\prime) = -E$.

To ensure that $u^*$ is nondivergent, a two-dimensional streamfunction will sometimes be used,

$$u^* = (u^*_x, u^*_y) = (\psi_{1x}, \psi_{2y} - \psi_{1y} - \psi_{2y}) = (\Psi_z, -V_2 \cdot \Psi),$$

with reference to Cartesian coordinates, where subscript 2 denotes horizontal differentiation; without loss of generality, the streamfunctions vanish on the floor $[z = -B(x,y)]$ and have zero horizontal divergence at the surface $[z = 0]$. This is required either for a rigid lid or for a steady mean free surface. Although $E$ is not fully arbitrary because of additional equations such as the buoyancy variance equation, for eddies of finite amplitude these equations do not place limits on $E$ (Canuto and Dubovikov 2006). We therefore place no restrictions on $E$ henceforth, other than that it is a divergence.

We use density coordinates in what follows. In density coordinates in the vertical,\(^2\) neglecting the bar on density for clarity when it is used as a coordinate, we have

$$-E = u^* \cdot \nabla \rho = -\frac{u^*_x \cdot \nabla \rho}{z_\rho} + \frac{w^*}{z_\rho},$$

so that

$$w^* = u^*_x \cdot \nabla \rho - Ez_\rho,$$

where the suffix $d$ denotes a derivative in a density surface $X, Y$ are the horizontal directions holding the density constant.

Using the streamfunction definition for $w^*$, we have

$$w^* = -\psi_{1x}^\rho - \psi_{2y}^\rho = -\psi_{1x}^\rho + \frac{z_X}{z_\rho} \psi_{1p}^\rho - \psi_{2y}^\rho + \frac{z_Y}{z_\rho} \psi_{2p}^\rho$$

$$= -v_d \cdot \Psi + u_H \cdot \nabla_d z,$$

so that comparison with the previous expression yields

$$\nabla_d \cdot \Psi = Ez_\rho,$$

Mass divergence then yields

$$u^*_x + u^*_y + w^* = 0; \quad \text{hence} \quad \nabla_d \cdot u^* z_\rho - (Ez_\rho)_\rho = 0.$$ (2.2)

An alternative expression uses the streamfunctions

$$u^*_x z_\rho = \Psi_\rho,$$

$$\nabla_d \cdot \Psi + (Ez_\rho)_\rho = 0.$$ (2.3)

The bolus flow satisfies boundary conditions. At vertical walls there is no normal flow:

$$u^* \cdot n = 0,$$

where $n$ is the normal to the boundary. At a density outcrop, the condition is more complicated. We suppose that top and bottom densities are denoted by $\bar{\rho}_T$, $\bar{\rho}_B$, respectively. At the bottom ($z = -B$), for example,

$$w^* + u_H \cdot \nabla B = 0,$$

so that from the definition of $w^*$ we find

$$u_H \cdot \nabla_d (z + B) = Ez_\rho,$$

$$\rho = \bar{\rho}_B.$$ (2.4)

Similarly

$$u_H \cdot \nabla_d z = Ez_\rho,$$

$$\rho = \bar{\rho}_T.$$ (2.5)

We need to define the normal to a density surface at an outcrop, in density coordinates. At the surface, the normal is parallel to $v_H \bar{\rho}_B(X,Y)$. Now

$$z[X,Y,\bar{\rho}_T(X,Y)] = 0,$$

so that

$$v_d z + z_\rho v_H \bar{\rho}_T = 0$$

at the surface. So the (horizontal) normal to a density surface outcropping at the surface or floor is $n = \nabla_d z/|\nabla_d z|$, $n = \nabla_d (z + B)/|\nabla_d (z + B)|$, respectively. We retain the use of $n$; there should be no confusion because the context should always be clear. No normal flow thus implies

\(^2\) Ignoring issues of mixed layers and other regions where density cannot be defined easily as a vertical coordinate, for now.
\[
\mathbf{u}^\ast \cdot \mathbf{n} = Ez_\rho/|V_d z|, \quad Ez_\rho/|V_d(z+B)|, \quad (2.4)
\]

respectively. Then, integration of (2.2) across a density surface requires (denoting an area element by \(dA\) and a tangential line element by \(ds\))

\[
\int V_d \cdot (\mathbf{u}^\ast z_\rho) \, dA = \int z_\rho \mathbf{u}^\ast \cdot \mathbf{n} \, ds = \int Ez_\rho/|V_d z| \, ds = \left(\int Ez_\rho \right) dA.
\]

(2.5)

Other boundary contributions vanish at solid walls. In general, the last two expressions in (2.5), which must hold for any density surface and only involve the flux divergence and the density geometry, will not be equal (although the work of McDougall and McIntosh (2001) both derives similar results. The problem is simple: A bolus velocity that is not divergent cannot be expressed with bolus velocities. This result, as noted previously, is not new; Nurser and Lee (2004) and Eden et al. (2007a) both derive similar results. The problem is simple: A bolus velocity that is not divergent cannot account for diapycnal mixing.

Instead, we now ask how closely eddy flux divergences can be approximated by bolus terms. Returning to Cartesian coordinates, we seek to minimize the expression

\[
\int dV(\mathbf{u}^\ast \cdot \nabla \phi + E)^2 = \int dV\left[\psi_{1x} \bar{\rho}_x - \psi_{1z} \bar{\rho}_z + \psi_{2x} \bar{\rho}_y - \psi_{2z} \bar{\rho}_z + E\right]^2.
\]

Denoting small perturbations by \(\delta\), we must have

\[
\int dV[\delta \psi_{1x} \bar{\rho}_x - \delta \psi_{1z} \bar{\rho}_z + \delta \psi_{2x} \bar{\rho}_y - \delta \psi_{2z} \bar{\rho}_z] \phi + \int dV[\delta \psi_{2x} \bar{\rho}_y - \delta \psi_{2z} \bar{\rho}_z] \phi = 0,
\]

where

\[
\phi = \psi_{1x} \bar{\rho}_x - \psi_{1z} \bar{\rho}_z + \psi_{2x} \bar{\rho}_y - \psi_{2z} \bar{\rho}_z + E.
\]

Integrating by parts,

\[
\int dV[\delta \psi_{1x} (\phi \bar{\rho}_x) + \delta \psi_{1z} (\phi \bar{\rho}_z) + \delta \psi_{2x} (\phi \bar{\rho}_y) + \delta \psi_{2z} (\phi \bar{\rho}_z)] + \int dA[\delta \psi_{1x} \phi + \delta \psi_{1z} \phi ]_{z=0} = 0.
\]

Now, at the surface \(z = 0\), we have \(w^\ast = \psi_{1x} + \psi_{2y} = 0\). Therefore, we may write

\[
\delta \psi_1 = -\delta \chi_y, \quad \delta \psi_2 = \delta \chi_x, \quad z = 0.
\]

Then the surface integral above can be written

\[
\int dA[\delta \chi_y \bar{\rho}_x \phi + \delta \chi_x \bar{\rho}_z \phi] = \int dA [\delta \chi_y \bar{\rho}_x \phi - \bar{\rho}_x \phi]_{z=0}.
\]

This is zero for arbitrary \(\delta \chi\) only if \(\phi(z = 0) = \phi(\bar{\rho})\).

The interior part of the integral vanishes when

\[
\phi_x \bar{\rho}_x - \phi_x \bar{\rho}_z = \phi_y \bar{\rho}_y - \phi_y \bar{\rho}_z = 0
\]

or

\[
\phi = \phi(\bar{\rho}).
\]

Thus, we have

\[
\nabla \cdot (\mathbf{u}^\ast \bar{\rho}) + E = \phi(\bar{\rho}), \quad (2.6)
\]

although we do not know the value of \(\phi(\bar{\rho})\).

In other words, while there is no representation of \(E\) by a bolus term, there may be a representation provided that \(E\) is adjusted by some function of density. To find this function, we integrate (2.6) over all fluid more dense than some density \(\bar{\rho}_0\). The integral of \(\nabla \cdot (\mathbf{u}^\ast \bar{\rho})\) is zero, because it reduces to an integral over the outcrop region of the normal component of \(\mathbf{u}^\ast \bar{\rho}_0\), which vanishes because \(\nabla \cdot \mathbf{u}^\ast\), and hence \(\nabla \cdot (\mathbf{u}^\ast \bar{\rho}_0)\), is zero. Therefore,

\[
\int_{\rho > \bar{\rho}_0} E \, dV = D(\bar{\rho}_0) = \int_{\rho > \bar{\rho}_0} \phi(\bar{\rho}) \, dV,
\]

so that

\[
\phi(\bar{\rho}_0) = \frac{dD/d\rho(\bar{\rho}_0)}{dV/d\rho(\bar{\rho}_0)},
\]

where

\[
V(\bar{\rho}_0) = \int_{\rho > \bar{\rho}_0} dV;
\]

and so

\[
dV/d\rho(\bar{\rho}_0) = \int_{\rho > \bar{\rho}_0} z_\rho \, dA.
\]

Because

\[
dD/d\rho(\bar{\rho}_0) = \int_{\rho = \bar{\rho}_0} Ez_\rho \, dA,
\]

we have

\[
\phi(\bar{\rho}_0) = \bar{E}(\bar{\rho}_0),
\]

where the bar on \(E\) denotes a thickness-weighted average on a density surface.

Defining

\[
\bar{E}(\bar{\rho}_0) = \bar{E}(\rho_0),
\]

The proof that follows is due to G. Nurser (2007, personal communication) and is much simpler than my original.
so that from the definition of \( \Psi \) we find
\[
\mathbf{u}_H \cdot \nabla \rho = -F(z_B) = \mathbf{u}_d \cdot \nabla \rho = \rho - \rho_B;
\]
similarly
\[
\mathbf{u}_H \cdot \nabla \zeta = -F(z_B) = \mathbf{u}_d \cdot \nabla \zeta = \rho - \rho_T.
\]
We now seek the smallest bolus velocity that solves the problem. From the previous section, we have
\[
\mathbf{u} = (u^x, u^y, w^z) = (\psi_1, \psi_2, -\psi_1 - \psi_2);
\]
\[
\Psi = (\psi_1, \psi_2);
\]
where the bolus velocity satisfies
\[
\nabla_z \cdot (u^*_z) = \nabla_z \cdot \Psi = (Fz_B) = -\mathbf{u}_d \cdot \nabla \zeta, \quad \rho = \rho_B.
\]
At outcrops,
\[
\mathbf{u}_H^* \cdot \nabla(z + B) = Fz_B = \mathbf{u}_d \cdot \nabla = -\mathbf{u}_d \cdot \nabla(z + B), \quad \rho = \rho_B;
\]
similarly
\[
\mathbf{u}_H^* \cdot \nabla z = Fz_B = \mathbf{u}_d \cdot \nabla = -\mathbf{u}_d \cdot \nabla z, \quad \rho = \rho_T.
\]
Note that (3.6) is a stringent condition: as much mean transport crosses a density outcrop locally as is removed by the bolus velocity.4 Also, integration of (3.5) across a density surface and use of (3.6) are mutually consistent.

We seek to minimize the kinetic energy of the bolus flow,

$$\int \left\{ (u^a)^2 + (v^a)^2 \right\} \rho \, d\rho \, dx \, dy \, dz \, d\theta + 2\mu \left[ \nabla_d \cdot (u\cdot z_\rho) - (Fz_\rho \cdot \rho) \right] \, d\rho \, dx \, dy$$

(3.7)

where $\mu$ is a Lagrange multiplier and we require (3.5) to hold. Note that the first term in (3.7) is converted with a depth factor and the second is an unweighted Lagrange multiplier (as that term is already depth weighted).

Both $u^a$ and $v^a$ can be varied simultaneously, giving

$$u^a = \mu_X, \quad v^a = \mu_Y,$$

$$\nabla_d \cdot z_\rho \cdot \nabla_d \mu = (Fz_\rho \cdot \rho) = -\nabla_d \cdot \nabla_\rho, \quad (3.8)$$

which is a thickness-weighted Poisson equation for $\mu$.5

On vertical walls, we have

$$\nabla \cdot \nabla_d \mu = 0. \quad (3.9)$$

At outcrops, (3.6) requires

$$\nabla_d \mu \cdot \nabla_d z = Fz_\rho, \quad \nabla_d \mu \cdot \nabla_d (z + B) = Fz_\rho,$$

that is,

$$\nabla_d \mu \cdot \nabla_\rho = -\nabla \cdot \nabla_\rho, \quad \rho = \overline{\rho}_T, \overline{\rho}_B, \quad (3.10)$$

because, for example, at the surface

$$n = \frac{\nabla_d z}{|\nabla_d z|}.$$

Equation (3.10) can alternatively be written

$$z_\rho \nabla_d \mu \cdot \nabla_\rho = -\nabla_\rho \cdot \nabla_\rho. \quad (3.11)$$

This has two important consequences. First, the Neumann boundary conditions are such that (3.8) is consistent with them so that the problem must have a solution. Second, if $u^a / \nabla_\rho$ is to be small, $\nabla \cdot \nabla_\rho \ll \nabla_\rho$ at all density outcrops; this is a stringent criterion.

The form (3.8) is not surprising, as (put loosely) an arbitrary curl can be added without affecting the divergence but that will increase the kinetic energy of the bolus flow.

It is interesting to note that the solution to (3.8) has no vertically integrated divergence (i.e., $u^a = 0$ at the surface), although we cannot require that each component of the vertically integrated horizontal bolus velocity vanishes independently. It is possible to add this constraint; Killworth (1997) and Canuto & Dubovikov (2006) discuss this condition in some detail. Appendix A shows the changes that this requirement would add.

In the simple case of one horizontal direction, the solution is identically the trivial solution, as can be seen by a simple integral of (3.8) with regard to $X$ from a solid wall. In two horizontal directions, the vorticity of the solution provides a degree of freedom and the solution is nontrivial.

4. A simple example

Consider uniform stratification $z_\rho = \text{constant},$ and uniform $x$ gradient of density as in the Eady problem confined to a square box of size $L$. Specifically, $z = ax - b\overline{\rho}, a, b > 0$ for definiteness. The simplest $F$, or, more usefully, $Fz_\rho$ would be large-scale, and is a horizontal divergence [cf. (3.1)]:

$$Fz_\rho = -\left( \psi_1 x + \psi_2 y \right) = -A_1 \left\{ (m_1 \pi / L) \cos m_1 \frac{\pi X}{L} \sin n_1 \frac{\pi Y}{L} \sin p_1 \frac{\pi (aX - b\overline{\rho})}{B} \right\}$$

$$\sin n_1 \frac{\pi Y}{L} - A_2 \left\{ (n_2 \pi / L) \cos n_2 \frac{\pi Y}{L} \sin p_2 \frac{\pi (aX - b\overline{\rho})}{B} \right\} \sin m_2 \frac{\pi X}{L}. \quad (4.1)$$
for \( z_\rho \) uniform, the problem reduces to a pure Poisson
equation:

\[
\nabla \rho \cdot \nabla (\nabla \mu) = F_\rho, \quad (4.2)
\]

There are boundary conditions at outcrops that occur when

\[
ax - b\bar{p} = 0 \quad \text{or} \quad -B;
\]

that is,

\[
\rho = \frac{1}{B} \{ax + ax + B\}.
\]

A numerical solution of this problem with \( A_1 = 1, A_2 = 1.5, m_i = n_i = p_i = 1 \) is shown in Fig. 1. The strongest bolus flows occur at surface and floor outcrops where the mean flow is strongly advecting diapycnally. The ratio \( |\mathbf{u}^\theta|/|\mathbf{u}| \) can take large values.

5. Size estimates

We estimate \( F \) by

\[
F = \mathbf{u}_3 \cdot \nabla \bar{p} = \varepsilon O(\bar{U}) \cdot O(\nabla \bar{p}).
\]

Here, the \( \varepsilon \) term measures the degree to which \( \mathbf{u} \) and \( \nabla \bar{p} \) are nonaligned, so that \( \varepsilon = 0 \) would correspond to flow along isopycnals, and \( \varepsilon = 1 \) would imply no alignment at all. The argument in section 3 shows that \( \mathbf{u} \) must be mainly along isopycnals at outcrops. Now, from thermal wind,

\[
\mathbf{u}_T = \frac{g \nabla \bar{p}}{\bar{f} \bar{\rho}_0}, \quad \text{so} \quad \mathbf{u}_T \approx \frac{gh \nabla \bar{p}}{\bar{f} \bar{\rho}_0},
\]

where \( h \) is a scale depth in the (stratified) fluid.

Then,

\[
F = \mathbf{u}_3 \cdot \nabla \bar{p} \approx \varepsilon \frac{gh \nabla \bar{p}}{\bar{f} \bar{\rho}_0}.
\]

Now,

\[
u^\theta = \psi_1 z = \frac{\psi_1 \rho}{z_\rho} = \mu_X,
\]

and (3.8) implies that where \( L \) is a typical horizontal gyre scale,

\[
\mu_X \approx \frac{\mu}{L} \approx LF_\rho \approx \frac{LF_z}{\bar{\rho}_z}.
\]

Then,

\[
\frac{u^\theta}{\bar{U}} \approx \frac{\varepsilon \frac{gh \Delta \bar{\rho}}{\bar{f} \bar{\rho}_0} L(\Delta \nabla \bar{p})}{\bar{f} \bar{\rho}_0}.
\]

We also have the above estimate for \( \bar{U} \) itself, giving

\[
\frac{u^\theta}{\bar{U}} \approx \varepsilon \frac{\Delta \bar{\rho}}{\nabla \bar{p}}.
\]

So, we expect \( u^\theta/\bar{U} \) to be small when one or both of the following conditions holds: \( \varepsilon \) is small and \( \Delta \bar{\rho} \) is small compared with \( \nabla \bar{p} \).

It is unclear what definition should be used for “density” here. If neutral density is chosen, and we examine the northern half of WOCE A23, then \( \Delta \bar{\rho} \) is of order \( 0.5 \) to \( 1 \) kg m\(^{-3}\) in the top \( 500 \) m over a gyre scale of a few thousand kilometers, and much smaller at greater depths, and \( \nabla \bar{p} \) is of order \( 1 \) kg m\(^{-3}\) in the same area. Results from the World Ocean Circulation Experiment (WOCE) A16 are similar. Thus, there must be a great degree of alignment of mean flow along isopycnal contours for the bolus velocity to be small compared with the mean flow.

6. The North Pacific

a. An eddy-permitting model

To test the order of magnitude of the minimum bolus velocities, we use \( 1/4^\circ \times 1/4^\circ \) eddy-permitting model data from Ocean Circulation and Climate Advanced Model (OCCAM; Coward and de Cuevas 2005). While a coarse-resolution model could be used, we choose one that is eddy permitting in order to have fields that are reasonably realistic. The OCCAM model was run globally for 18 yr and the fields were stored every 5 days.

The mean flow has been computed on 72 density surfaces by Lee (2005, personal communication); details of this are given for a 1/12\(^\circ\) model by Lee et al. (2007). (This is not strictly consistent with the Eulerian averaging used previously, but it will suffice for our purposes.) “Density” here is defined as \( \sigma_2 \); this is a convenient compromise. The mixed layer is treated no differently from the rest of the fluid. (The mixed layer is considered in more detail in appendix B, which shows that in general a consistent bolus velocity cannot be found for the mixed layer.) The mean is taken over 3 yr; this is adequate because we are only considering the mean and not the eddy components. As confirmation, a similar calculation using a 1-yr average gave almost identical results.

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\(^6\) Mean velocities on density surfaces may be slightly divergent; however, because the problem is posed using divergences on density surfaces, this difficulty should be small.

\(^7\) The reader who is unhappy with the use of a 3-yr average may prefer to think of it as a set of dynamics that is consistent with climatological temperature and salinity data.
We use data from the North Pacific in what follows. The area taken is rectangular, with notional rigid boundaries at the edge of the rectangle, where the mean normal flow is set to zero. (Some choice must be made about boundaries in anything other than a global diagnosis; very little noise is induced by this.) The calculation uses an area north of 58° and south of 60°N, and between 120°E and 100°W. The area is shown in Fig. 2.

The C grid is used, with $\mu$ and $z_p$ stored at central points, and $\psi_1$, $\psi_2$ stored to the east and north of $\mu$, respectively. Implementation of the outcrop boundary conditions inevitably suffers from the zigzag nature of such boundaries in finite differences, but from visual inspection it does not appear to cause problems.

The problems (3.8) and (3.11) are solved by relaxation at all density levels simultaneously; because the

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**FIG. 1.** Solution to simple square box problem. Vectors show bolus flow in the indicated density surface (value above plot); red contours show the flux divergence in (3.5).
levels do not interact, the problem is ideally suited for parallelization. The iterations are terminated when the maximum change in $H_m$ is less than $10^{-2}$. For comparison, we shall also show a traditional estimate of the bolus velocity given by the GM formulation, using a constant thickness diffusion $\nu$ of 1000 m$^2$ s$^{-1}$. We choose this formulation because it has been widely used in the literature, and its users are comfortable with bolus velocities produced by such values. As GM note, their bolus streamfunction is given by the slope of a density surface $C_{GM} = 5/C_0 \kappa \frac{dz}{r}$; so that the GM bolus velocity is $u_{GM} = -\kappa \frac{V_d z_r}{r} = -\kappa V_d (\ln |z_r|)$.

Three points can be made. First, this does not take into account any tapering of the streamfunction near the surface or floor, which normally induces strong bolus flows (indeed, in the Eady problem, there is no GM flow except that induced by the tapering). Second, the GM velocity is always noisy, even when computed on very smooth density fields. Third, the GM velocity is a pure gradient in density space, just like the minimum bolus.

Figure 2 shows the minimum bolus velocity on three density levels, corresponding to light, medium, and dense water. Speeds are of order 1 cm s$^{-1}$ within the light fluid, but are almost everywhere less than 1 mm s$^{-1}$ in the denser layers. The bolus flow has no preferred direction and is strongly divergent. Figure 3 shows meridional sections at 180$^\circ$ of the mean eastward velocity, the GM bolus eastward velocity, and the minimum eastward bolus velocity. These are all drawn with the same contour interval, although of course the mean flow exceeds the 1 cm s$^{-1}$ cutoff in many places and is quite noisy, as would be expected of an eddy-permitting model. The minimum bolus $u$ field is small (of order 1 mm s$^{-1}$) except near the surface where it becomes much larger. In contrast, the GM $u$ field is small except near the floor, where large values are found.

The same features are found in Fig. 4, which shows the $\nu$ fields at 30$^\circ$N in a similar way. Both these diagrams suggest a broad description of the minimum bolus flow as being weak at depth and rising near the surface. This is borne out by Figs. 5 and 6, which show area- and thickness-weighted kinetic energies of the mean, GM, and minimum bolus flows as functions of density and average depth of a density surface, respectively. In both diagrams, it is evident that the minimum bolus energy is strongly correlated with the mean energy. It is noteworthy that, in fact,

*Layer-averaged minimum bolus kinetic energy (KE) = 0.0087 layer average mean KE, $r = 0.99,*

when computed over the density levels. It is not clear why such a good correlation should be found. The fit suggests that a suitable level of bolus flow is of order one-tenth of the mean level. It is also noteworthy that the GM level is far less than the minimum near the surface (while hardly surprising in the top 50 m, where processes other than eddies occur, the shortfall oc-
Fig. 3. Meridional sections at 180° of the mean eastward velocity, the GM bolus eastward velocity, and the minimum eastward bolus velocity. The contour interval is 0.1 cm s⁻¹, and values beyond ±1 cm s⁻¹ are indicated by ±1 cm s⁻¹.
FIG. 4. Zonal sections of the northward mean velocity, the GM northward velocity, and the minimum northward bolus velocity at 30°N. Contours as in Fig. 3.
pies 200 m) and is far more than the minimum at depth (exceeding the mean energy near the floor). This would suggest for these data at least that higher thickness diffusivities are needed near the surface. Yet the GM formulation worked well in practice, so that the large bolus flow induced by the rapid change in streamfunction near the surface, even if reduced by tapering, may be offsetting this effect. Note that the layer-average minimum bolus KE is not well correlated with the area-averaged buoyancy frequency \( r = 0.22 \).

It is possible that large values of speed (e.g., in western boundary currents) might be dominating the above averages. This is not the case, as the histograms in Fig. 7 show. In light density layers, the GM flow is everywhere too small, and in the denser layers, the minimum bolus flow is everywhere very small. Neither the average cosine nor the average sine between the directions of the bolus and mean flows differed significantly from zero at any density level, so the mean and minimum bolus flows are not correlated in direction.

Not only is the bolus KE correlated with the mean KE over the whole density range but it is well correlated on any single density surface. Figure 8 shows the correlation of bolus speed with mean speed for each density level, together with the best-fit amplitude \( A \) for a fit of the form bolus speed = \( A \times \text{mean speed} + B \). The correlations are of order 0.4 and are highly significant, even if the \( O(80\,000) \) points are reduced drastically in number to reflect losses in degrees of freedom caused by lateral homogeneity. The amplitude \( A \) is of order 0.08, consistent with the KE figures above. Both the correlations and \( A \) are smaller for layers of light density.

There is a simple heuristic argument concerning the size of the bolus speed. For uniform layer thickness, the problem to be solved is

\[
V_{b}\mu = -V_{b} \cdot \mathbf{u}.
\]

In midlatitudes, \( \mathbf{u} \) is mainly geostrophic and thus rotational. Indeed, its divergence is of order \( \beta \sigma / f \), so that

\[
|V_{b}\mu| \approx \left| \frac{\beta L}{f} \right| \mathbf{u}.
\]

Here, \( L \) is the length scale of the mean flow. If this is of order 1000 km, the factor \( \beta L/f \) is of order 0.1 as found above. However, this argument does not explain the correlations between bolus and mean speed, though the large rotational component of the mean flow would explain the lack of correlation between the two current directions.

Thus, for these data, we have shown that very small bolus flows are needed at depth, while much larger flows—of order one-tenth of the mean flow—are required to balance the mean flow near the surface. These values are not well related to those suggested by thickness diffusion with a constant \( \kappa \).

b. Coarse-resolution model

We now repeat this calculation with a coarse-resolution model. Rather then use the results of such a
model directly, which would be as unrealistic as any other such model, we choose here to average the eddy-permitting model onto a $1^\circ$ grid. Thus, the fields are smoother but still realistic.

We would expect the minimum bolus velocities to be larger than before, and this turns out to be the case. Figure 9 shows the minimum bolus velocities on the same three density surfaces as Fig. 2. In the buoyant surface layers, the minimum flow is similar to the previous case, but in the lower layers the flow is larger than before, and rather noisy. This is quantified in Figs. 10 and 11, which show the equivalent of Figs. 5 and 6. Below the surface layers, the minimum bolus KE is typically an order of magnitude larger than before. Once again the minimum bolus KE is highly correlated with that of the mean flow; the minimum layer-average bolus KE is now 0.02 $\times$ the mean KE, with a correlation of 0.96. Again, the GM KE is much smaller than required near the surface, down to about 900 m and much larger than the minimum below there (it is itself smaller than the eddy-permitting case because the small-scale noise is not present).

Thus, the same findings hold for coarse-resolution data.

7. Discussion

This paper has examined the concept of the representation of eddy buoyancy fluxes by bolus velocities from an unusual angle. In general, bolus velocities cannot be a complete representation while satisfying conditions of no normal flow at boundaries. However, if the role of the eddy fluxes is to balance a mean flow, then a bolus velocity does exist and is not unique. We
Fig. 9. The minimum bolus velocity on the three density layers shown for the 1° model. Gray shading shows where the layers do not exist. The velocities are shown as follows: a maximum speed is first applied (1, 0.1, 0.1 cm s\(^{-1}\) reading downward) and then the vector is scaled so that a vector length of 10° corresponds to the maximum. Vectors are shown every 3°, i.e., every three grid points.
have solved for the smallest possible bolus velocity in this case. Using data from an eddy-permitting model, we find that very small bolus velocities are needed in dense fluid layers, but in light fluid layers, even below the top 50 m, the minimum bolus speed is much larger than values used by the community. Remarkably, the minimum bolus speed is about one-tenth of the mean flow, although its orientation is not correlated with the direction of the mean flow.

Formulations for bolus flows are normally chosen to reduce the available potential energy of the system. In the steady state considered here, the bolus terms simply negate the tendency due to the mean flow, which can be of either sign.

Repeating the calculation with artificial but realistic coarse-resolution model data gives similar results but with minimum bolus speeds about 50% larger. We have also shown that a bolus velocity is not able to describe arbitrary eddy buoyancy fluxes in the mixed layer. The oceanographic community is still struggling with how best to represent the broad spectrum of processes that affect the mixed layer (Boccaletti et al. 2007).

Although this work has given some guidance as to the size of bolus speeds, a formulation for the bolus flow remains unavailable.

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**APPENDIX A**

**Baroclinic Bolus Velocities**

Previously, the expression minimized was (3.7), namely,

\[
\int \{(u^2 + v^2)z_\rho + 2\mu [\Delta_d \cdot (u^z z_\rho) - (Fz_\rho)_\rho] \} \, d\rho \, dX \, dY.
\]

To make the bolus velocities baroclinic, we also need to apply the requirement that

\[
\int_{\rho_b}^\rho u^z z_\rho \, d\rho = 0
\]

everywhere. Adding in another Lagrange multiplier, then, we minimize

\[
\int \{(u^2 + v^2)z_\rho + 2\mu [\nabla_d \cdot (u^z z_\rho) - (Fz_\rho)_\rho] \} \, d\rho \, dX \, dY
\]

\[-2 \int dX \, dY \int_{\rho_b}^\rho u^z z_\rho \, d\rho.
\]
Here, \( f = [f_1(X, Y), f_2(X, Y)] \). Both \( u^* \) and \( \nu^* \) can be varied simultaneously, giving
\[
\begin{align*}
u^* &= \mu_X + f_1, \\
u^* &= \mu_Y + f_2,
\end{align*}
\]
(A.1a)
\[
V_d \cdot z \rho (V_d \mu + f) = (Fz \rho)_\rho.
\]
(A.1b)

Again, on vertical walls, we have
\[
n \cdot (V_d \mu + f) = 0.
\]

The \( f_i \) are awkward and play the role of removing the barotropic component, so that \( \psi_{1,2} \) can vanish at the surface and floor. Indeed, the additional requirement comes from integrating (A.1a) from bottom to top and requiring the streamfunctions to vanish there. This gives
\[
\int_{\rho_B}^{\rho_T} z \rho V_d \mu \, d\rho + Bf = 0,
\]
(A.2)
where \( \rho_B, \rho_T \) are again bottom and surface densities. So if we define a pseudovertical averaging operator (the bar and the gradient do not commute)
\[
\bar{V}_d \mu = \frac{1}{B} \int_{\rho_B}^{\rho_T} z \rho V_d \mu \, d\rho,
\]
then
\[
V_d \cdot z \rho (V_d \mu - \bar{V}_d \mu) = (Fz \rho)_\rho,
\]
\[
n \cdot (V_d \mu - \bar{V}_d \mu) = 0
\]
on vertical walls, and so on. This operator is therefore really three-dimensional, “knowing” about the baroclinic requirement of the bolus velocity. We have not found a suitable numerical scheme for this version of the problem.

**APPENDIX B**

**The Mixed Layer**

The solution in the main text assumes that there is no mixed layer. This appendix examines the equivalent problem, including the mixed layer. We first treat here the isolated problem of minimizing bolus flows within a simple mixed layer, and we show that with a general diapycnal forcing, there may be no bolus velocity that can reproduce the observed density.

Techniques for treating bolus effects in the mixed layer remain under debate (Ferrari et al. 2008), with both Killworth (2005) and Canuto and Dubovikov (2006) arguing for a diffusivity representation of bolus effects, with the diffusivity related to that used in defining the bolus velocity from, for example, the density slope in the interior. We can treat both bolus and diffusion terms.

**a. Bolus velocities in the mixed layer**

For the present purposes, we assume that eddy effects continue to be produced by an additional bolus velocity, now termed \( u^+ \) for clarity, as in the stratified portion of the flow. In the mixed layer, whose density will be denoted by \( \bar{\rho}_S \), to distinguish it from the (possibly different) \( \bar{\rho}_T \) at the top of the stratified fluid, we have \( \bar{\rho}_T = 0 \), and we write
\[
V \cdot (u^+ \bar{\rho}_S) = Q/h - \partial \bar{\rho}_S / \partial t,
\]
where \( Q \) is the net surface buoyancy forcing (i.e., diabatic), which may include mean entrainment fluxes, whose area integral must be zero for a steady state, and \( h \) is the local mixed layer depth. We let
\[
V \cdot (u^+ \bar{\rho}_S) - Q/h + \partial \bar{\rho}_S / \partial t = F_s,
\]
so that
\[
V \cdot (u^+ \bar{\rho}_S) = -F_s
\]
is to be solved. Note that by construction, \( F \) may not be continuous across the mixed layer base; we assume it is independent of depth in the mixed layer. Also, \( \int F_s \, dA = 0 \) must hold, so that \( F_s \) must vanish somewhere. In the mixed layer, the trivial solution \( u^+ = -\bar{u} \) cannot hold.

It is clear that (B.1) will not in general possess solutions within a closed basin. The simplest counterexample is
\[
\bar{\rho}_S = \alpha x,
\]
\[
V \cdot (u^+ \bar{\rho}_S) = u^+ \bar{\rho}_S = -F_s,
\]
\[
u^+ = -F_s / \alpha.
\]

In general, the normal component of the bolus velocity will be nonzero at one vertical boundary at least. The same difficulty occurs if one seeks to minimize rms errors in (B.1). So consistent solutions can only exist under special circumstances.

The above argument has treated the mixed layer in isolation, whereas in fact it is intimately tied to the interior problem. Including both the mixed layer and the interior in the minimization does not improve matters. To see this, consider the minimization in Cartesian coordinates.

We minimize (for a steady state)
\[
\int \{ (\psi_{1z}^2 + \psi_{2z}^2) + 2\lambda [\psi_{12}\bar{p}_x + \psi_{22}\bar{p}_y - (\psi_{1x} + \psi_{2y})\bar{p}_z + F] \} \, dV.
\]

Small perturbations to this must vanish for a minimum, so that

\[
\int \{ \delta \psi_{12}\psi_{1z} + \delta \psi_{22}\psi_{2z} + \lambda [\delta \psi_{12}\bar{p}_x + \delta \psi_{22}\bar{p}_y - (\delta \psi_{1x} + \delta \psi_{2y})\bar{p}_z] \} \, dV = 0.
\]

Integrating by parts gives

\[
\int dA [\delta \psi_{12}\psi_{1z} + \delta \psi_{22}\psi_{2z} + \lambda \delta \psi_{12}\bar{p}_x + \lambda \delta \psi_{22}\bar{p}_y] \, dz - 0
\]

\[+ \int [-\delta \psi_{12}\psi_{1zz} - \delta \psi_{22}\psi_{2zz} - \delta \psi_{12}(\lambda \bar{p}_x)_z - \delta \psi_{22}(\lambda \bar{p}_y)_z + \delta \psi_{12}(\lambda \bar{p}_x)_x + \delta \psi_{22}(\lambda \bar{p}_y)_y] \, dV = 0.
\]

Now, at the surface, \( \psi_{1x} + \psi_{2y} = 0 \), so \( \psi_1 = -\chi_y, \psi_2 = \chi_x \) holds at the surface again. Also \( \psi_{1x}\bar{p}_x + \psi_{2y}\bar{p}_y + F = 0 \) at the surface. Therefore, the surface contribution becomes, integrating by parts,

\[
\int dA \delta \chi [(\psi_{1z} + \lambda \bar{p}_x)_y - (\psi_{2z} + \lambda \bar{p}_y)_x] \, dz = 0,
\]

which vanishes only when

\[ (\psi_{1z} + \lambda \bar{p}_x)_y - (\psi_{2z} + \lambda \bar{p}_y)_x = 0, \quad z = 0. \]

Because \( \psi_1 \) and \( \psi_2 \) can be varied independently in the interior, calculus of variations implies

\[ -\psi_{1zz} - \lambda \bar{p}_x + \lambda_x \bar{p}_z = 0, \]

\[ -\psi_{2zz} - \lambda \bar{p}_y + \lambda_y \bar{p}_z = 0, \]

or

\[ \psi_{1zz} = J_{xZ}(\lambda, \bar{p}), \]

\[ \psi_{2zz} = J_{yZ}(\lambda, \bar{p}), \]

where \( J_{ab}(c, d) = c_a d_b - c_b d_a \). These are to be solved together with the original equation

\[ \psi_{1x}\bar{p}_x + \psi_{2y}\bar{p}_y - (\psi_{1x} + \psi_{2y})\bar{p}_z + F = 0, \]

or

\[ J_{xZ}(\psi_1, \bar{p}) + J_{yZ}(\psi_2, \bar{p}) = F. \]

To proceed, we move into density coordinates in the vertical, assuming for the moment that there is no mixed layer, so that

\[ \frac{\partial}{\partial \rho} \left( \frac{\psi_{12}}{\bar{z}_p} \right) = \lambda_X, \quad (B.2) \]

\[ \frac{\partial}{\partial \rho} \left( \frac{\psi_{22}}{\bar{z}_p} \right) = \lambda_Y, \quad (B.3) \]

\[ \psi_{1X} + \psi_{2Y} = FZ_\rho, \quad (B.4) \]

Equation (B.4) is just the expression of the diapycnal flux in density coordinates. If we set

\[ \lambda = \mu_\rho, \]

then (B.2) and (B.3) can be integrated as

\[ \psi_{1p} = z_p(\mu_X + f_1(X, Y)), \]

\[ \psi_{2p} = z_p(\mu_Y + f_2(X, Y)). \]

We first apply the surface boundary condition, assuming that there is no mixed layer. In density coordinates, this becomes

\[ (\psi_{1z} + \lambda \bar{p}_x)_y - (\psi_{2z} + \lambda \bar{p}_y)_x = 0, \quad z = 0. \]

\[ \left( \frac{\psi_{1p}}{\bar{z}_p} \right)_y = \frac{z_y}{\bar{z}_p} \left( \frac{\psi_{1p}}{\bar{z}_p} \right)_X - \frac{z_x}{\bar{z}_p} \left( \frac{\psi_{1p}}{\bar{z}_p} \right)_Y + \frac{z_y}{\bar{z}_p} \left( \frac{\psi_{2p}}{\bar{z}_p} \right)_Y - \frac{z_x}{\bar{z}_p} \left( \frac{\psi_{2p}}{\bar{z}_p} \right)_X \]

\[ = z = 0, \quad f_{1Y} - f_{2X} = 0. \]

so that \( f_1 = \sigma_X, f_2 = \sigma_Y \) for some function \( \sigma(X, Y) \). Without loss of generality, \( f_i \) may be subsumed into \( \mu \). Then,

\[ u^+ = \mu_X, \]

\[ v^+ = \mu_Y, \]

and substituting in \( \partial/\partial \rho \) (B.4) gives

\[ \nabla \cdot z_p(\nabla \mu) = (Fz_\rho)_\rho, \quad (B.6) \]

which is the same problem as already discussed.

Now, we examine the effect of adding a mixed layer whose density is continuous at its base.\(^9\) Within the mixed layer \( \bar{p}_z = 0 \), and so we may integrate to get

\[ \psi_{1z} + \psi_{2y} = FZ_\rho, \]

where the density may jump at the mixed layer base, and there may be active entrainment. These options are not considered here.
\[ \psi_{1z}(z = 0) - \psi_{1z}(z = -h) = -\bar{p}_z[\lambda(z = 0) - \lambda(z = -h)], \]
\[ \psi_{2z}(z = 0) - \psi_{2z}(z = -h) = -\bar{p}_z[\lambda(z = 0) - \lambda(z = -h)], \]
so that, applying the surface boundary condition we have
\[ \frac{\partial}{\partial y} \psi_{1z}(z = -h) - \frac{\partial}{\partial x} \psi_{2z}(z = -h) \]
\[ = -\bar{p}_z \frac{\partial}{\partial y} \lambda(z = -h) + \bar{p}_y \frac{\partial}{\partial x} \lambda(z = -h). \]

We assume that \( \lambda \) is continuous across the base of the mixed layer, so that \( \psi_{1z} \) is also continuous. Then, the above equation can be written as
\[ \frac{\partial}{\partial y} \psi_{1z}(x, y, -h(x, y)) - \frac{\partial}{\partial x} \psi_{2z}(x, y, -h) \]
\[ = -\bar{p}_z \frac{\partial}{\partial y} \lambda(x, y, -h) + \bar{p}_y \frac{\partial}{\partial x} \lambda(x, y, -h) \]
or
\[ \frac{\partial}{\partial y} [\mu_x(X, Y, \bar{p}_S) + f_1] - \frac{\partial}{\partial x} [\mu_y(X, Y, \bar{p}_S) + f_2] \]
\[ = -\bar{p}_{xy} \frac{\partial}{\partial y} \lambda(X, Y, \bar{p}_S) + \bar{p}_y \frac{\partial}{\partial x} \lambda(X, Y, \bar{p}_S), \]
which may be rewritten as
\[ \bar{p}_{xy} \mu_x + f_1 - \bar{p}_{xy} \mu_y - f_2 = -\bar{p}_{xy} \lambda \]
or \( f_1 - f_2 = 0, \)
which is the same result as in the case of no mixed layer. Integrating again within the mixed layer gives
\[ \psi_{1z} - \mu_x(X, Y, \bar{p}_S) = -\bar{p}_{xy}[\lambda - \mu_p(X, Y, \bar{p}_S)] \]
and similarly in the \( y \) direction, so that
\[ \bar{p}_{xy} \{ \mu_x(X, Y, \bar{p}_S) - \mu_p(X, Y, \bar{p}_S) \} \]
\[ + \bar{p}_{xy} \{ \mu_y(X, Y, \bar{p}_S) - \mu_p(X, Y, \bar{p}_S) \} + F = 0, \]
which specifies \( \lambda \) in the mixed layer as a function of quantities at the top of the interior stratified region:
\[ \lambda(\bar{p}_S^2 + \bar{p}_{xy}^2) = \bar{p}_{xy}[\mu_x(X, Y, \bar{p}_S) + \mu_y(X, Y, \bar{p}_S)] \]
\[ + \bar{p}_{xy} [\mu_y(X, Y, \bar{p}_S) + \mu_p(X, Y, \bar{p}_S)] + F \]
or
\[ [\lambda - \mu_p(X, Y, \bar{p}_S)](\bar{p}_S^2 + \bar{p}_{xy}^2) = \mu_x(X, Y, \bar{p}_S)\bar{p}_{xy} \]
\[ + \mu_y(X, Y, \bar{p}_S)\bar{p}_{xy} + F \]
\[ \text{u}^* - \nabla\mu(X, Y, \bar{p}_S) \]
\[ = -\frac{\nabla\bar{p}_S}{|\nabla\bar{p}_S|^2} [\nabla\mu(X, Y, \bar{p}_S) \cdot \nabla\bar{p}_S + F]. \quad (B.7) \]

Note that at present there is no reason for the bolus flow to vanish normal to a boundary. This appears difficult to satisfy. In the interior,
\[ \nabla\mu \cdot n = 0 \]
at a boundary. We need
\[ \text{u}^* \cdot n = 0 \]
at the same location in the mixed layer. From (B.7), this requires either
\[ \nabla\bar{p}_S \cdot n = 0 \quad \text{or} \quad \nabla\mu(X, Y, \bar{p}_S) \cdot \nabla\bar{p}_S + F = 0 \]
at a boundary. The first of these does not happen in general, and because \( \mu \) already satisfies a boundary condition, the second cannot hold either.

Thus, in general, a bolus velocity cannot represent arbitrary flux divergences in the mixed layer and still satisfy no normal flow at a boundary.

b. Diffusion terms in the mixed layer

Instead, we now assume that the eddy terms are given by a lateral diffusion term satisfying
\[ \nabla \cdot (\kappa \nabla \bar{p}_S) - F = 0 \quad (B.8) \]
and we seek, presumably, to minimize some quantity over the mixed layer. We first choose the quantity to be \( \kappa^2 \). (We shall assume for convenience that \( \nabla \bar{p}_S \cdot n \) vanishes at lateral boundaries.) Thus, we seek the minimum of
\[ \int_{-h}^{0} dx \int_{-h}^{0} dy \int_{-h}^{0} dz \left\{ \kappa^2 + 2\lambda(x, y) [\nabla_2 \cdot (\kappa \nabla \bar{p}_S) - F] \right\}. \]
This gives
\[ \kappa = \nabla \bar{p}_S \cdot \nabla \lambda, \]
and substitution into (B.8) gives an equation for \( \lambda \):
\[ \nabla_2 \cdot [(\nabla \bar{p}_S \cdot \nabla \lambda) \nabla \bar{p}_S] = F. \]
The second-order terms in \( \lambda \) are
\[ \lambda_{xy} \bar{p}_S^2 + 2\lambda_{xy} \bar{p}_S \bar{p}_{xy} + \lambda_{yy} \bar{p}_{xy}^2 \]
so that the equation for \( \lambda \) is everywhere parabolic. A natural coordinate for this is normal to density, so that \( \lambda \) “diffuses” normal to density.
This is not surprising. Using characteristics, (B.8) can be solved directly:

\[
\frac{dx}{ds} = \bar{p}_{Sx}, \quad \frac{dy}{ds} = \bar{p}_{Sy}; \quad \text{then} \quad \frac{d\kappa}{ds} = F_s - \kappa \bar{V}_2^2 \bar{p}_S
\]

defines \(\kappa\) entirely apart from "initial" values along lines of constant density. Adjusting these are the only degrees of freedom of the problem.

If instead we minimize the diffusive fluxes (i.e., minimize \(\kappa^2 |\bar{V}_2 \bar{p}_S|^2\)), we merely find

\[
\kappa = \frac{\bar{V}_2 \bar{p}_S \cdot \bar{V}_2 \lambda}{|\bar{V}_2 \bar{p}_S|^2},
\]

and now \(\lambda\) satisfies

\[
\bar{V}_2 \cdot \left( \frac{\bar{V}_2 \bar{p}_S \cdot \bar{V}_2 \lambda}{|\bar{V}_2 \bar{p}_S|^2} \right) - F_s = 0,
\]

and this remains identically parabolic as before.

In both cases, the parabolic nature of the problem means that there will be no sensible solution under realistic boundary conditions.

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