An Analysis of Klemp–Wilhelmson Schemes as Applied to Large-Scale Wave Modes

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ABSTRACT

The use of Klemp–Wilhelmson (KW) time splitting for large-scale and global modeling is assessed through a series of von Neumann accuracy and stability analyses. Two variations of the KW splitting are evaluated in particular: the original acoustic-mode splitting of Klemp and Wilhelmson (KW78) and a modified splitting due to Skamarock and Klemp (SK92) in which the buoyancy and vertical stratification terms are treated as fast-mode terms. The large-scale case of interest is the problem of Rossby wave propagation on a resting background state. The results show that the original KW78 splitting is surprisingly inaccurate when applied to large-scale wave modes. The source of this inaccuracy is traced to the compressible vertical adjustment—and more precisely, to the splitting of the hydrostatic balance terms between the small and large time steps. The splitting errors can be reduced somewhat through implicit biasing, but large biasing coefficients are needed for acceptable error levels—and even then the time steps are limited to moderate values. The errors in the KW78 splitting are shown to be largely absent from the SK92 scheme. Two versions of the SK92 splitting are considered in particular: the original leapfrog splitting (SK92-LF) of Skamarock and Klemp and the third-order Runge–Kutta splitting (SK92-RK) proposed by Wicker and Skamarock. The mixed cubic (on the large time step) and quadratic (on the small step) behavior of the SK92-RK scheme is described in detail and is compared with the strictly quadratic behavior of the SK92-LF method.

1. Introduction

Atmospheric motions feature a broad range of frequencies, with the lowest-frequency modes often being the most important for prediction. Unfortunately, the largest time step allowed by an explicit integration scheme is ultimately limited by the highest frequencies, thereby making the explicit integration of the system very computationally expensive. A common method for relaxing this computational burden is to use split-explicit (or time-split) schemes. The basic idea behind these schemes is to use two time steps: the terms associated with fast motions are integrated on a small time step, while the remaining terms are integrated on a longer time step to increase efficiency.

The most common time-splitting approach used in mesoscale modeling is the partial-splitting method, first introduced by Klemp and Wilhelmson (1978, hereinafter KW78) and later updated by Skamarock and Klemp (1992, hereinafter SK92). This Klemp–Wilhelmson (KW) method differs from a conventional additive-splitting approach (e.g., Marchuk 1974; Strang 1968) in that the fast- and slow-mode operators are never completely split [see Durran (1999, section 7.3.2) for discussion]. Instead the fast- and slow-mode terms are integrated simultaneously, with the slow terms updated less frequently than the fast terms. This KW method and its extensions are used in a wide range of mesoscale research and forecasting models, including the widely used Weather Research and Forecasting Model (WRF) and the fifth-generation Pennsylvania State University–National Center for Atmospheric Research Mesoscale Model (MM5) (among many others).

The KW approach was originally developed in the context of mesoscale cloud modeling—specifically, for horizontal grid spacings on the order of 1 km or so and with typical time steps on the order of 10 s or less. However, in recent years the method has increasingly been used for the simulation of large-scale flows as well. Global and planetary atmosphere (particularly Mars based) versions of the KW models now exist [see, e.g., Dudhia and Bresch (2002) for the global MM5; a global WRF is under development] and the method is now...
routinely also used for regional climate simulation (e.g., Leung et al. 2003; Duffy et al. 2006). The grid spacings and time steps for these simulations are often much larger than those used at the mesoscale, with typical grid spacings on the order of 100 km and with time steps on the order of 100 s or more.

a. Historical context: The KW78 and SK92 splittings

The first consideration in any splitting algorithm is to define an operator splitting—that is, to identify terms associated with the fast modes and to split the full equations into the resulting small-step and large-step parts. In cloud modeling the fast motions are typically acoustic waves and the slower modes of interest are primarily gravity driven. The operator splitting proposed by KW78 thus treated only the acoustic modes—specifically the pressure gradient and divergence terms—as part of the small step cycle. This original KW78 splitting was adopted by a number of mesoscale community models, some of which are still widely used. Examples of widely used KW78 models include the MM5 (Dudhia 1993) and global MM5 (Dudhia and Bresch 2002) and the Colorado State University Regional Atmospheric Modeling System (RAMS) model (Pielke et al. 1992).

One potential disadvantage of the KW78 splitting is that the scheme becomes unstable to fast gravity wave motions once the large time step becomes sufficiently large (at least in principle—but see comments below). To avoid this problem, SK92 proposed an alternative operator splitting in which the buoyancy and vertical stratification terms are treated as fast-mode terms in addition to the pressure gradient and divergence. This modified splitting integrates gravity waves on the small time step, thus stabilizing the scheme for larger $\Delta t$. The computational cost of this enhanced stability turns out to be modest, and most recently developed KW models have thus used the SK92 splitting in place of the KW78 scheme. Some examples of SK92 models include the WRF model (Klemp et al. 2007), the University of Oklahoma Advanced Regional Prediction System (ARPS) model (Xue et al. 2000), and the Navy Coupled Ocean–Atmosphere Mesoscale Prediction System (COAMPS) model (J. D. Doyle 2007, personal communication), among several others.

As already mentioned, the original motivation for the SK92 splitting was the nominal instability of the KW78 scheme at large scales—specifically, for time steps exceeding $N\Delta t > 1$ (where $N$ is the Brunt–Väisälä frequency and $\Delta t$ is the large time step). However, in practical applications this KW78 stability cutoff has rarely proved to be a limitation. The reason is that the $N\Delta t > 1$ cutoff occurs only in the most nonhydrostatic modes, and at large horizontal grid spacings—specifically, for grid spacings on the order of the domain depth or larger—these modes are generally not present. Indeed, a brief survey of the literature shows that the KW78 community models are routinely run on synoptic-scale grids, and that the large time steps on these grids routinely exceed the nonhydrostatic stability cutoff (often by factors of 3 or more).

b. Study overview

The purpose of the present study is to provide a formal analysis of the KW78 and SK92 splitting methods as applied in the large-scale modeling context. Three different versions of the splittings are addressed in particular: the original KW78 leapfrog (LF) splitting proposed by Klemp and Wilhelmson (1978), the SK92 version of the leapfrog splitting (SK92-LF) suggested by Skamarock and Klemp (1992), and the newer SK92 third-order Runge–Kutta scheme (SK92-RK) presented by Wicker and Skamarock (2002). For each scheme a set of von Neumann accuracy and stability analyses is carried out for the problem of large-scale Rossby wave propagation on a resting background state.

A brief outline of the study is as follows. The following section introduces a simplified, constant-coefficient form of the Rossby wave problem and describes the von Neumann methodology as applied for the two leapfrog schemes (i.e., KW78 and SK92-LF). The results for the two leapfrog schemes are given in section 3. The original KW78 method is shown to produce significant Rossby-mode phase-speed errors, whereas the SK92-LF splitting is essentially error free. The source of the KW78 errors is explored further in section 4. Ultimately the errors are traced to the compressible vertical adjustment—or more precisely, to the failure of the scheme to maintain the hydrostatic balance. Section 5 presents the SK92-RK analysis and compares the RK and LF schemes at varying small and large time steps. The final section gives a summary of results.

It should be noted for later reference that a similar (but somewhat more involved) analysis has also been developed for the Eady baroclinic wave problem. However, the Eady results largely echo those of the Rossby problem, and thus only the Rossby case will be discussed in detail.

2. The Rossby problem: Basic formulation

The present section develops the Rossby problem formulation and discusses the discretizations used for
the two leapfrog splittings (KW78 and SK92-LF). Results for the leapfrog splittings are given in section 3.

a. Theoretical problem setup

As a starting point, consider a two-dimensional (2D) compressible-Boussinesq system on an f plane (the β effect is added below) as linearized about a resting background state; specifically

\[ u_t + P_x = f_0v, \] (1)
\[ v_t = -f_0u, \] (2)
\[ w_t + P_z = b, \] (3)
\[ b_t = -N^2w, \quad \text{and} \]
\[ P_t + c_s^2(u_x + w_z) = 0, \] (5)

where (1)–(5) are the horizontal and vertical momentum equations, the thermodynamic equation, and the pressure equation, respectively; \( u, v, \) and \( w \) are the horizontal and vertical velocity components in Cartesian coordinates; \( P \) is the Boussinesq disturbance pressure; \( b \) is the buoyancy; \( f_0 \) is the Coriolis parameter; \( c_s \) is the speed of sound; and \( N \) is the Brunt–Väisälä frequency. Subscripts denote partial derivatives with respect to the given coordinate. The parameters \( f_0, c_s, \) and \( N \) are all taken to be constants.

As is well known, the propagation of Rossby waves on a resting background state depends on the meridional variation of the Coriolis parameter. This meridional dependence renders the Rossby wave a nonconstant-coefficient problem, which in turn complicates the analysis. Fortunately, however, an equivalent constant-coefficient system can be formulated by simply adding the appropriate driving term to the 2D problem (1)–(5). The details are given in appendix A, but the end result is that (2) is replaced by the modified form:

\[ v_t = -\int_x^{x'} \beta v dx' - f_0u, \] (6)

where \( \beta \) is the meridional gradient of \( f \) and where it is implicitly assumed that \( v \to 0 \) as \( x \to -\infty \).

The system consisting of (1), (6), and (3)–(5) supports three wave types: high-frequency acoustic modes, intermediate-frequency gravity modes, and low-frequency Rossby modes. In the appropriate limits, the dispersion relation for the system reduces to the standard dispersion relation for each of the three wave types (see appendix A). This existence of realistic fast and slow modes makes the modified system useful for testing multitime-scale numerics.

Substituting a Fourier mode of the form

\[ \psi = \hat{\psi}(t) \exp[i(kx + mz)], \] (7)

where \( \hat{\psi} = (\hat{u}, \hat{v}, \hat{w}, \hat{b}, \hat{P})^T \) is the Fourier amplitude, leaves

\[ \hat{u}_t + ik\hat{P} = f_0\hat{u}, \] (8)
\[ \hat{v}_t = i \frac{\beta}{k} \hat{v} - f_0\hat{u}, \] (9)
\[ \hat{w}_t + im\hat{P} = \hat{b}, \] (10)
\[ \hat{b}_t = -N^2\hat{w}, \quad \text{and} \]
\[ \hat{P}_t + c_s^2(ik\hat{u} + im\hat{w}) = 0. \] (12)

To simplify the notation, the carat (^) over the Fourier variables will be dropped for the remainder of the study.

b. Analytic solution

Leaving only the time derivatives on the left hand side, (8)–(12) can be written in matrix form as

\[ \frac{d\psi(t)}{dt} = M\psi(t), \] (13)

where \( M \) is a \( 5 \times 5 \) matrix of constant coefficients. Standard methods then show that the frequencies of all modes supported by the system are determined completely by the eigenvalues of \( M \). In the present study these eigenvalues are computed numerically. The Rossby mode is then identified by finding the solution with frequency most closely matching the corresponding quasigeostrophic (QG) frequency [see (A13)]. In all cases this Rossby solution is also the slowest of the five modes supported by (13).

c. Discretized solution: KW78 splitting

In the original KW78 splitting the terms integrated on the small time step \( t_{ns} \) are those included on the left-hand side (lhs) of (8)–(12), while those integrated on the large step are included on the right-hand side (rhs). Here the slow-mode terms include the Coriolis, buoyancy, vertical stratification, and Rossby driving terms. In the following it is assumed that the large time step \( \Delta t \) is a factor of \( ns \) larger than the small step \( \Delta t_s \); that is, \( \Delta t = \Delta t_s/\text{ns} \), where \( ns \) is an integer.

The KW78 scheme uses LF time differencing for the large time-step forcings and forward-backward (FB) differencing for the small step. In practice this means that the large time-step forcings are held fixed at time level \( t \) while the remaining terms are integrated from \( t - \Delta t \) to \( t + \Delta t \) using \( 2ns \) FB small steps. The vertical
pressure gradient and vertical divergence terms are computed trapezoidally so as to improve the stability of the scheme at small grid aspect ratios. Letting the time on the large and small time steps be denoted by \( t \) and \( \tau \), respectively, the fully discretized versions of (8)–(12) are given by

\[
u^{r+\Delta r} = u^{r} + \Delta r (-ikP^{r+\Delta r} + F_{w}^{r}),
\]

(14)

\[
u^{r+\Delta r} = v^{r} + \Delta r F_{w}^{v},
\]

(15)

\[w^{r+\Delta r} = w^{r} + \Delta r (-imP^{r} + F_{w}^{r}),
\]

(16)

\[b^{r+\Delta r} = b^{r} + \Delta r F_{b}^{r}, \quad \text{and}
\]

(17)

where the time averaging operator is defined by

\[\mathbf{q}^{r} = \frac{\mathbf{q}^{r-\Delta r} + \mathbf{q}^{r+\Delta r}}{2},\]

(19)

and where the \( F_{j}^{r} \) terms represent the fixed large-step forcings. The solution is advanced from \( t - \Delta t \) to \( t + \Delta t \) by applying (14)–(18) \( 2\pi \) times.

Note that (14)–(18) are not yet prognostic as they include terms that involve implicit time differencing. However, these implicit terms are easily resolved through straightforward algebraic manipulation. The details are given in appendix B, but the end result is that (14)–(18) corresponds to the equivalent explicit system:

\[\mathbf{A}^{r} \mathbf{v}^{r} = \mathbf{S}^{r} \mathbf{v}^{r} + \mathbf{L}^{r} \mathbf{v}^{r},\]

(20)

where \( \mathbf{v}^{r} = (u^{r}, v^{r}, w^{r}, b^{r}, P^{r})^{T} \) is the discretized approximation to \( \mathbf{v} \) at time \( r \) and where \( \mathbf{S} \) and \( \mathbf{L} \) include the terms on the small and large time steps, respectively.

The solutions to (20) are obtained by first defining a two-level solution vector:

\[\mathbf{A}^{r} \mathbf{v}^{r+\Delta r} = \mathbf{R}^{2\pi s} \mathbf{v}^{r},\]

(22)

where \( \mathbf{R} \) is a \( 10 \times 10 \) matrix that advances the solution forward a single small step, while

\[\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\]

(24)

is a reordering matrix. The matrix \( \mathbf{I} \) in (23) and (24) is the \( 5 \times 5 \) identity matrix. According to (20), a single application of \( \mathbf{A} \) in (22) advances the upper half of \( \mathbf{v} \) forward a single small time step while leaving the lower half unchanged. The \( \mathbf{R} \) operator then reorders the variables at the end of the small-step cycle in preparation for the next series of small steps.

As with the continuous problem, the frequencies of the wave modes in (22) are determined completely by the eigenvalues of \( \mathbf{RA}^{2\pi s} \). Specifically, given a discretized eigenvalue of the form \( \lambda = \exp(-i\omega\Delta t) \), the associated complex frequency is obtained as \( \omega = \omega_{r} + i\omega_{i} = \ln(\lambda)/\Delta t \) (using the standard branch for \( \ln \)). The discretized Rossby mode is then selected as the mode with the complex frequency that most closely matches the analytic frequency described in section 2b. As shown in section 3b, this discretized frequency approaches the analytic frequency in the small \( \Delta t \) limit, thus confirming that the mode selected is in fact the Rossby mode.

d. Discretized solution: SK92-LF splitting

The SK92-LF splitting differs from the KW78 splitting in that the buoyancy and vertical stratification terms are updated on the small step. To be specific, (16) and (17) are replaced by

\[w^{r+\Delta r} = w^{r} + \Delta r (-imP^{r} + \bar{b}^{r} + F_{w}^{r}) \quad \text{and}
\]

(25)

\[b^{r+\Delta r} = b^{r} + \Delta r (-N^{2}w^{r} + F_{b}^{r}),\]

(26)

where in the present case \( F_{w}^{r} = F_{h} \) = 0. As before, the implicit dependence is resolved through straightforward algebraic manipulation. Given the modified \( \mathbf{S} \) and \( \mathbf{L} \) operators, the solution then follows as in section 2c.

e. An unsplit scheme

The following sections also make reference to an unsplit scheme. The specific scheme considered is

\[u^{r+\Delta t} = u^{r-\Delta t} + 2\Delta t (-ikP^{r+\Delta t} + F_{u}^{r}),\]

(27)

\[v^{r+\Delta t} = v^{r-\Delta t} + 2\Delta t F_{v}^{r},\]

(28)

\[w^{r+\Delta t} = w^{r-\Delta t} + 2\Delta t (-imP^{r} + \bar{b}^{r}),\]

(29)

\[b^{r+\Delta t} = b^{r-\Delta t} + 2\Delta t (-N^{2}w^{r}), \quad \text{and}
\]

(30)

\[P^{r+\Delta t} = P^{r-\Delta t} + 2\Delta t (-ic^{2}ku^{r} - \Delta t - ic^{2}m\bar{w}^{r}),\]

(31)
Table 1. Parameter values for the Rossby problem.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(c_s) (m s(^{-1}))</th>
<th>(f_0) (s(^{-1}))</th>
<th>(\beta) (m s(^{-1}))</th>
<th>(N) (s(^{-1}))</th>
<th>(\lambda_x) (km)</th>
<th>(\lambda_z) (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference value</td>
<td>330</td>
<td>10(^{-4})</td>
<td>1.65 \times 10(^{-11})</td>
<td>0.01</td>
<td>4800</td>
<td>20</td>
</tr>
<tr>
<td>Range</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.001-0.02</td>
<td>2000-12 500</td>
<td>12-150</td>
</tr>
</tbody>
</table>

where

\[
q' = \frac{q' - \Delta t + q' + \Delta t}{2}
\]

and where \(F'_d\) and \(F'_a\) are as described previously. Note that (27)–(31) is exactly the SK92-LF method with \(ns = 1/2\) (so that \(\Delta t = 2\Delta t\)). Comparing the split and unsplit schemes for the same \(\Delta t\) then measures the error introduced by simply splitting off a small-step cycle.

3. The Rossby problem: Results

For the Rossby problem, the discretization error is measured in terms of the fractional error in \(\omega_s\)—that is

\[
\epsilon_p = \frac{\omega_{sd} - \omega_{sa}}{\omega_{sa}},
\]

where the subscripts \(d\) and \(a\) refer to discretized and analytic, respectively. Since the phase speed of the mode is given by \(c = \omega_s/k\), (32) is equivalent to the fractional phase-speed error.

a. Parameter ranges

The eigenvalues of the analytic coefficient matrix \(\mathbf{M}\) in (13) are determined completely by five parameters, all with units of frequency.\(^1\) \(N, f_0, \beta/k, c_s k\) and \(c_s m\). The full parameter space for \(\epsilon_p\) then consists of these five physical parameters plus \(\Delta t\) and \(ns\). In the following, this parameter space will be explored in two ways. First, 2D cross sections are mapped by varying two of the five physical parameters while keeping the other three physical parameters and the two time steps held fixed. (There is one exception to this, in that \(\beta\) is actually held fixed rather than \(\beta/k\).) The second set of computations involves keeping all five physical parameters and \(ns\) held fixed while varying only \(\Delta t\). Three values of \(ns\) are considered: \(ns = 1, 2,\) and \(3\).

The phase-speed error (32) was found to be most sensitive to changes in \(N, c_s k\) and \(c_s m\). The 2D cross-section results are thus shown only for these three parameters. The ranges shown in the cross sections are as given in the final row of Table 1, with \(\lambda_x = 2\pi/k\) and \(\lambda_z = 2\pi/m\) being the horizontal and vertical wavelengths, respectively (with \(c_s\) held fixed). Parameters not varied in a given cross section are fixed at reference values, as shown in the middle row of Table 1.

The time step for all cross sections using the KW78 scheme is set at \(\Delta t = 100\) s, which is characteristic of the outer-grid time steps used in most regional-scale modeling. This is also similar to the time step used by Duddia and Bresch (2002) in their global MM5 calculations. By contrast, the cross sections computed with the SK92 splitting are all set at \(\Delta t = 500\) s, which for typical \(ns\) (say \(ns\) between 2 and 6) implies a large time step similar to those used in general circulation models. But as will be seen, the \(\Delta t\) dependence for the SK92 scheme is effectively quadratic. The results for \(\Delta t = 100\) s can thus be inferred easily from the \(\Delta t = 500\) s results (so as to allow comparison with the KW78 case). Note that even for \(\Delta t = 500\) s the Rossby mode is very well resolved in time (\(|\omega_s \Delta t| \leq 1 \times 10^{-3}\)).

b. KW78 results

Figure 1 shows the phase-speed error (32) for the Rossby mode as computed using the KW78 splitting with \(\Delta t = 100\) s. Column 1 of the figure shows the error as a function of \(c_s k \Delta t\) and \(c_s m \Delta t\), with column 2 showing the \(N \Delta t\) and \(c_s m \Delta t\) dependence. Column 3 shows the spectral radius of the discretized amplification matrix \(\mathbf{RA}^\text{ns}\) defined by (22). The first row of the figure shows the \(ns = 2\) case, with row 2 showing the \(ns = 3\) case. A cross in each panel shows the characteristic reference parameter values listed in Table 1. Shading indicates phase-speed errors greater than 10% in columns 1 and 2 and spectral radii greater than 1 in column 3. Since the analytic solution is nonamplifying, a spectral radius exceeding 1 indicates numerical instability.

Figures 1a,b show that for \(ns = 2\) the error throughout much of the relevant parameter space is greater than 10%. Figure 1a shows that the error is greatest for small \(k\) and large \(m\)—that is, for small aspect ratios \(k/m\). At fixed \(k\) and \(m\) the error increases as \(N\) increases (Fig. 1b), with the error exceeding 100% for large values of \(N\) and \(m\). As \(ns\) is increased the error distribution at large \(m\) remains roughly unchanged, suggesting that the error in this case is determined primarily by \(\Delta t\) and not \(\Delta t\). However, at smaller \(m\) a set of complicated error

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\(^1\) To see this, substitute the modified buoyancy \(\theta = b/N\) and modified pressure \(p = Pc_s\) into (8)–(12) (as done, e.g., by SK92). The resulting system then depends only on the five parameters listed.
bands appears. These latter error bands coincide with the numerical instability bands seen in column 3.

It should be noted that the instability bands in column 3 are due to the acoustic modes and not the Rossby mode. As shown in appendix C, this acoustic instability is similar to that found by SK92 for the acoustic-advection problem, except that here the instability reflects a coupling with buoyancy rather than with advection. As in SK92, the instability occurs only where the acoustic-mode frequency as sampled onto the large time step is either \( \omega_{ac} = \pm \pi/2\Delta\tau, \pm \pi/\Delta\tau, \) or zero—hence, the banded structure. Further details can be found in appendix C. The correspondence between this instability and the Rossby mode phase-speed error apparently reflects the acoustic nature of the Rossby mode small-step adjustment.

Figure 2 shows both \( \epsilon_p \) and the spectral radius for the KW78 scheme considered as functions of \( \Delta\tau \). The physical parameters for these calculations are fixed at the reference values shown in Table 1. The error curve for \( ns = 1 \) (Fig. 2a) shows that the error vanishes at small \( \Delta\tau \), but quickly becomes large as \( \Delta\tau \) is increased, reaching 27% at \( \Delta\tau = 100 \) s. The steady increase in the error shifts to larger \( \Delta\tau \) for \( ns = 2 \) and 3, but instability bands and associated error peaks appear at smaller \( \Delta\tau \) (as in Fig. 1—see appendix C). In all three cases the error at \( \Delta\tau = 100 \) s is clearly unacceptable.

c. Filtering the KW78 scheme

The KW78 scheme is typically used with numerical filters to damp the acoustic modes. Some commonly used filters include the Robert–Asselin time filter (Durran 1999, see his section 2.3.5), the 3D divergence damping of SK92, and the implicit biasing of Durran and Klemp (1983). Of the three, only implicit biasing was found to have a nonnegligible impact on the KW78 Rossby results.

With implicit biasing, the time-averaging operator in (19) is replaced by the weighted time average:

\[
\bar{q}^r = \frac{(1 - \epsilon)q^r + (1 + \epsilon)q^{r+\Delta\tau}}{2},
\]  (33)
where $\epsilon$ is the implicit biasing coefficient. Setting $\epsilon > 0$ acts to bias the vertical pressure gradient and divergence terms toward backward time differencing, thus damping and slowing the vertically propagating acoustic modes. As suggested by Durran and Klemp (1983), setting $\epsilon = 0.2$ is effective at filtering the acoustic modes without noticeably affecting the gravity modes.

Figure 3 shows the KW78 Rossby phase-speed errors
and spectral radius for the case $n_s = 2$ (cf. Figs. 1b,c) with implicit biasing coefficients of $\epsilon = 0.1, 0.2, \text{and} 0.4$. Adding the implicit biasing significantly improves the accuracy and stability of the scheme. However, the errors are still significant—for $\epsilon = 0.1$ and 0.2 the errors still exceed 10% over much of the parameter space.

d. SK92-LF results

The phase-speed errors for the SK92 version of the leapfrog scheme are as shown in Fig. 4. Figures 4a,b show the SK92-LF cross-section results for $\Delta \tau = 500 \text{ s}$ with $n_s = 3$. Comparison with Figs. 1d,e shows that switching to the SK92 scheme has dramatically reduced the error, even with $\Delta \tau$ increased by a factor of 5. (The contours in Fig. 4 are in units of $10^{-3}$, so that the largest error shown is just less than 1%. To get the errors for $\Delta \tau = 100 \text{ s}$, divide by 25.)

The $\Delta \tau$ dependence for the SK92-LF splitting is shown in Fig. 4c (cf. Fig. 2a). Note that the upper limit for $\Delta \tau$ has been increased to $\Delta \tau = 1000 \text{ s}$, so as to better show the error at large time steps. But even with this increased $\Delta \tau$ the largest error shown is still less than 0.5%. Consideration of the spectral radius shows that the instability bands have completely disappeared as well (not shown).

Inspection of the errors in Fig. 4c shows that the SK92-LF error behaves roughly quadratically, both in terms of increasing $\Delta \tau$ (at fixed $n_s$) and increasing $n_s$ (at fixed $\Delta \tau$). However, comparing these errors to those of an equivalent unsplit scheme shows that to a large extent the errors stem from the splitting rather than from the individual small-step and large-step operators.

As an example, the unsplit method in (27)–(31) with $\Delta \tau = 3000 \text{ s}$ (analogous to the $n_s = 3$ case with $\Delta \tau = 1000 \text{ s}$) produces an error of 0.0005%, which is almost $10^3$ times smaller than the equivalent error in Fig. 4c. So while the errors in the SK92-LF method are generally small, the split scheme does in a relative sense produce more error than an analogous unsplit scheme.

4. Analysis of the KW78 errors

The size of the KW78 Rossby errors is somewhat surprising, given the widespread use of this method on synoptic-scale grids. The present section explores the source of these errors in greater detail.

a. A vertical adjustment problem

The KW78 error distributions in Fig. 1—specifically, the increasing errors at small $k/m$ and large $N$—suggest that the source of the errors is most likely the compressible vertical adjustment—or stated differently, the adjustment to hydrostatic balance. To demonstrate this, consider a 1D vertical adjustment model with a slow oscillatory driving force added to the thermodynamic equation. Specifically,

$$w_t + imP = b,$$

$$b_t = -N^2w - i\omega_b b,$$

and

$$P_t + c_s^2(imw) = 0,$$

where the $\omega_b$ term in (35) forces an oscillation with natural frequency $\omega_b$ [cf. (10)–(12)]. The dispersion re-
lation for this system implies three modes: two fast acoustic modes and a driven mode produced by the $\omega_0$ term. It can be shown that for $\omega_0 \ll c_m$, the driven mode is effectively in hydrostatic balance.

The KW78 method as applied to the vertical adjustment problem is evaluated below. The discretized and analytic solutions are again obtained as in section 2, with the specific time discretization following (16)–(18) (with $k = 0$), except that the driving term in (35) is added to the large time-step forcing in (17). The error is again measured using (32) but with the frequency of the slow driven mode used in place of the Rossby frequency. The natural frequency of the driven mode is held fixed at $\omega_0 = 1$ day$^{-1}$. All other necessary parameter values are as in the Rossby problem, section 3a.

b. Results

Figures 5a,b show the slow-mode phase-speed error and the spectral radius of the KW78 amplification matrix as functions of $N\Delta \tau$ and $c_m\Delta \tau$ for $ns = 3$ (recall that $k = 0$). The contours and shading in the figure are the same as in Fig. 1. Figure 5c shows the error as a function of $\Delta \tau$ for $ns = 1, 2, \text{and } 3$. Comparison with Figs. 1e,f and 2a shows that the current adjustment results are surprisingly similar to those of the previous Rossby model. This similarity suggests that the errors and instability in the two problems stem from the same source—namely, from the vertical adjustment dynamics described by (34)–(36) and (10)–(12).

The evolution of the hydrostatic balance under the KW78 splitting is considered in Fig. 6. The results shown in the figure are for the $N = 0.02$ s$^{-1}$ and $\lambda_z = 15$ km case with $ns = 3$, which has a phase-speed error of roughly 300% (cf. Fig. 5a). Figure 6a shows the vertical pressure gradient as computed on the small-step cycle, as well as the corresponding buoyancy term as computed on the large time step. (Note that the pressure gradient is shown with two curves, since the leapfrog small-step cycles overlap.) The sum of the pressure gradient and buoyancy as averaged over a small-step cycle is shown in Fig. 6d.

As seen in the figure, the splitting of the pressure gradient and buoyancy terms between the small and large time steps leads to significant acoustic noise and a failure to maintain hydrostatic balance (Figs. 6a,d). The relative imbalance then in turn leads to an acceleration of the oscillation—first downward (increasing the buoyancy) and then upward (decreasing buoyancy). The balance can be restored to some extent through implicit biasing (as described in section 3c), which damps the acoustic modes so that the buoyancy and pressure gradient remain more closely coupled (Figs. 6b,e). However, a much better representation of the balance comes from the SK92-LF scheme, in which the buoyancy and pressure gradient are no longer split (Figs. 6c,f). For reference, the error in the SK92-LF case is less than 0.0001%.

5. Third-order Runge–Kutta differencing

As shown by Wicker and Skamarock (1998, 2002), the KW splitting can also be stably applied to several

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2 Note that the adjustment problem does not support gravity modes, since $k = 0$. 
forward-in-time schemes, particularly to some Runge–Kutta variants. Here the third-order Runge–Kutta (RK3) method of Wicker and Skamarock (2002, hereafter WS02) is evaluated as implemented using the SK92 operator splitting (SK92-RK). This SK92-RK method is the scheme currently used in the Advanced Research WRF model (WRF-ARW; Klemp et al. 2007).

a. Discretized solution: SK92-RK splitting

The particular RK3 method used by WS02 can be defined (in unsplit scalar form) as

\[ q^* = q' + \frac{\Delta t}{3} F(q'), \]

\[ q^{**} = q' + \frac{\Delta t}{2} F(q^*), \] and

\[ q'^{+\Delta t} = q' + \Delta t F(q^{**}). \]

where \( F \) represents the right-hand-side forcing terms. As compared with the LF method, the RK3 scheme allows a larger stable Courant number (roughly 1.73 times larger) but also requires more function evaluations per time step (see, e.g., Durran 1999, his section 2.3). The end result is that in time-split form the two methods have similar overall efficiencies (see the discussion in Wicker and Skamarock 1998; WS02).

In the split RK3 scheme, each of the three stages in (37)–(39) is replaced by an equivalent FB small-step cycle. In a given cycle the large time-step terms are held fixed at the times indicated in (37)–(39) while the FB small-step system is advanced over the appropriate time range. For instance, in the first stage the large-step forcings are evaluated at time \( t \) while the small-step system is advanced through \( ns/3 \) small steps to arrive at time \( t^* \). The \( t^* \) values of the fields are then used as the large-step forcings for the second stage, which is advanced through \( ns/2 \) small steps to arrive at time \( t^{**} \). The \( t^{**} \) fields then provide the large-step forcings for

![Fig. 6. Evolution of the hydrostatic balance in the vertical adjustment model. Vertical pressure gradient on the small step (gray) and buoyancy on either the large or small time step (black) as functions of time at fixed \( z \) (values normalized by the largest \( \partial P/\partial z \)). Shown are the (a) KW78, (b) KW78 with implicit biasing (\( \varepsilon = 0.2 \)), and (c) SK92-LF schemes. (d)–(f) \( -\partial P/\partial z + b \) (units of \( 10^{-8} \text{ m s}^{-2} \)), where \( \langle \cdot \rangle \) indicates an average over the small-step cycle. Cases shown are as in (a)–(c). All results are for \( N = 0.02 \text{ s}^{-1} \), \( \lambda_z = 15 \text{ km} \), and \( ns = 3 \) with \( \Delta t = 100 \text{ s} \).]
the final stage. For consistency, \( ns \) must be a multiple of 6 (so as to be divisible by both 2 and 3). Alternatively, WS02 suggest using a single small step of length \( \Delta t/3 \) in the first stage regardless of \( ns \), in which case \( ns \) just needs to be even.

To express the split RK3 scheme in matrix form, first define the three-stage solution vector

\[
\phi' = \begin{pmatrix} \xi' \\ \xi'' \\ \xi''' \end{pmatrix},
\]

where \( \xi' \) is the discretized approximation to \( \psi(t) \) as described previously. The analog to (23) is then

\[
\begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix} \begin{pmatrix} S & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi' \\ \phi'' \\ \phi''' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

being the appropriate reordering-copying matrices. The eigenvalues of the amplification matrix \( B \) in (42) are then used to find the error in the scheme, as described previously in section 2.

b. Parameter ranges

The RK3 scheme in unsplit form allows a larger maximum stable time step than the unsplit LF scheme (by roughly a factor of 2). But when combined with time splitting, the constraint on the smaller time step in the two schemes is roughly identical (since both use FB small-step differencing). The net implication then is that the larger stability region for the unsplit RK3 scheme translates into larger stable \( ns \) values for a given \( \Delta t \). Consistent with this interpretation, the analysis below uses \( ns = 2, 4 \) and 6 rather than the \( ns = 1, 2 \), and 3 cases considered previously.

Apart from \( ns \), the parameter ranges and parameter values considered below are identical to those considered for the SK92-LF case.

c. Results

The errors for the SK92-RK scheme are as shown in Fig. 7. Figures 7a,b show the two parameter-space cross-section plots for the case \( ns = 6 \) with \( \Delta t = 500 \) s. Comparison to Figs. 4a,b shows that the RK errors in this case are generally similar to but somewhat less than those of the LF scheme, despite \( ns \) being twice as large. The \( \Delta t \) dependence for the SK92-RK errors is shown in Fig. 7c. Comparing to Fig. 4c for the case \( ns = 2 \) shows that for given \( ns \) the RK scheme is considerably more accurate. When the RK \( ns \) is doubled (relative to the LF case) the errors are then similar to the LF errors but still somewhat less. And as found for the LF scheme, the RK method is stable over the full range of parameters considered (not shown).

Inspection of Fig. 7c shows that the RK error depends roughly quadratically on \( \Delta t \) and roughly cubically on \( ns \) (at least for the smaller \( ns \) values). The practical consequence is that for given \( \Delta t \), the RK scheme can be used with larger \( ns \) (and hence larger \( \Delta t \)) than the LF scheme without significantly compromising accuracy. On the other hand, with increasing \( \Delta t \) at fixed \( ns \) the errors in both schemes increase quadratically.

As found by WS02, replacing the first stage of the method with two small steps (as opposed to a single step of length \( \Delta t/3 \)) for the case \( ns = 6 \) produced only small changes to the error.

6. Summary

A series of von Neumann accuracy and stability analyses have been presented for the problems of KW78 and SK92 time splitting as applied at large scales. The particular problem of interest has been the case of large-scale Rossby wave propagation on a rest-
The Rossby phase-speed errors exceed 10% over much of the parameter space and are in some cases as large as 100%. The source of the errors was traced to the compressible vertical adjustment—and more precisely, to a failure of the method to maintain hydrostatic balance due to the splitting of the balance terms between the small and large time steps. The errors can be reduced somewhat through implicit biasing, but large biasing coefficients are needed—and even then the time steps are limited to moderate values.

The errors in the KW78 scheme are in large part absent from the SK92 methods, as these latter methods treat the entire vertical adjustment process on the small step. Indeed, with the SK92 splitting the time steps can be an order of magnitude larger than the time steps used for the KW78 analysis without significantly compromising large-scale accuracy. The third-order Runge–Kutta version of the SK92 scheme was shown to have a mixed quadratic (with increasing $\Delta \tau$) and cubic (with increasing $ns$) error dependence, whereas the leapfrog method is strictly quadratic. The practical consequence is that for given fixed $\Delta \tau$, the SK92-RK method can be used with larger $ns$ than the LF scheme while maintaining similar overall accuracy.

It is worth noting that the Rossby problem introduced here is strictly heuristic, since the mode of interest results from a heuristic forcing term added to the equations of motion. However, a similar analysis was also developed for the Eady baroclinic wave problem—which lacks the added forcing term—and for the most part the Eady results echo those of the Rossby problem described above. For instance, at $\Delta \tau = 100$ s the KW78 Eady growth-rate errors are as large as 25% in some cases, and the overall error patterns are generally similar to those seen in Fig. 1. And as in the Rossby problem, the Eady errors in the KW78 scheme are largely absent from the SK92 methods.

Finally, it should be restated that most of the newer KW community models have already adopted the SK92 splitting out of stability concerns. However, a few of the older models—most notably the MM5 and RAMS models—still use the original KW78 splitting. Fortunately, the standard configurations for MM5 and RAMS both use relatively large implicit biasing coefficients ($\epsilon = 0.4$ for MM5 and $\epsilon = 1.0$ for RAMS), which likely moderates the associated errors somewhat (at least for $\Delta \tau \leq 100$ s or so—see Fig. 3). Even so, careful consideration of the large-scale accuracy of these older models is still probably in order.

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APPENDIX A

The Rossby-Restoring Model: Derivation

Analyzing difference approximations in the large-scale context raises one particular challenge: virtually all the large-scale modes of interest (i.e., the Rossby modes, the baroclinic instability modes, etc.) stem from nonconstant-coefficient problems. In principle these problems are tractable numerically, but in practice the
analysis tends to be cumbersome and the relevant computations are often involved. To simplify matters, this appendix is devoted to finding an appropriate test problem with constant coefficients. The particular mode of interest will be the Rossby (or planetary) wave on a resting background state.

As a starting point consider the linearized compressible-Boussinesq system on a β plane, as described by

\[ u_t = -P_x + (f_0 + \beta y)u, \quad (A1) \]
\[ v_t = -P_y - (f_0 + \beta y)v, \quad (A2) \]
\[ w_t = -P_z + b, \quad (A3) \]
\[ b_t + N^2 w = 0, \quad \text{and} \]
\[ P_t + c_s^2 \nabla \cdot \mathbf{u} = 0, \quad (A5) \]

where \( f_0 \) and \( \beta \) are constants and where the remaining variables are as described in section 2a. Recall that to within the QG approximation the approximate governing equation for Rossby waves allows strictly 2D modes [i.e., modes with no \( x \)-dependence; see, e.g., Holton (2004), his section 12.3.1]. The strategy here will thus be to find a 2D version of (A1)–(A5) that maintains the basic mechanism for Rossby wave propagation. This 2D model will then also be constant-coefficient, since the \( y \)-variation of the parameters will then be immaterial.

To begin, recall that in the QG theory the zonal wind \( u \) becomes ageostrophic in the 2D limit. The \( \beta yu \) term is then small in this limit (it is in fact neglected in the QG model) and (A2) can therefore be approximated by

\[ u_t = -P_y - f_0 u. \]

Next the heuristic assumption is made (to be justified a posteriori) that all variables other than \( P \) are strictly 2D. From (A1) it then follows that

\[ \frac{\partial}{\partial y} ( -P_x + f_0 v + \beta y v ) = -P_{xy} + \beta v = 0, \]

which can be satisfied by letting

\[ P = y \int_{-\infty}^{x} \beta v \, dx' + P'(x, z, t), \quad (A6) \]

where it is implicitly assumed that \( v \to 0 \) as \( x \to -\infty \).

The \( y \)-dependent term in (A6) is recognized as the \( y \)-varying part of the total geostrophic pressure—that is, the part associated with the \( \beta y \) dependence of \( f \). This term is smaller than the \( P'(x, z, t) \) term—the \( y \)-varying term is in fact zero at the reference latitude—but it nonetheless introduces a small meridional pressure gradient that would otherwise be absent. In the present context this meridional pressure gradient serves as the driving force for the Rossby wave. As shown in Fig. A1, north of \( y = 0 \) the \( y \)-dependent pressure is positive to the east of positive \( v \) and negative to the west. South of \( y = 0 \) this phase relationship switches. The resulting north-south pressure gradient then in turn drives the velocity pattern—and by extension the pressure distribution—westward and results in the westward propagation of the wave. Note for reference that this driving mechanism is not new—essentially the same driving force was identified by Durran (1988).

Finally, it is noted that for small Rossby number \( (R_0) \), the gradients of \( P \) in the \( x, z \), and \( t \) directions are all dominated by the \( P' \) term in (A6). That is, assuming a QG scaling such that \( P' \sim L f_0 |v| \), it holds that

\[ \frac{y \int_{-\infty}^{x} \beta v \, dx'}{P'_z} \sim \frac{|\beta y|}{f_0} \ll 1 \]

and similarly for the gradients in \( x \) and \( t \). Neglecting the relevant terms in (A3) and (A5) [the term in (A1) cancels exactly] then leaves the 2D system:

\[ u_t = -P'_{x} + f_0 v, \quad (A7) \]
\[ v_t = -\int_{-\infty}^{x} \beta v \, dx' - f_0 u, \quad (A8) \]
\[ w_t = -P'_{z} + b, \quad (A9) \]
\[ b_t + N^2 w = 0, \quad \text{and} \]
\[ P'_t + c_s^2 (u_x + w_z) = 0, \quad (A11) \]
which is indeed seen to be a constant-coefficient problem. A spatial Fourier decomposition then leads to (8)–(12) in the text. The model system (A7)–(A11) supports all three standard wave types—namely, the acoustic modes, the internal gravity modes, and the Rossby modes. To see this, note that the dispersion relation for the system is given by

\[ \omega^2 = \left( \frac{v_a^2}{v_a^2 k^2 - \beta^2} \right) \left( \frac{N^2 + f_0^2}{v_a^2} + 1 \right) - \omega^2 \frac{\beta}{k} \left( \frac{N^2}{v_a^2} + 1 \right), \]

where \( v_a = \sqrt{c_i'(k^2 + m^2)} \) is the acoustic frequency and where the remaining parameters are as defined in section 2a. For the sake of discussion it will be assumed that \( \beta/k \ll f_0 \ll N \ll v_a \).

Two of the modes described by (A12) are fast in the sense that \( |\omega| \gg N \). For these modes (A12) reduces to

\[ \omega^2 \approx v_a^2 = c_i'(k^2 + m^2), \]

which is of course the standard dispersion relation for acoustic waves. If intermediate waves are considered in the sense that \( v_a \gg |\omega| \gg \beta/k \), then (A12) becomes

\[ \omega^2 \approx \frac{N^2 k^2}{k^2 + m^2} + \frac{f_0^2 m^2}{k^2 + m^2}, \]

which describes the incompressible inertia‐gravity modes. Finally, assuming slow modes with \( |\omega| \ll f_0 \) leads to

\[ \omega \approx \frac{-\beta k}{k^2 + \frac{f_0^2 m^2}{N^2}}, \]

which is the dispersion relation for 2D QG Rossby waves.

The system (A7)–(A11) should in a strict sense be understood as heuristic, since one of the key assumptions of the model—namely, that \( P \) is the only field with \( y \) dependence—cannot be formally justified. Even so, the end result supports a realistic Rossby wave mode in addition to the standard acoustic and inertia‐gravity modes. [For the range of parameters considered in section 3a the Rossby frequency given by (A7)–(A11) never differs from the corresponding QG frequency by more than 5%. This is of course well within the error due to finite-\( R_0 \) and nonhydrostatic effects.] In this sense the system provides a realistic and convenient test case for multiscale numerics.

**APPENDIX B**

**The Rossby L and S Operators**

The implicit dependence in (14)–(18) is resolved by simply solving the system for the time level \( \tau + \Delta \tau \) variables algebraically. Writing out the \( F'_t \) forcings in terms of the time level \( t \) variables then leads to an explicit system of the form (20).

To specify the KW78 \( L \) and \( S \) operators, first define

\[ \lambda_{\text{cx}} = c_x k \Delta \tau, \quad \lambda_{\text{cz}} = \frac{1}{2} c_z m \Delta \tau, \quad \lambda_f = f_0 \Delta \tau, \quad \lambda_N = N \Delta \tau, \quad \text{and} \quad \lambda_\beta = \frac{\beta}{k} \Delta \tau. \]

Then

\[ S = \begin{pmatrix} 1 - \lambda_{\text{cx}} s_3 & 0 & -\lambda_{\text{cx}} s_2 & 0 & -i \lambda_{\text{cx}} s_1 / c_x \\ 0 & 1 & 0 & 0 & 0 \\ -\lambda_{\text{cp}} s_3 & 0 & s_1 & 0 & -i s_2 / c_x \\ 0 & 0 & 0 & 1 & 0 \\ -i c_x s_3 & 0 & -i c_z s_2 & 0 & s_1 \end{pmatrix} \]

and

\[ L = \begin{pmatrix} 0 & \lambda_f & 0 & -\lambda_{\text{cx}} \lambda_{\text{cz}} l_1 / N & 0 \\ -\lambda_f & i \lambda_\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 / N & 0 \\ 0 & 0 & -N \lambda_N & 0 & 0 \\ 0 & 0 & 0 & -i c_x \lambda_{\text{cx}} l_1 / N & 0 \end{pmatrix}, \]

where

\[ s_1 = \frac{1 - \lambda_{\text{cz}}^2}{1 + \lambda_{\text{cz}}^2}, \quad s_2 = \frac{2 \lambda_{\text{cz}}}{1 + \lambda_{\text{cz}}^2}, \quad s_3 = \frac{\lambda_{\text{cx}}}{1 + \lambda_{\text{cz}}^2}, \quad \text{and} \quad l_1 = \frac{\lambda_N}{1 + \lambda_{\text{cz}}^2}. \]
The $L$ and $S$ operators for the SK92 splitting are derived similarly.

APPENDIX C

Instability of the Acoustic Modes: Further Details

The instability described in section 3b is evidently the same instability found by SK92 in their analysis of the acoustic-gravity wave system (see their section 4d and their Fig. 6). The instability is also similar to that found for the acoustic-advection problem (see section 2a of SK92), except that here the coupling of the acoustic modes with buoyancy drives the instability rather than the coupling with advection.

As with the acoustic-advection coupling, the instability seen in Figs. 1 and 2 is closely tied to the use of leapfrog time differencing on the large time step. Recall that for the unsplit case, the instabilities of the leapfrog scheme always take the form of $4\Delta t$ oscillations as seen on the large step (see the discussion in SK92). Instability is also favored whenever the small-step terms are not identical—indeed, in the limit of $n\Delta t$ smaller than those shown in Fig. 1. On the top axis is the equivalent small time step $\Delta t$ corresponding to the particular values of $c_s$ and $m$ assumed in Fig. 2.

The use of overlapping leapfrog cycles for (C1) and (C2) demands four solutions: two physical modes and two computational modes. To help distinguish these modes, assume as in previous sections that $\omega_{rd}\Delta t = \pm \pi/2$ dis- crete oscillations (see, e.g., Durran 1999, his section 2.3.4). As a result, the time-split leapfrog scheme is susceptible to instabilities whenever fast acoustic modes from the small time step alias onto $\omega_{rd}\Delta t = \pm \pi/2$ oscillations as seen on the large step (see the discussion in SK92). Instability is also favored whenever the small-step modes alias onto $\omega_{rd}\Delta t = 0$ or $\pm \pi$ oscillations, as demonstrated below.

To address the instability more concretely, first consider the vertical acoustic mode problem:

$$w^{r+\Delta t} = w^r - im\Delta tPw^r$$

and

$$P^{r+\Delta t} = Pr - ic_s^2m\Delta tPw^r,$$

where it should be understood that the system is to be integrated using overlapping leapfrog small-step cycles of length $2\Delta t$. However, in the absence of large-step terms, the two overlapping small-step cycles are completely independent (see the schematic in Fig. C1). That is, given the starting values $\xi^0 = (w^0, P^0)^T$ and $\xi^\Delta t = (w^{\Delta t}, P^{\Delta t})^T$ and the small-step operator $S$, the solutions on the odd and even cycles as observed on the large time step are

$$\xi^{2\Delta t} = S^{2\Delta t}\xi^0,$$

and

$$\xi^{2\Delta t+\Delta t} = S^{2\Delta t}\xi^{\Delta t},$$

where $j$ is an integer.

Figure C2 shows the discretized large-time-step frequency $\omega_{rd}$ for (C1) and (C2) as integrated with leapfrog time splitting for the case $n\Delta t = 3$. The results are given in terms of the phase change $|\omega_{rd}\Delta t|$ as seen over one large time step. As described in section 2c, this phase change is determined by first computing the eigenvalues for the appropriate $RA^{2\Delta t}$ matrix as formu-
lated for (C1) and (C2) [see discussion below (24)]. Note that the eigenvalues and associated phase changes in this case are functions of $c_m\Delta t$ and $ns$ only (and not of $c_m$ and $\Delta t$ independently).

As expected, for small $c_m\Delta t$ the physical-mode phase change is roughly $c_m\Delta t \times ns$, while the phase change for the computational modes is roughly $\pi - c_m\Delta t \times ns$. As in the unsplit case, the crossing point for the two modes occurs at $\omega_{rd}\Delta t = \pm \pi/2$. The physical mode reaches its maximum frequency of $\omega_{rd}\Delta t = \pm \pi$ at roughly $c_m\Delta t = 1.1$, with larger $c_m\Delta t$ then leading to aliased slower modes when sampled to the large time step. For this same range of $c_m\Delta t$ the computational-mode frequency decreases to zero and then begins to increase.\(^{C1}\)

Further increases in $c_m\Delta t$ lead to oscillations in the two frequencies, with the crossing points at $\omega_{rd}\Delta t = \pm \pi/2$. Inspection of growth rates shows that for all $c_m\Delta t$ the solutions are absolutely stable (not shown).

Now suppose that buoyancy is included so that the system to be integrated becomes

\[
\begin{align*}
    w^{r+\Delta t} &= w^r - im\Delta \tau P^r + \Delta \tau b^r, \\
    b^{r+\Delta t} &= b^r - \Delta \tau N^3 w^r, \quad \text{and} \\
    P^{r+\Delta t} &= P^r - ic_m\Delta \tau \pi^r,
\end{align*}
\]

where the buoyancy and vertical stratification terms are differentiated on the large step. Analysis of (C3)–(C5) shows that introducing these buoyancy terms causes instability in the acoustic modes, with the instability closely matching that shown in Figs. 1f and 2b. And cross-referencing with Fig. C2 shows that this instability is present only when the acoustic frequency as aliased onto the large time step has particular values—specifically, $\omega_{rd}\Delta t = \pm \pi/2, \pm \pi$, or 0. (Note that the minimum $c_m\Delta t$ in Fig. 1f is roughly 1.4 rather than 0. Also, an instability is in fact present at $\Delta \tau \approx 5.1$ s in Fig. 2b, although the instability in this case is weak and thus not easily visible in the figure.)

It is thus apparent that the instability shown in Figs. 1 and 2 results from the coupling of buoyancy and vertical stratification terms on the large time step with the acoustic modes as subsampled every $ns\Delta t$. And the instability is present only when the aliased acoustic frequency has particular values. Fortunately, integrating the buoyancy on the small step completely removes this coupling and thereby stabilizes the scheme, as indicated in section 3d.

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\(^{C1}\) Note that the interpretation given here differs slightly from that given by SK92. Specifically, SK92 only consider solutions satisfying $\omega_{rd}\Delta t \leq \pm \pi/2$ (effectively through their choice of branch on $\sin^{-1}$) and apparently assume that all solutions in this range belong to physical modes. (See their Figs. 2 and 4 and the associated discussion.)