Evolution of a Random Directional Wave and Freak Wave Occurrence

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ABSTRACT

The evolution of a random directional wave in deep water was studied in a laboratory wave tank (50 m long, 10 m wide, 5 m deep) utilizing a directional wave generator. A number of experiments were conducted, changing the various spectral parameters (wave steepness \( \varepsilon = 0.05 \), with directional spreading up to \( 36^\circ \) and frequency bandwidth \( 0.2 < \delta k / k < 0.6 \)). The wave evolution was studied by an array of wave wires distributed down the tank. As the spectral parameters were altered, the wave height statistics change. Without any wave directionality, the occurrence of waves exceeding twice the significant wave height (the freak wave) increases as the frequency bandwidth narrows and steepness increases, due to quasi-resonant wave–wave interaction. However, the probability of an extreme wave rapidly reduces as the directional bandwidth broadens. The effective Benjamin–Feir index (BFI\(_{\text{eff}}\)) is introduced, extending the BFI (the relative magnitude of nonlinearity and dispersion) to incorporate the effect of directionality, and successfully parameterizes the observed occurrence of freak waves in the tank. Analysis of the high-resolution hindcast wave field of the northwest Pacific reveals that such a directionally confined wind sea with high extreme wave probability is rare and corresponds mostly to a swell–wind sea mixed condition. Therefore, extreme wave occurrence in the sea as a result of quasi-resonant wave–wave interaction is a rare event that occurs only when the wind sea directionality is extremely narrow.

1. Introduction

A number of freak wave studies have been conducted theoretically, experimentally, and observationally in the last decade or so, which has significantly advanced our knowledge of ocean waves. A number of important discoveries were made in the middle of the twentieth century: the fetch law for the growing wind sea, the equilibrium wave spectrum, four-wave resonance interaction, the wind wave growth mechanism, and the statistical description of wave height distribution. These findings seem robust and to date are still a good description of the observed ocean wave field. Seafarer stories about giant waves seemed ludicrous and, even if they existed, the half-century-old conception of the ocean waves did not seem to need to change significantly because freak waves are so rare. But, this picture may not be true any longer. Recent freak wave research has forced researchers to challenge our existing understanding of ocean waves and to carefully validate theories against observations in search of a better understanding of giant waves. Freak waves are defined in this paper as wave exceeding twice the significant wave height of a given wave record. The basis of this definition is that the occurrence of those waves is very rare—about once every 3000 waves according to linear theory. Other definition exists such as 2.2 times the significant wave height and its occurrence is once every 16 000 waves.
In retrospect, there were hints about the possible existence of freak waves in the ocean. A uniform wave train of finite amplitude (the Stokes wave) is known to be unstable to a pair of sideband perturbations (Benjamin and Feir 1967; Zakharov 1968). The stability analysis suggested that the growth of the perturbation is largest for a collinear disturbance, whereas the growth rate rapidly reduces as the direction of the perturbation deviates from the energetic wave (e.g., McLean 1982).

The long-term evolution of an unstable wave train can be described within the framework of a weakly nonlinear narrow banded wave system. With use of the nonlinear Schrodinger equation, Lake and Yuen (1978) have shown that the wave train undergoes a cycle of modulation and demodulation, analogous to the Fermi–Pasta–Ulam recurrence. Under certain conditions, as the modulation increases, the individual wave becomes steep enough and the wave breaks in a wave group (Melville 1982; Tulin and Waseda 1999). The steepness of the breaking wave is quite high but it is less than the Stokes limiting wave height as previous work has indicated (Su and Green 1984; Tanaka 1990; Tulin and Waseda 1999). However, the geometry of the breaking wave is such that a strong radar signal is expected at low grazing angle (Fuchs et al. 1999; Tulin and Landrini 2001). Breaking waves are imaged as strong radar backscatter that moves at the wave phase speed and their repeat pattern propagates at the group speed (Poulter et al. 1994; Lamont-Smith et al. 2003). The regularity of the repeat pattern suggested that modulation instability is at work in the open ocean, but this is not necessarily related to the freak wave generation mechanism.

As interest in understanding freak waves has grown, the view has changed and many now believe that quasi-resonant interaction is at work in the random directional ocean wave (Janssen 2003; Onorato et al. 2004; Mori and Janssen 2006). The theory of the instability of a discrete wave system (i.e., the Benjamin–Feir instability) was extended to describe the evolution of random directional waves by Alber (1978), who has shown theoretically that a wave system is unstable when the oblique perturbation wave angle \( \delta \theta / \delta k \) is less than 35° or so. Therefore, the theory suggests that wave group formation due to instability is likely to occur only for a directionally confined spectrum. On average, observed ocean waves have a broad distribution (e.g., Hwang and Wang 2001) and, when the directionality is sufficiently broad, then four-wave interaction (Hasselmann et al. 1985) becomes dominant and the directional wave spectrum is self-stabilized (Young and Van Vledder 1993). Consequently, a sufficiently narrow wave spectrum for the quasi-resonant interaction to be dominant is very rare, even under rapidly changing wind fields and under the influence of ocean currents (Tamura et al. 2008). An alternative approach is to describe the ocean wave field as a superposition of wave groups. For instance, Donelan et al. (1972) have identified the existence of wave groups in the ocean from visual observation of the whitecap frequency. In the spectral domain a wave group can be represented by a narrow banded spectrum and the evolution of such a wave group can be studied using the nonlinear Schrodinger equation (NLS hereafter). Yuen and Lake (1982) have suggested the use of the NLS in describing the ocean wave. Tulin and Landrini (2001) has advanced the idea further and suggested the possibility that for each direction an independent wave group can form and its growth is governed by independent fetch laws. Donelan et al. (1996) have previously suggested the idea of superposing wave groups propagating at different angles to describe ocean waves. Observation of breaking waves in the ocean by Gemmrich and Farmer (1999) reinforces their suggestion since the accumulated directional distribution of the individual breaking waves with their unique propagating angle reproduced the conventional spreading of the directional ocean wave spectrum. The mathematical representation suitable for such a wave field is a projection onto local orthogonal basis functions (e.g., wavelet transforms) rather than onto global orthogonal basis functions (e.g., the Fourier transform). Donelan et al. (1996) developed a directional wave detection method that utilizes wavelet transformation instead of a Fourier transformation. Given the success of their analysis, and considering the inhomogeneity of the ocean wave field, the premise that the representation of ocean waves by Fourier modes may be debatable: nevertheless, it is convenient to use the Fourier representation of the wave field and consider the nonlinear interaction among Fourier modes. The evolution of the random directional wave is best described by Hasselmann's kinetic equation (Hasselmann et al. 1985); the Fourier modes interact among each other when the resonance condition is met at the third order in wave steepness. The interaction, therefore, would require thousands of wave periods (e.g., Badulin et al. 2007). The resulting nonlinear transfer of energy downshifts the spectral peak and simultaneously the energy cascades toward high frequency and is dissipated through whitecapping.

A deterministic evolution of the directional wave can be studied using Zakharov's equation (Zakharov 1968). Unlike Hasselman's kinetic equation that averages the random phases, Zakharov's equation explicitly resolves the amplitudes and phases, and allows interaction among combinations of waves that do not satisfy the resonance condition. The effect of this frequency mismatch from
exact resonance is emphasized when the resonance de-
tuning annuls the amplitude dispersion (Phillips 1974). A
typical case is the Benjamin–Feir (BF) instability of the
Stokes wave in which the quasi-resonant interaction oc-
curs among the carrier wave and the pair of sideband
waves. Note that, in the case of BF instability, the space
and time are conveniently switched so that frequencies of
the carrier wave (counted twice) and the pair of sideband
waves satisfy the resonant condition, whereas the corre-
sponding wavenumbers do not. Thus in physical space, as
the amplitude modulation grows in space, a wave group
is formed and the largest wave in the group breaks for a
sufficiently energetic wave train (Su and Green 1984;
has related the modulational instability to the occurrence
of freak waves for a random directional wave; the nu-
erical solution to Zakharov’s equation indicated that
the extreme wave occurrence (say, waves that exceed
twice the significant wave height) increases as the wave
steepness increases and the frequency bandwidth nar-
rows. From Monte Carlo simulation runs, Janssen has
shown that for a unidirectional wave spectrum the in-
crease of kurtosis is related to the quasi-resonant inter-
action parameterized by the Benjamin–Feir index, which
is the ratio of steepness to frequency bandwidth. As a
result of the quasi-resonant interaction, the conven-
tional Rayleigh distribution for the wave height should
be modified. Onorato et al. (2004) have experimentally
shown that by altering the peak enhancement of a Joint
North Sea Wave Observation Project (JONSWAP)
type spectrum (and therefore cleverly controlling the
BFI) the exceedence probability increased and so did
the kurtosis. Mori and Janssen (2006) have further pa-
rameterized the freak wave occurrence probability by
the kurtosis. Of course, as Alber (1978) has indicated, the
modulational instability is most effective for unidirec-
tional wave systems. Soquet-Juglard et al. (2005) and
Onorato et al. (2002) have also indicated that the effi-
ciency is reduced with increased directionality, and more
explicitly the effect of crest length was investigated by
Gramstad and Trulsen (2007). As the spectrum becomes
sufficiently broad, Hasselman’s four-wave interaction
(i.e., the resonant interaction) is most efficient in trans-
ferring energy among wave components, and therefore,
the extreme wave probability is reduced.

The purpose of this study is to investigate experi-
mentally the evolution of the random directional wave
field including breaking effects. The basis of the deter-
ministic evolution of a wave train is shown in the sim-
plest case for the unstable Stokes wave, so, in section 2,
we review the initial instability and the long-term evol-
uation of the Stokes wave including the impact of
breaking dissipation. Since numerous studies suggest
that the BFI does not correlate well with extreme wave
statistics in the open ocean where directionality is broad
(Babanin and Soloviev 1998), experiments have been
conducted that have ranges of directional spreading.
The details of the experimental conditions are pre-
sented in section 3. In section 4, the experimental results
of the directional wave evolution are reported. In sec-
section 5, the implication of the experimental findings for
identifying extreme waves in the ocean is discussed
utilizing the wave hindcast from the high-resolution
wave–current coupled model. Conclusions follows.

2. Evolution of the unstable Stokes wave

The works by Janssen (2003), Onorato et al. (2002,
2004), and Soquet-Juglard et al. (2005) suggest that the
probability of freak wave occurrence is high when the
wave spectrum is narrow and quasi-resonant interaction
becomes significant. The simplest model for the spec-
trally narrow wave system is the unstable Stokes wave
train, whose evolution is well studied and documented.
In this section, a brief review is given of the instability of
the Stokes wave and its long-time evolution, including
the effects of wave breaking.

a. Instability of the Stokes wave

From Zakharov’s equation, the linearized evolution
equation of the sideband wave system can be readily
derived:

\[
\frac{\partial b_+}{\partial t} = k_0^2 a^2 b_- \left( \frac{\omega_0}{2k_0} \right)^{1/2} \left( \frac{\omega - \omega_+}{2k_+} \right)^{1/2} T_{1123} \sin \phi
\]

\[
\frac{\partial b_-}{\partial t} = k_0^2 a^2 b_+ + \left( \frac{\omega_0}{2k_0} \right)^{1/2} \left( \frac{\omega - \omega_-}{2k_-} \right)^{1/2} T_{1123} \sin \phi
\]

\[
\frac{\partial b}{\partial t} = -2 \left( \frac{\omega_0}{2k_0} \right) k_0^2 a^2 T_{1111}
+ 2 \left( \frac{\omega_0}{2k_0} \right) k_0^2 a^2 T_{1212} + \left( \frac{\omega_0}{2k_0} \right)^{1/2} \left( \frac{\omega - \omega_+}{2k_+} \right)^{1/2} T_{1123} \cos \phi
+ 2 \left( \frac{\omega_0}{2k_0} \right) k_0^2 a^2 T_{1313} + \left( \frac{\omega_0}{2k_0} \right)^{1/2} \left( \frac{\omega - \omega_-}{2k_-} \right)^{1/2} T_{1123} \cos \phi - \Delta \omega:
\]
where the subscripts $i,j,k,l$ denote the combination of waves that satisfy $\mathbf{k}_i + \mathbf{k}_j = \mathbf{k}_k + \mathbf{k}_l$. In this special case of unstable Stokes waves, the carrier wave is represented by $i = j = 1$, and the sideband waves are represented by $k = 2; j = 3$, whose wavenumbers are $\mathbf{k}_2, \mathbf{k}_3, \ldots$. The corresponding wave frequency does not satisfy the exact resonance condition and the resonance detuning is introduced: $2\omega_0 = \omega_+ + \omega_- + \Delta\omega$. To this order of approximation, the carrier wave amplitude remains constant, $\partial a / \partial t = 0$, whereas the sideband wave amplitudes, $b_+, b_- << a$, grow at an exponential rate when the combined phase $\phi = 2\alpha - \beta_+ - \beta_- - \Delta\omega t$ is fixed. Here $\alpha, \beta_+, \beta_-$ are the phases of the participating waves: the carrier ($\omega^2 = g k_0$), the upper sideband [$\omega^2 = g k_+ = g(k_0 + \delta k)$], and the lower sideband [$\omega^2 = g k_- = g(k_0 - \delta k)$] waves, respectively. A straightforward integration of the evolution of Eq. (1) leads to the derivation of the initial growth rate of the sideband perturbations:

$$\beta / \omega_0 = \frac{1}{2} \varepsilon^2 T_{1123} \sin \phi,$$

$$\cos \phi = \left[ \frac{\Delta \omega / \omega_0}{\varepsilon^2} - \left( -T_{1111} + T_{1212} + T_{1313} \right) / T_{1123} \right] / T_{1123}.$$  

(2)

The second equation of (2) represents the instability condition when the resonance detuning ($\Delta \omega t = (2\omega_0 - \omega_+ - \omega_-) t$) annuls the amplitude dispersion [terms involving $T_{1111}, T_{1212}, T_{1313}$ in (1)]. The normalized growth rate is proportional to the steepness squared, $\varepsilon^2$ ($\varepsilon = a k_0$); the magnitude of the interaction among the carrier and the two sideband waves $T_{1123}$; and the effectiveness of the instability represented by $\sin \phi$. Comparing this growth rate to that originally derived by Benjamin and Feir (1967), $\beta = \varepsilon^2 \delta (2.0 - \delta^2)^{1/2}$, apparently the $\sin \phi$ term depends on the normalized spectral bandwidth; the normalized frequency bandwidth is $\tilde{\delta} = (\delta \omega / \omega)_/\varepsilon$. For the unidirectional case, the normalized interaction coefficient $T_{1123} = V_{0,1,2,3} / k_0^3$ is 1. When the sideband waves are oblique to the carrier wave, both $T_{1123}$ and $\sin \phi$ depend on the directional perturbation wavenumber ($\delta k, \delta l$); the growth rate in the perturbed wavenumber space is shown in Fig. 1. Similar growth diagrams were derived by Crawford et al. (1981) and McLean (1982). The unstable region is bounded by the four-wave resonance locus that goes through the origin; there, the locus intersects at an angle of 35.26°. Thus, the perturbation wavenumber is bounded by ($\delta k, \delta l = \sqrt{2} \delta k$) near the origin, which is the condition found by Alber (1978) for the stability of a random two-dimensional wave. To third order the bounding locus can be approximated as

$$\delta k_{\text{bound}} = \frac{2 \sqrt{2} \delta l_{\text{bound}} / 4 - 15(\delta l_{\text{bound}} / k_0)^2.}$$  

(3)

The interaction coefficient $T_{1123}$ monotonically decreases with distance along the resonance locus (figure not shown). As in the unidirectional case, the maximum growth condition is attained when the difference of the perturbation wavenumber from the resonance locus is

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1 The interaction coefficient $V_{0,1,2,3}$ is from Krasitskii (1994) and the reader should refer to the original paper for the exact mathematical representation.
the same order as the wave steepness [i.e., ($\delta k - \delta k_{\text{bound}})/k = O(\varepsilon)$]. The growth rate decreases as the perturbation deviates from the maximum growth condition offset from the resonance locus at $O(\varepsilon)$. Thus, in the growth diagram (Fig. 1) a ridge forms an order of initial wave steepness away from the resonance locus.

In the case of unidirectional perturbation (i.e., Benjamin–Feir instability), the growth rate is a function of normalized frequency bandwidth, $\delta = (\delta\omega/\omega)/\varepsilon$. Janssen (2003) introduced the Benjamin–Feir index, which is a reciprocal of $\delta$,

$$BFI = \frac{1}{\delta} = \frac{\varepsilon}{\delta\omega/\omega}, \quad (4)$$

and demonstrated its usefulness in sorting out the various statistical parameters characterizing the evolution of the random unidirectional wave. As stated earlier, $\delta$ or the BFI represents the balance of resonance detuning and amplitude dispersion, or equivalently the balance of nonlinearity and dispersion. The Benjamin–Feir solution $\beta = \sqrt{2(2.0 - \delta^2)^{1/2}}$ imposes an inevitable boundary of the instability, $0 \leq \delta \leq 1.4$, while a more sophisticated analysis slightly reduces the upper-end value, $\delta \approx 1.2$ (e.g., Waseda and Tulin 1999 and references therein). Thus, the instability occurs when BFI = $1/\delta > 0.8$ or when the normalized wavenumber bandwidth is used instead, $BFI_k = \varepsilon(\delta k/k) > 0.4$. The growth rates including oblique sideband perturbations are plotted against BFI$\_k$ in Fig. 2a. The large scatter of the growth rates results because the unstable region is bounded by the resonance locus, which does not allow a straightforward coordinate transformation of the perturbed wavenumber to the spectral bandwidth equivalent to $\delta$. The transformation introduced by Alber (1978) is adopted here, which maps the perturbation wavenumber vector to a coordinate that takes into account the effect of the bounding locus at an angle $35.26^\circ$:

$$BFI_{\text{eff}} = \frac{\varepsilon_{\text{eff}}}{\sqrt{(\delta k/k)^2 - 2(\delta l/l)^2}}. \quad (5)$$

This approximation is valid only in the neighborhood of the origin of the resonance locus wherein the infinitesimal perturbation wavenumber lies along a straight line at an angle of $35.26^\circ$. The growth rates are less scattered when the effective Benjamin–Feir index, $BFI_{\text{eff}}$, is used (Fig. 2b). The newly introduced index (5) takes into consideration the effective spectral bandwidth $\sqrt{(\delta k/k)^2 - 2(\delta l/l)^2}$, which to the lowest order is zero along the resonance locus. Along the locus, the interaction coefficient monotonically decreases, $T_{1123} \approx 1 - \delta k/4 - (5/8)\delta k^2$, with the magnitude of the perturbation wavenumber, $\delta k = \sqrt{\delta k^2 + \delta l^2}$, and therefore the effective wave steepness, introduced in (5), is $\varepsilon_{\text{eff}} \approx [1 - \delta k/4 - (5/8)\delta k^2]^{1/2}$. 

b. Long-term evolution of the unstable wave train

The long-term evolution of the unstable Stokes wave undergoes the so-called Fermi–Pasta–Ulam recurrence, which is a repetition of the growth of the sideband energy and its decay (Lake and Yuen 1978). The process becomes irreversible when the waves break and the energy is lost from the wave system (Melville 1982). The loss of the energy is associated with a permanent downshifting of the spectral peak, which can be accounted for by the imbalance of the lost energy and momentum (Tulin and Waseda 1999). Because the resonant quartet interaction is impossible in the unidirectional case, the wave spectrum evolves due to quasi-resonant interaction. Without breaking dissipation, evolution is recursive, so the wave spectrum, on average, remains unchanged.

Tulin and Waseda have shown that when the initial wave steepness, $\varepsilon = a_k k_0$, is less than 0.1 or so, there will be no wave breaking and the wave train undergoes recurrence. For an initial steepness greater than 0.1, regardless of the frequency bandwidth (or the sideband frequencies) $\delta = (\Delta\omega/\omega)/\varepsilon$, there will be wave breaking and the evolution becomes irreversible. Therefore, the wave parameters that determine the initial instability also represent the eventual state of the wave train. The connection of the instability to the long-term evolution is not necessarily apparent but the study by Stiassnie and Kroszynski (1982), for example, suggests that the recurrence period is well characterized by the wave steepness $\varepsilon$, and the normalized wavenumber bandwidth $\delta$. The study of unstable Stokes waves is extended to consider the directional random wave in the next section.

3. Facility and directional waves

a. Ocean engineering tank

The experiments were conducted using the Ocean Engineering Tank of the Institute of Industrial Science, University of Tokyo (Kinoshita Laboratory and Rheem Laboratory). The tank is 50 m long, 10 m wide, 5 m deep, and is equipped with a towing carriage, a wind blower, a multidirectional wave maker, and a current generator. The multidirectional wave maker has 32 triangular plungers, 31cm wide, that are digitally controlled to generate waves of various periods ($0.5 \sim 5$ s) propagating at prescribed angles. Regular as well...
as irregular directional waves can be generated for any prescribed spectral shape. Typical arrangements of the wave wire arrays are shown in Fig. 3. The array of wave wires 2.3 m away from the sidewall is arranged at 5-m intervals to monitor the evolution of the wave spectrum and other statistical parameters down the tank. At around 14 m, an array of four wires is shown to form a regular triangle where three wires are located on the vertices and one at the center of gravity. The array was used to detect the directional distribution of wave energy.

In this study, the directional wave spectrum is stated in a variable separation form:

\[ S(\omega, \theta) = F(\omega)G(\theta, \omega). \]  

The control signal can be constructed from an arbitrary frequency spectrum \( F(\omega) \) and the directional spreading function \( G(\theta, \omega) \). A directional wave can be generated by either the single summation (SS) method or by the double summation (DS) method. The SS method assigns a single direction for each frequency component (1024 frequencies were used) and selects an angle
randomly by treating the directional spreading function as a probability density function. The SS method is used when the directional spreading function is independent of the frequency,

\[ G(u; v) = G(u); \]

a linear superposition of 1024 waves with random initial phases \( e_n \) gives the wave signal as

\[ h(x, y, t) = \frac{1}{C^2} \sum_{n=1}^{1024} a_n \cos \left( \frac{\omega_n t - \mathbf{k}_n \cdot \mathbf{x} + e_n}{C} \right), \]

where

\[ a_n = \sqrt{2F(\omega_n)\Delta\omega}. \]  

(7)

The DS method assigns multiple directions for each frequency so the wave signal is given as

\[ h(x, y, t) = \frac{1}{C^2} \sum_{n=1}^{N} \sum_{m=1}^{M} a_{n,m} \cos \left( \frac{\omega_n t - \mathbf{k}_m \cdot \mathbf{x} + e_{n,m}}{C} \right), \]

\[ a_{n,m} = \sqrt{2F(\omega_n)G(\theta_m, \omega_n)\Delta\omega\Delta\theta}, \]  

where \( N \times M = 1024 \) and \( N \) and \( M \) are each \( O(10) \). For both SS and DS methods, the wave energies of each of the wave modes are kept equal, so the selected frequencies are not uniformly distributed.

In Fig. 4, evolution of the various wave parameters (which will be described later in more detail) is compared between the DS and SS methods. The significant wave height (Fig. 4a) is nearly identical and other parameters such as the maximum wave height (Fig. 4b) show random variation between the SS and DS methods. The difference, however, is likely due to cross-tank variation inherent in the wave tank. The difference in the wave generation methods (SS and DS) is therefore irrelevant in this study, so it is not referred to in the rest of the paper, unless necessary.

b. Experimental conditions

For all of the experiments conducted, the JONSWAP frequency spectrum with the equilibrium tail proportional to \( \frac{\omega}{C_0^2} \) was used:

\[ S(\omega) = \alpha g^2 (2\pi)^{-\frac{5}{2}} \omega^{-5} \exp \left[ -\frac{5}{4} \left( \frac{\omega}{\omega_p} \right)^{-4} \exp \left( \frac{(\omega - \omega_p)^2}{2\sigma^2} \right) \right] \]  

where

\[ \omega_p = \frac{a g}{2\pi}, \]

\[ \sigma = \frac{a g}{2\pi}, \]

\[ \frac{a g}{2\pi} = 1.2389 \text{ Hz}, \]

or 7.784 rad s\(^{-1}\) (wavelength 1 m, \( T_{1/3} = T_p/1.05 = 0.7626 \text{ s} \)) so that the domain of wave propagation bounded by the tank walls is approximately 10 by 50 wavelengths. The control parameters are \( \alpha \) and \( \gamma \); the former is related to
the significant wave height and the latter to the frequency bandwidth. The wave steepness $e$ and the peakedness parameter $\gamma$ were determined as

$$e^2 = 6.84 \times 10^{-6} (2\pi)^3 \times 4\beta,$$
$$\gamma = 4.42 \left( \omega g U_{10} \right)^{3/7} = 4.42 \times (2\pi)^{-3/7} \beta^{-3/7},$$

(10)

where the wave age is defined as $\beta = (c_p/U_{10})$. To derive (10), the JONSWAP-type fetch law was utilized:

$$\frac{E\omega g^3}{(2\pi)^3 g U_{10}} = 6.84 \times 10^{-6},$$

(11)

in which $E = (0.5H_{1/3})^2$. By using the fetch law as a constraint, the relationship between significant wave height and significant wave period is not independent and will likely satisfy the Toba 3/2 law. In this way, we assure that the choice of the base parameters is such that the initial wave spectrum is in local equilibrium with the wind. The directional spreading function was selected from among the Mitsuyasu, Hwang; and bimodal type. The Hwang type (Hwang and Wang 2001) is an empirical fit to the high-resolution directional spectrum obtained by a scanning lidar in the open ocean and is characterized by the bimodal directional spreading at high frequency (see Fig. 4f in Hwang and Wang 2001). A section of the Hwang distribution at three times peak frequency was selected and used as the bimodal distribution function. Both the Hwang and bimodal distribution cases were generated by the DS method.

<table>
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<th>Parameter</th>
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<td>6.583</td>
<td>3.2068</td>
<td>1.1637</td>
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</table>

Table 1. Parameters values for wave age ($\beta$) range from 1.0 to 0.2.

**FIG. 5.** Frequency spectra averaged for three significant wave height categories (3.5, 4.0, and 5.0 cm): dotted line is $\omega^{-5}$. 

*Fig. 5. Frequency spectra averaged for three significant wave height categories (3.5, 4.0, and 5.0 cm): dotted line is $\omega^{-5}$.***
Now, since the wave age dependence of the directional distribution is given roughly as (Mitsuyasu 1975)

\[ n = 11.5 \beta^{2.5}. \]  

(10) and (13) determine the appropriate parameter range for the experiment (Table 1). Since we are not interested in a mature sea (small steepness), wave age ranges of 0.2 \( \leq \beta \leq 0.6 \) corresponding to a significant wave height between 3 and 6 cm were selected \((0.03 \leq H_s \leq 0.06 \text{ m})\). For cases exceeding \( H_s = 5.5 \text{ cm} \) or so, an energetic breaker was observed; as the magnitude and frequency of the breaker decreased, \( H_s \) decreased. Frequency spectra averaged over various directional spreads are shown for three significant wave height ranges (around 3.5, 4.0, and 5.0 cm) in Fig. 5. It is remarkable that the high frequency equilibrium tail develops naturally beyond 2.5 Hz due to nonlinear wave–wave interaction cascading the energy from the wave-maker-generated gravity waves. The exponent of the tail between around 2 and 7 Hz seems to be close to or slightly steeper than \( \omega^{-5} \).

c. Directional spectrum

The directional spectra were estimated from the four wave wires at 14-m fetch using various methods: the maximum likelihood method (MLM), the maximum entropy method (MEP), and the wavelet detection method (WDM). Only the results using the WDM are presented here. WDM is a nonstationary analysis method using wavelet transformation to detect the directionality of the random waves (Donelan et al. 1996). The method assumes the wave field to be a superposition of wave groups propagating in a single direction at each instance in time. The phase differences detected by the three wave wires are inverted to provide the wavenumber vector. The method works best in detecting a single regular wave train, but fails to differentiate two waves of similar amplitude and frequency crossing at an angle (in which case, the average wavenumber vector is falsely detected). The success of the method, therefore, implies that, to a good approximation, the random waves generated in the tank (whether the DS or SS method is used) can be decomposed into a bunch of wave groups. The directional spreads among different cases were clearly distinguished by WDM (Fig. 6), whereas by MLM and MEP they were indistinguishable. The recommended configuration of the wave wire array is to arrange the wave wires on the vertices of a pentagon (A. Babanin and M. Donelan 2007, personal communication). The pentagon array provides smoother spectra for shorter wave records but the spectral resolution did not seem to change compared to the triangular array configuration. Thus, for most of the experimental cases, we have used the directional spectra obtained from the triangular array.

Babanin and Soloviev (1998) introduced a convenient directional distribution parameter that inverts the integrand of the normalized directional distribution:

\[ A^{-1} = \pi \int_{-\pi}^{\pi} K(\theta; f) d\theta; \quad \pi \int_{-\pi}^{\pi} A K(\theta; f) d\theta = 1. \]  

Here \( AK(\theta; f) = G(\theta; f) \), where \( G(\theta; f) \) is the directional distribution function and \( \max [K(\theta; f)] = 1 \). The \( A \) parameters from the measured directional spectrum at peak frequency have been estimated and compared to the theoretical value of the Mitsuyasu-type directional distribution (12) (Fig. 7a). The disagreement in the case of a broad spectrum (say, \( n \) less than 75 or so) could be
due to the directional resolution of the WDM and/or the failure of the wave generator to generate broad enough spectra. On the other hand, the saturation of the parameter in the narrow spectrum indicates the directional resolving power of the wave-wire array configuration. The cross-correlation between two wires separated by 32 cm (at fetch 14 m) was used as an alternative to detect the directional spreading. The values and the cross-correlation \( R_{23} \) have one to one correspondence to a good approximation (Fig. 7b). Therefore, either parameter can be used to represent the measured directional spread. For convenience, a directional spreading angle \( \theta_{1/2} \), defined as the angle where half of the wave energy is contained,

\[
0.5 = \int_{-\theta_{1/2}}^{\theta_{1/2}} G(\theta) \, d\theta / \int_{-\pi/2}^{\pi/2} G(\theta) \, d\theta,
\]

has been introduced and will be used in the rest of the paper.

4. Evolution of the random directional wave

a. Exceedence probability

The distribution of wave height follows the Rayleigh distribution if the evolution of the random wave is linear. Onorato et al. (2004) have shown empirically that for a unidirectional random wave the distribution of extreme waves is enhanced due to quasi-resonant interaction. Figure 8 summarizes the exceedence probability for different directional spreading \( n = 125, 75, 3 \) and frequency bandwidth \( \gamma = 30.0, 3.0, 1.0 \) for cases with relatively large wave steepness \( \varepsilon \sim 0.11 \). As the spectrum becomes narrower in direction and frequency bandwidth (the upper-left corner), deviation from the Rayleigh distribution (solid line) becomes large. On the other hand, as the spectrum becomes broader \( (\gamma = 3, n = 75) \), the distribution tends to fit better to the Rayleigh distribution. In the broadest case \( (\gamma = 1, n = 75) \), the distribution is overestimated by the Rayleigh distribution, consistent with observations in the North Sea by Forristall (1978), who has shown that the distribution is better described by a Weibull distribution with reduced probability at large wave heights.

b. Evolution of kurtosis

As Onorato et al. (2004) have shown experimentally, the exceedence probability evolves from its initial distribution as the waves propagate down the tank. Following Onorato et al. (2005), the fourth moment of the surface elevation can be expressed as a summation of the linear term, the correction due to nonresonance interaction (free waves), and the correction due to the distortion of the wave shape (Stokes correction; bound waves):

\[
\langle \eta(x)^4 \rangle = 3 \langle \eta(x)^2 \rangle^2 + 16 \int M_{1,2,3,4} T_{1,2,3,4} N_{1} N_{2} N_{3} \times \frac{1 - \cos(\Delta\omega t)}{\Delta\omega} \delta_{1234} \, dk_{1234} + 12 \int K_{1,2,3,4} N_{1} N_{2} N_{3} dk_{1234}.
\]

The kurtosis, \( \kappa_4 = \langle \eta^4 \rangle / \langle \eta^2 \rangle^2 \), is 3 when the higher order terms can be neglected. The second term evolves in time (or equivalently in space) and has been numerically studied for the unidirectional case by Janssen (2003) and experimentally by Onorato et al. (2004). Figure 9 presents an example of the evolution of the kurtosis from our experiment for a relatively small directional spread \( (\gamma = 3.0, n = 125) \). The kurtosis immediately
starts to grow and at around 15 wavelengths the growth slows down. Janssen (2003) has shown that the intensity of the second term at the fully developed state can be parameterized by the Benjamin–Feir index (4):

$$k_4 = 3.0 + \frac{\pi}{\sqrt{3}} \text{BFI}^2.$$  \hspace{1cm} (16)

The inverse of the BFI is closely related to the growth rate of the sideband wave of the unstable Stokes wave and that the wave train is unstable for $$\delta < 1.2$$, as discussed in section 2. The BFI from the observed spectrum can be estimated following Janssen and Bidlot (2003):

$$\text{BFI} = k_0 m_0^{1/2} Q_p \sqrt{2\pi}, \quad \text{where}$$

$$Q_p = 2/m_0^2 \int \omega F^2(\omega) d\omega.$$ \hspace{1cm} (17)

Here $$m_0 = \int F(\omega) d\omega.$$ According to (17), the BFI in our experiment is in the range $$0.28 < \text{BFI} < 1.0$$. The estimated kurtosis is compared against the estimated BFI in Fig. 10. The solid line corresponds to (16), which does not fit the observation. There is some arbitrariness in the estimation of the BFI from the spectrum that may account for this discrepancy, but the overall scatter of the observation clearly suggests that the directionality and the breaking dissipation, which are not considered in the derivation of (16), may be significant. The breaking dissipation imposes an upper limit on the maximum attainable wave height and correspondingly affects the estimation of the kurtosis; kurtosis can take much higher values when weakly nonlinear theory is used.

To exclude the effects of strong nonlinearity from the dependency of the kurtosis on parameters such as the frequency bandwidth and directional spread, selected cases that do not involve energetic breakers have been analyzed. The frequency bandwidth of the spectrum is defined as $$1/Q_p$$ [see Eq. (17)] and was estimated at the initial stage (fetch 14–15 m). In Fig. 11, the kurtoses at the developed stage (fetch 35–40 m) for the Hwang directional distribution cases are plotted.
Fig. 9. Evolution of kurtosis with fetch: $\gamma = 3.0, n = 125$.

Fig. 10. Kurtosis plotted against BFI. Curved line corresponds to (16).
FIG. 11. Kurtosis plotted against frequency bandwidth.

FIG. 12. Kurtosis plotted against directional spreading (°), circle: 35–40 m; square: 14–15 m.
against the frequency bandwidth (γ varied between 1 and 30). The two symbols (x and o) are for Hs around 5 cm (ak = 0.081) and 4 cm (ak = 0.073), respectively. For both cases, the value of the kurtosis decreases as the frequency bandwidth increases: Beyond 1/Qp ~ 0.4, or equivalently γ = 5 of the JONSWAP peakedness parameter, the kurtosis becomes insensitive to the frequency bandwidth as it becomes asymptotic to the linear solution, 3.0. The BFI of the narrowest case shown here is about 1.0. According to (16) the kurtosis is estimated to be about 3.8, which is much higher than the experimental result. Undoubtedly, this discrepancy is due to the broad directional spread of the cases presented in Fig. 11, which are for the Hwang directional distribution, whose approximate directional spreading is about 15°. We next observe the kurtosis as a function of directionality.

For cases with the JONSWAP peakedness parameter γ = 3.0, the kurtoses at their developed stages are plotted against directional spreading (14) 1/A (Fig. 12, circles). For fixed BFI ~ 0.36, defined by Eq. (4), the kurtosis decreases with directionality and above 7° or so (1/A ~ 0.25), which corresponds to n = 25 of the Mitsuyasu-type distribution (12), the kurtosis remains unchanged. This result suggests that, when the directional spreading is sufficiently large (> 7°), the quasi-resonant interaction becomes inactive; however, the kurtosis is still higher than 3. The increase is about 10%.

For the cases with directionality larger than 10° or so (1/A > 0.25), the kurtosis remains more or less constant down the tank (figure not shown), so the second term on the rhs of (15) should vanish. Therefore, the slight increase in the kurtosis from 3 indicates the distortion of the wave shape or the generation of bound waves [third term in (15)]. For a typical ocean wave with broad directional spread (i.e., cases of the Hwang distribution in this study) the increase of the kurtosis is mostly due to the existence of bounded waves.

Finally, we make an attempt to unify the two diagrams in Figs. 12 and 13 using the effective Benjamin–Feir index (5) introduced in section 2. Recall that the BFIeff takes into consideration the unstable region of the oblique perturbation bounded by the locus of the resonant interaction. The effective bandwidth can be derived by combining the frequency bandwidth 1/Qp and the directional spread 1/A. Since these estimates have empirical uncertainties and biases, these values were calibrated by mapping 1/Qp → δk/k and 1/A → tan φ = δl/k (see the appendix for details). As the BFIeff increases, the kurtosis tends to increase up to around BFIeff ~ 1.0; whereupon the value starts to diminish (Fig. 13 a). This tendency is quite consistent with how the growth rate depended on BFIeff (see Fig. 2). Because the perturbation wavenumber (δk, δl) optimum for the quasi-resonant interaction is not a straightforward function of δk alone, the kurtosis does not monotonically increase with BFIeff. The scatter of (δk, δl) for the case Hs = 4.15 cm is shown in Fig. 13b. The data points tend to lie mostly beneath the resonance trajectory (solid line). The leftmost four symbols are for cases in which BFIeff > 1.0 and is expected to stabilize; the corresponding values of the kurtosis indicate just that and decrease as BFIeff increases (see Fig. 13b, rightmost four points). However, with the conventional definition of BFIk = κ/(δk/k), the frequency bandwidth (δk) is much narrower than for other cases and can mistakenly be treated as being more unstable. As the (δk, δl) approaches the resonance trajectory, BFIeff becomes infinitely large. For small (δk, δl), this bounding resonance trajectory is approximately linear with an angle of 35.26°. With the new definition (5),
BFI_{eff} \sim 1 \) is an optimum condition that gives the maximum growth rate of the instability. In this study, it has been shown that such a condition is also related to the increase of the kurtosis, which is an indication of the likelihood of a freak wave.

5. Discussion

The results from the tank experiment suggest that the spectral evolution of a directionally confined random wave is governed by quasi-resonant wave–wave interaction. As a consequence, the occurrence of extreme waves becomes more likely than expected by linear wave theory. On the other hand, when the spectrum is sufficiently broad, the quasi-resonant interaction is inactive. Instead, Hasselmann’s resonant interaction governs the evolution of the directionally broad random wave (reference to the short paper). Since the evolution of the ocean wave is successfully predicted by numerous operational wave forecasting models, that is, the third-generation wave models, which do not include the quasi-resonant interaction, it is anticipated that the directionally confined wind sea is rare. To investigate how often such conditions with a narrow directional spread occur in the real world, newly derived wave hindcast data is analyzed.

**Occurrence of narrow spectrum in the real ocean**

One of the most important and robust features of ocean waves is the increase in wavelength and amplitude as they grow with fetch and duration. The primary driving mechanism for this spectral downshifting is postulated as being due to the nonlinear energy transfer by resonant wave–wave interaction. The nonlinear source term is an integral of infinite combinations of a quartet of waves that satisfies the resonance condition. Hasselmann et al. (1985) have shown that this nonlinear energy transfer can be approximated without loss of the principle feature by a single combination of waves. The approximate angle of the most effective wave combination is about 45°. Therefore, the wave spectrum tends to broaden or narrow if the directional spreading is less or more, respectively, than the equilibrium spectrum; that is, the directional wave self-stabilizes when perturbed (Young and Van Vledder 1993). The conventional view has been that, when the waves become sufficiently narrow, then the resonant nonlinear energy transfer becomes ineffective, and the waves transition to swell.

The experimental findings in this study provide us with a different perspective. When the wave spectrum becomes sufficiently narrow, quasi-resonant interaction becomes the main driving mechanism for its evolution. The spectral shape changes accordingly and, when breaking dissipation is absent, the time-averaged wave spectrum maintains its original shape. However, when breaking dissipation is present, the spectral change becomes irreversible and the spectral peak permanently shifts to lower frequency due to the imbalance of the wave energy and momentum. Therefore, although the energy transfers due to resonant wave–wave interaction...
are weak, the combination of quasi-resonant interaction and breaking dissipation results in a downshifting of the wave spectrum.

In order for the quasi-resonant interaction to be effective, we have shown that both the frequency bandwidth and the directional spreading need to be sufficiently narrow; in terms of the JONSWAP peakedness parameter $\gamma > 5.0$ and of the Mitsuyasu distribution from (12) with $n > 25$. How likely is this condition realized in the real ocean? The output of a coupled wave–current model computed near Japan will next be considered. The model is based on an improved third-generation wave model coupled with a realistic ocean prediction model, the Japan Coastal Ocean Predictability Experiment (JCOPE; Tamura et al. 2008). The simulated one-month record (October 2004) includes two passages of a sizable typhoon. A sample distribution of the frequency bandwidth and the directional spreading is shown in Fig. 14 for a typhoon approaching Japan from the southwestern end of the displayed domain. The regions that satisfy the condition ($\gamma > 5.0$ and $n > 25$) are circled for clarity. Evidently, the regions are highly restricted in the case of frequency distribution (Fig. 14a); they are located at the front of a developing wind sea region where the significant wave height is increasing (not shown). Similarly, the directional distribution tends to be narrow at the frontal region (Fig. 14b). In addition, a wide area with relatively small wave height but with narrow directionality exists in the Sea of Japan, suggestive of a swell-dominated region. These regions overlap only in a restricted area and, according to the experimental findings, it is this area where a dangerous sea might occur with freak waves. Since there is no sea-truth data available for the postulated dangerous sea, some observations are needed to prove this hypothesis. The overall distribution of frequency bandwidth and directional spreading was obtained from the one-month simulation output (Fig. 15). The histograms seem to be of different character. The distribution of the frequency bandwidth is peaked around $\gamma = 2.0$ and drops rapidly for larger $\gamma$, the occurrence of instability is quite rare. On the other hand, the distribution of the directionality is double peaked: the first peak around $n = 25$ and the second peak around $n = 3$. The latter is a typical of wind sea, whereas the former corresponds to either a swell transition or swell–wind sea mixed region. The region of possible instability ($n > 25$), therefore, is much larger than the region constrained by the frequency spreading. It should be noted here that the actual spectral shape is highly complicated, so a closer look at the spectral parameter is needed.

6. Conclusions

A laboratory experiment studying the initial instability of a wave train has been extended to study random directional waves. The long-term evolution was thoroughly investigated to better understand the mechanism of freak wave generation. Over the course of this
experiment, it has been confirmed that the spectral parameters relevant for the initial instability are also relevant for the long-term evolution. Stiassnie and Kroszynski (1982) have shown theoretically that the parameters relevant for the Benjamin–Feir instability determine two time scales: the modulation period and the modulation–demodulation cycle. However, the modulation–demodulation cycle and the maximum attainable wave height are not necessarily known to correlate. Thus, it is not obvious whether the parameters relevant for the initial instability are related to the resulting freak wave occurrence after the long-term wave evolution. This was numerically demonstrated by Janssen (2003) for the unidirectional case and, likewise, experimentally by Onorato et al. (2004). In this study we have extended the experimental study to include directionality. As soon as the directionality is introduced, difficulty in interpreting the results arises because of the distorted region of instability in the disturbance wavenumber space. We therefore started out, in this paper, discussing the instability of a Stokes wave for exploring the disturbance wavenumber space. That led us to modify the conventional Benjamin–Feir index to map the regular wavenumber space to the unstable region bounded by the resonance trajectory. This parameter is named the effective Benjamin–Feir index.

The result suggests that the increase in extreme wave probability is related to the increase of kurtosis. On the basis that the kurtosis is a relevant parameter to represent freakness of the wave train, the parameter dependence of the kurtosis was studied further. The kurtosis increases as the frequency bandwidth narrows, confirming the result of Onorato et al. (2004). Then, with the change of the directional spreading, the kurtosis decreased when the spectrum was directionally narrowed. This result is consistent with the report by Soquet-Juglard et al. (2005) and more recently by Gramstad and Trulsen (2007). We are also aware of the recent experimental and numerical works suggesting similar results (M. Onorato, P. A. E. M. Janssen, and N. Mori 2007, personal communication; Onorato et al. 2009). Thus, regardless of limitations coming from waveguide effects of the narrow channel and the capacity of the directional wave maker, we are confident that directionality does play a significant role in stabilizing the random directional wave. Different wave cases were then classified against the effective Benjamin–Feir index and it was discovered that there is an optimum value for the largest kurtosis that roughly corresponds to the maximum growth condition of the initial instability of a Stokes wave.

The experimental findings therefore suggest that freak waves are expected in a statistical sense when the spectral directionality is narrow. The next important step is to find out how often such special conditions occur in the sea. Since there is a significant lack of directional information over the open sea, the recent numerical simulation of waves near Japan has been utilized that includes the effects of the weather system and ocean currents (Tamura et al. 2008). Qualitatively, it can be understood that the “dangerous” seas occur ahead of the patch of strong wind, perhaps a transition region from wind wave to swell. The overall statistics indicate that regions of swell–wind sea mixed conditions are not negligibly small, except that the spectra are not necessarily narrow in frequency. In combination, therefore, this type of dangerous sea is extremely rare in the real ocean, and so is the occurrence of a freak wave.

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APPENDIX

Estimation of the Effective Benjamin–Feir Index

The effective Benjamin–Feir index (BFIeff) is estimated from the observed directional wave spectrum \( S(\omega, \theta) = E(\omega)G(\theta; \omega) \) following the steps below.

a. Step 1: Estimate \( \delta k / k \)

First, the Goda parameter, \( Q_p = 2/m_0^2 \int \omega E^2(\omega) d\omega \), is estimated from the frequency spectrum, where \( m_0 = \int E(\omega) d\omega \). Next, the JONSWAP spectrum is empirically fit to the frequency spectrum, \( E(\omega) \), to determine the peakedness of the observed spectrum \( \gamma \). For each \( \gamma \), the frequency bandwidth can be determined using the following definition:

\[
\left( \frac{\delta f}{f} \right)^2 \approx \frac{\int f^{-3} \exp \left\{ -\frac{5}{4} \frac{f^{-4}}{\gamma} \right\} \exp \left\{ \frac{(\gamma - 1)\gamma}{\gamma} \right\} df}{\int f^{-5} \exp \left\{ -\frac{5}{4} \frac{f^{-4}}{\gamma} \right\} \gamma \exp \left\{ \frac{(\gamma - 1)\gamma}{\gamma} \right\} df}.
\]
FIG. A1. Empirical fit used to determine the linear tendency of spectrum and was converted to the analyses, where the constant is determined as the angle within which half of the wave energy is contained: 

$$\tan \theta_{1/2} = 0.82 \times A^{-1} - 0.11.$$  \hspace{1cm} (A2)

There is no basis for \( \delta l/k = \tan \theta_{1/2} \), so we have further introduced an arbitrary parameter \( b \delta l/k = b \tan \theta_{1/2} \). We have chosen \( b = 0.45/\sqrt{2} \), which effectively corrects the poor directional resolution of the wave array and that the BFI\(_{\text{eff}}\) is nearly 1 at maximum growth condition.

c. Step 3: Estimate the effective wave steepness

The steepness is estimated from the observed frequency spectrum as \( \varepsilon = k_0 m_{0}^{1/2}/\sqrt{2} \). The perturbation wavenumber is estimated using the results from steps 1 and 2 as \( \delta k = \sqrt{\delta k^2 + \delta l^2} \). The effective steepness is

$$e_{\text{eff}} \equiv \sqrt{1 - \delta k^4 / (5/8) \delta k^2}.$$  \hspace{1cm} (A3)

d. Step 4: Estimate the effective Benjamin–Feir index

Finally, the parameters from the analysis steps 1, 2, and 3 are combined:

$$\text{BFI}_{\text{eff}} = \frac{e_{\text{eff}}}{\sqrt{(\delta k/k)^2 + 2(\delta l/k)^2}}.$$  \hspace{1cm} (A4)

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