Formation of Jets through Mixing and Forcing of Potential Vorticity: Analysis and Parameterization of Beta-Plane Turbulence

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ABSTRACT
Formation of multiple jets in forced beta-plane turbulence is studied from the perspective of nonuniform nonconservative arrangement of potential vorticity (PV). Numerical simulations are analyzed to show that mixing and forcing reinforce jets by concentrating PV gradients at the axes of prograde jets. Based on the formalism developed in the companion paper, the nonconservative driving of jets is diagnosed and parameterized through the diffusive flux of PV and the source of wave activity. It is found that the two terms nearly balance on a long time scale, and they are both strongly anticorrelated with the PV gradient, which suggests that PV controls the nonconservative processes and that these processes could be parameterized as functions of the PV gradient. The flux is modeled using the effective diffusivity formula recently obtained by Ferrari and Nikurashin. Consistent with the PV barrier concept, the nonlinear diffusivity is a decreasing function of the squared PV gradient and agrees well with the diffusivity diagnosed from the numerical simulation. The source term is assumed to be inversely proportional to the PV gradient. The parameterization gives rise to a nonlinear partial differential equation (PDE) for the mean flow. A finite-difference model of the PDE predicts formation of a piecewise linear PV (staircase) and piecewise parabolic jets from a near-uniform initial condition when anisotropy and mixing of the flow are sufficiently strong. The origin of the discontinuities is antidiffusive instability of PV gradients, and although nonlinearity allows the discrete model to integrate stably, the solution is sensitive to the initial condition and resolution. The emerging jets in the 1D model have similar characteristics to those in the numerical simulation, but the details of the transient behavior are distinct. Similar discrete models of ill-posed PDEs in which discontinuities form also appear in image processing and granular matter dynamics.

1. Introduction
Starting with the work of Rhines (1975), it is now well known that macroturbulence on a rotating sphere is fundamentally different from isotropic 2D turbulence. Eddies become anisotropic as the inverse cascade of energy makes them aware of the meridional gradient in the background potential vorticity (PV), at which point zonal jets begin to form. On the beta plane, this occurs when the energy-containing eddy reaches the Rhines scale $L_{\beta} \sim (EKE)^{1/4} \beta^{-1/2}$, where EKE denotes the mean eddy kinetic energy, and $\beta$ is the meridional gradient of planetary vorticity. Rhines’ work has been extended to baroclinic systems (Salmon 1978; Panetta 1993; Held and Larichev 1996) and spherical geometry (Yoden and Yamada 1993; Cho and Polvani 1996). For a comprehensive review of the literature, see Vallis (1992, 2006, section 9) and Danilov and Gurarie (2000). Although the details of these simulations vary, they all produce bands of persistent, nearly barotropic jets. Collectively referred to as geostrophic turbulence, this type of dynamics has been suggested as a driving mechanism of the observed jets on gas planets (e.g., Porco et al. 2003). Jetlike structures are also identified in the upper layer of the open ocean (Maximenko et al. 2008), and here again geostrophic turbulence may be at play.

While turbulence studies tend to focus on the flow of energy in the spectral space, an alternative perspective to jet formation is stirring of PV, since PV governs the flow under a balanced dynamics. The theory of atmospheric general circulation based on PV stirring dates back to Green (1970) and is discussed extensively by Vallis (2006, section 12). Jets form when stirring (eddy PV flux) is nonuniform, creating an inhomogeneous distribution of PV gradients. An idealization of Green’s
concept is the “PV staircase” (Marcus 1993; Dritschel and McIntyre 2008), a stepwise variation in PV with finite latitudinal intervals. A PV staircase supports, through invertibility, a series of piecewise concave jets. Several observations hint at the natural tendencies for the atmospheres to form staircaselike PV structures; stepwise variations of PV are observed at the edge of the stratospheric polar vortex and at the extratropical tropopause of the earth’s atmosphere, as well as in the atmospheres of gas planets. The formation of nonuniform PV gradients has been also observed in numerical simulations of betaplane turbulence (e.g., Dritschel and McIntyre 2008; Nakamura and Zhu 2010, hereafter NZ).

In discussing arrangements of PV, it is important to make a distinction between advective and nonadvective (nonconservative) arrangements. Although the advective arrangement can lead to a nonuniform zonal-mean PV, in most cases a highly nonuniform PV distribution involves a nonconservative (and irreversible) arrangement due to mixing. Yet, the nonconservative processes are also affected by, and in some cases slaved to, the advective process. For example, mixing in the upper troposphere and lower stratosphere is primarily driven by large-scale differential advection (stirring). Dritschel and McIntyre (2008) propose “Rossby elasticity” (McIntyre 1982, 1992) and straining at the flanks of jets as driving mechanisms for the staircase formation. The combined effects of sharp PV gradients as a suppressor of cross-stream excursion of air parcels and kinematics of jets lead to preferential mixing in the regions of weak PV gradients, accelerating homogenization there while concentrating gradients to narrow “barrier” regions at the axes of the prograde jets. This view is partially supported by the collocation of minimum effective diffusivity (Nakamura 1996; Shuckburgh and Haynes 2003) with the edge of the polar vortex and at the extratropical tropopause, where PV gradients are strong (Haynes and Shuckburgh 2000a,b). The negative correlation between effective diffusivity and PV gradients, however, does not necessarily prove causality—the concentrated PV gradients may well be the effect of nonuniform mixing rather than the root cause of it. Additional support for the active role of a PV barrier has been given by Haynes et al. (2007), who compare the persistence of a jet in a dynamically consistent model (in which PV governs the flow) and a kinematic model (in which PV is passive) that are comparably perturbed. The jet in the dynamically consistent model is found to be more persistent, attesting to the dynamical role of PV gradients reinforcing the resilience of the jet.

The primary goal of this paper is to quantify and model the interplay between the large-scale flow (jets) and its nonconservative driving (mixing and forcing) with PV as an agent of interaction. We choose numerical simulations of forced barotropic beta-plane turbulence as a platform of inquiry because of their well-known property (coexistence of jets and eddy) and the simplicity in the diagnostic framework. We shall first diagnose the nonconservative effects on the jet formation with the formalism developed by NZ and use this observation to parameterize the feedback of large-scale dynamics on the nonconservative effects. Though not a complete closure (some internal variables are prescribed), the parameterization highlights the role of PV gradients in this feedback. With the parameterized mixing and forcing we shall construct a simplified (1D) model for the mean flow and compare the self-organization of multiple jets in this model with that in the direct numerical simulation.

The next section describes and analyzes the direct numerical simulations of beta-plane turbulence. Section 3 introduces the parameterization of effective diffusivity. In sections 4 and 5, solutions of the 1D model with the parameterized mixing and forcing will be examined in detail. Section 6 provides summary and further discussion.

2. Numerical simulation of barotropic beta-plane turbulence

In this section we describe direct numerical simulations of beta-plane turbulence in which a staircaselike PV structure emerges from an initially uniform gradient. The model is based on the barotropic vorticity equation on a doubly periodic beta plane and solved with a spectral transform method. The physical domain is discretized with 256 × 256 grids. To limit our scope to mixing and forcing, no damping is applied other than hyperdiffusion $\nabla^6$ on relative vorticity. A statistically steady forcing is applied at small scales (total wavenumbers 41–45), with its phase randomized with a short temporal correlation. Because models of this sort are well documented in the literature (e.g., Rhines 1975; Vallis and Maltrud 1993), no further details are discussed here. Refer to the caption of Fig. 1 for the dimensions of the model.

The model is initialized with a zero zonal-mean wind and a very weak white noise in relative vorticity. Figure 1a shows an example of the simulated zonal-mean zonal wind $\bar{u}$ as a function of time and latitude. Although $\bar{u}$ is initially zero, energy injected at the forcing scales is transferred upscale and leads to a formation of multiple jets. The solution has yet to reach a steady state at the end of the simulation, but four pairs of robust prograde–retrograde jets emerge after mergers of weaker jets. Figure 1b shows the corresponding evolution of PV in equivalent latitude (Butchart and Remsberg 1986; Allen and Nakamura 2003). It is evident that the initially uniform PV gradient becomes gradually nonuniform, creating densely packed PV contours at the axes of the
four prograde jets. Figure 2 shows the profiles of PV and its gradient, as well as the zonal-mean zonal wind, averaged over the first, third, and last 1/5 periods of the simulation. The initially linear PV develops stepwise variations across at least three prograde jets (Figs. 2a–c), where the PV gradients are spiked (Figs. 2e,f). PV gradients adjacent to the spikes become smaller than the initial value but remain finite, unlike the idealized staircase in which they vanish. Note that in Figs. 1 and 2 the PV and its gradient are analyzed in equivalent latitude to...
emphasize the nonconservative effects: since equivalent latitude is a material coordinate in the absence of non-conservative processes, deviations from the initial profile in these figures are solely due to mixing and forcing. It is this altered PV profile that reinforces, through invertibility, the robust jet profile in Figs. 2h,i.

In NZ we derived a relationship between $\pi$, the diffusive flux of PV, and the finite-amplitude wave activity. Ignoring the subgrid diffusion of planetary vorticity and the mean friction, Eq. (27) of NZ may be written as

$$\frac{\partial}{\partial t} (\bar{u} + A) \approx -K_{\text{eff}} \frac{\partial Q}{\partial y} + \Delta \Sigma$$

where $K_{\text{eff}}$ and $\frac{\partial Q}{\partial y}$ are the effective diffusivity (Nakamura 1996) and gradient, respectively, of PV in equivalent latitude, $A(y,t)$ is the finite-amplitude Rossby wave activity, and $\Delta \Sigma$ is the source of wave activity as defined in NZ. The last identity in Eq. (1)
uses the relationship between PV, the zonal-mean PV \( q \), and \( A \):

\[
Q = \bar{q} - \partial A/\partial y = f_0 + \beta y - \frac{\partial}{\partial y} (\bar{\pi} + A) \tag{2}
\]

[see Eq. (18) of NZ]. Here \( Q(y, t) \) denotes the value of the PV contour at equivalent latitude \( y \) at time \( t \), and \( f_0 \) is the constant Coriolis parameter. The advantage of Eq. (1) over the traditional zonal-mean momentum equation is that it eliminates advective flux of PV, and the right-hand side contains only nonconservative processes. For a comparison, the profiles of \( \bar{\pi} + A \) are also plotted in Figs. 2g–i. Although the difference \( A \) has significant magnitude, especially around the retrograde jets, the profile of \( \bar{\pi} \) is qualitatively similar to that of \( \bar{\pi} \). In what follows we will use \( \bar{\pi} + A \) as a surrogate for the mean flow and quantify its nonconservative driving using Eq. (1) (we do not treat \( \bar{\pi} \) and \( A \) separately).

In NZ, we applied Eq. (1) to a freely decaying beta-plane turbulence (\( \Delta \Sigma = 0 \)) and demonstrated that most of the meridional structure in \( \bar{\pi} \) by the end of the simulation was attributable to the cumulative effect of the PV diffusive flux \(-K_{eff} \partial Q/\partial y\) [Eq. (32) of NZ]. In forced turbulence, however, a statistical steady state (or near–steady state) may be reached, where the diffusive flux is balanced by the source term on the right-hand side of Eq. (1). Figures 2a–c reveal that this is indeed the case throughout the simulation. These figures show the two terms integrated over the first, third, and last \( 1/5 \) period of the simulation (approximately 2000 days). In all three periods, the two curves are nearly identical. Their difference, which would give the change in \( \bar{\pi} + A \) during each period, is negligibly small compared to the magnitudes of the terms themselves. Furthermore, the two curves are highly inhomogeneous, and one notices a striking anticorrelation between Figs. 2b,c and the corresponding PV gradients in Figs. 2e,f: where PV gradient is large, the flux and source terms are both small and vice versa. Figures 3d–f show scatter diagrams of the reciprocal of time-mean PV gradient \( (\partial Q/\partial y)^{-1} \) and time-integrated \( \Delta \Sigma \) for each period using the values sampled at the same equivalent latitudes. Although there is some scatter, there seems to be a robust linear relationship particularly during the early stage of the simulation (Fig. 3d).
and for small values of \((\partial Q/\partial y)^{-1}\) and \(\Delta \Sigma\) (in the “PV step” regions) in general

\[
\Delta \Sigma \approx F_0 (\partial Q/\partial y)^{-1},
\]

where \(F_0\) is a constant source of enstrophy. Recalling that \(\Delta \Sigma\) is the source of wave activity [see Eq. (24a) of NZ], one can see that Eq. (3) is consistent with its
small-amplitude approximation \(\Delta \Sigma \approx \vec{q}'/\partial q/\partial y \approx \vec{q}'/\partial Q/\partial y\) [e.g., Eq. (7.23b) of Vallis 2006], where \(q'\) is the eddy PV, and \(\vec{q}'\) is the small-scale forcing term in the PV equation. The overbar denotes zonal averaging. At the forcing scales a significant fraction of \(q'\) retains a memory of forcing from the recent past; since \(\vec{q}'\) has a finite correlation time, the time average of \(\vec{q}' q'\) is proportional to the autocorrelation of \(\vec{q}'\). Since \(\vec{q}'\) is statistically steady and homogeneous, one expects the time average of \(\vec{q}' q'\) to be nearly uniform, hence \(\Delta \Sigma \propto (\partial Q/\partial y)^{-1}\). The coefficient \(F_0\) estimated from the least squares fit, however, decreases with time; it becomes almost two-thirds of the initial value by the last \(1/5\) period. This decrease in \(F_0\) is attributable to the decrease in the temporal correlation between \(\vec{q}'\) and \(q'\); as eddy energy in turbulence increases, \(q'\) is more scrambled and its memory of forcing becomes shorter. However, as a first approximation, we will ignore this time dependence and use Eq. (3) to parameterize the forcing term. [Although the diffusive flux has a nearly identical profile, we will not use Eq. (3) for its parameterization for reasons to be explained.]

That the right-hand-side terms in Eq. (1) are strongly correlated with the PV gradient suggests that the PV gradient regulates the nonconservative driving of the mean flow, which makes sense to the extent that the PV gradient governs the flow through invertibility, and the flow in turn regulates the kinematics of mixing and forcing. To test this idea, we have run an identical simulation except that the advection of eddy PV by the zonal-mean flow \(-\pi \partial q'/\partial x\) is artificially suppressed in the model, which weakens the influence of PV on the kinematics of the mean flow (the advecting flow excludes the part invertible from the zonal-mean PV). In turn, mixing of PV is significantly reduced at the flanks of the jets, because straining by \(\vec{u}/\partial y\) is suppressed. The result shown in Fig. 4 reveals that both the peak PV gradients and the speeds of the jets are approximately one-third of those in Fig. 2 (Figs. 4a and 4b correspond to Figs. 2f and 2i, respectively). This result demonstrates that the dynamical coupling of PV and the kinematics of the flow are critical in strengthening the jets, in agreement with Haynes et al. (2007).

We have repeated the foregoing analysis for many different realizations of beta-plane turbulence. Although the details of the solutions depend very much on the initial condition and the magnitudes of beta and forcing, the results shown above are representative: stepwise variations of PV occur commonly at mature prograde jets. The dependence of the source term on the PV gradient, Eq. (3), is also found to be robust.

3. Parameterization of effective diffusivity

The result of the previous section suggests that the right-hand-side terms in Eq. (1) may be parameterized
as functions of PV gradient. Based on Eq. (3), we assume that $\Delta \Sigma$ is inversely proportional to $\partial Q/\partial y$. Since $\Delta \Sigma \approx K_{\text{eff}} \partial Q/\partial y$ in the near-steady state, one expects

$$K_{\text{eff}} \approx \left(\frac{\partial Q}{\partial y}\right)^{-2}$$  \hspace{1cm} (4)

once that stage is reached; however, a complete balance of the two terms precludes evolution of $\pi + A$, including jet formation. Instead, we seek an independent parameterization of $K_{\text{eff}}$ and test a posteriori if it recovers Eq. (4) as a steady state is approached.

Recently Ferrari and Nikurashin (2010, hereafter FN) proposed a parameterization of effective diffusivity for a conservative tracer in the context of isopycnal mixing across an oceanic jet. They use a fluctuation–dissipation model for eddy streamfunction to derive their result of FN, who derive it explicitly with a fluctuation–dissipation model of $\Delta \Sigma$. (b),(c) As in (a), but with $\partial Q/\partial y$ for the third and last 1/5 period, respectively, as shown in Figs. 2e,f.

$$L_i^2 \approx O(\chi' \psi' J^{-1}).$$  \hspace{1cm} (7)

In the presence of strong anisotropy, however, the length of the cross-stream excursion is dictated by the first two linear terms in Eq. (6). Using their ratio, the length scale is expressed as

$$L_A^2(y, t) = O\left[\chi' \left(\frac{\partial \chi'}{\partial y}\right)^{-1}\right] = O[\psi'^2(\pi - c)^{-2}]$$

$$= O[\text{EKE}(k^2 + l^2)^{-1}(\pi - c)^{-2}],$$

where $k$ and $l$ are the local zonal and meridional wave-numbers of the eddy. As the flow grows increasingly anisotropic by spinning up jets, $L_A^2$ becomes smaller. In places where it becomes smaller than $L_i^2$ it should replace $L_i^2$ in Eq. (5). Such crossover may be approximated with (half of) a harmonic mean of $L_i^2$ and $L_A^2$. Thus, an anisotropic extension to Eq. (5) would be

$$K_\text{FN}(y, t) \approx \tau^{-1}(L_i^2 + L_A^2)^{-1}$$

$$= K_{\text{TP}}[1 + L_i^2(k^2 + l^2)(\pi - c)^2(\text{EKE})^{-1}]^{-1},$$

Note that $K_{\text{FN}} \approx K_{\text{TP}}$ when $L_A \gg L_i$. Although the above development is heuristic, Eq. (9) reproduces the result of FN, who derive it explicitly with a fluctuation–dissipation model of $\psi'$. Comparing Eq. (9) with a direct numerical calculation of effective diffusivity (Nakamura...
1996; Marshall et al. 2006), FN find reasonable agreements (up to constants) for mixing in the Southern Ocean. In particular, within the Antarctic Circumpolar Current where \((u/C_0)^2\) is large, \(K_{FN}\) captures a distinctive minimum.

Equation (9) is purely kinematic, and \(x\) can be any conservative tracer. Here, we adapt Eq. (9) for PV:

\[
x = q \frac{f}{\psi} \frac{1}{b}
\]

Since \(q = 2c\), Eq. (8) may be rewritten as

\[
K_{FN}(y, t) \approx K_{TP} \left\{ 1 + L^2_i \left( \frac{\partial q}{\partial y} \right)^2 \left( [k^2 + l^2] EKE \right)^{-1} \right\}^{-1} = K_{TP} \left\{ 1 + \mu \left[ \left( \frac{\partial q}{\partial y} \right) \beta \right]^2 \right\}^{-1},
\]

with

\[
\mu = \beta^2 L^2_i \left( [k^2 + l^2] EKE \right)^{-1} = L^2_i \left( [k^2 + l^2] L_{\beta}^2 \right)^{-1} \approx L^2_i L_{\beta}^{-2}.
\]

The nondimensional parameter \(\mu\) measures the anisotropy of the flow. The last approximation in Eq. (12) is due to the fact that the wavelengths of the internally generated Rossby waves are at least on the order of the Rhines scale \(k^2 + l^2 \approx L_{\beta}^2\). For a well-developed beta-plane turbulence, \(L_i^2/L_{\beta} \approx O(1)\), so \(\mu\) is a constant on the order of unity (on the other hand, in the isotropic limit \(L_i^2 \ll L_{\beta}^2\), so \(\mu \ll 1\)).

Equation (11) shows that the meridional variation in \(K_{FN}\) comes from the PV gradient: where the PV gradient

\[
L_A^2(y, t) \approx O \left( \left( q' \frac{\partial q}{\partial y} \right)^{-1} \right)^2 \approx O \left( (k^2 + l^2)^2 \psi \eta \left( \frac{\partial \psi}{\partial y} \right)^2 \right) = O \left( (k^2 + l^2) EKE \left( \frac{\partial \psi}{\partial y} \right)^{-2} \right).
\]

In essence Eq. (10) incorporates the local Rossby wave dispersion relation (in a Wentzel–Kramers–Brillouin sense) to Eq. (8). Here, \(L_A\) measures the meridional undulation of the PV contours associated with free Rossby waves. In beta-plane turbulence, these waves are generated internally through upscale energy cascade. With Eq. (10), Eq. (9) becomes
is large, $K_{FN}$ is small, and vice versa. Thus, to the extent that Eq. (11) is valid, large PV gradients hinder mixing, consistent with the Rossby elasticity idea. The ability of PV to affect its own mixing sets it apart from a passive tracer. We will use Eq. (11) to approximate $K_{eff}$ with one modification: we replace $\partial \bar{q} / \partial y$ with $\partial Q / \partial y$, so that Eq. (1) may be closed for $u_A$

$$K_{eff}(y, t) = K_{TP} \left\{ 1 + \mu \left[ \left( \frac{\partial Q}{\partial y} \right)^2 \right] ^{1/2} \right\},$$

(13)

where $K_{TP}$ and $\mu$ are assumed constant. It is clear from Eq. (13) that Eq. (4) is approximately recovered if $(\partial Q / \partial y)^2 \gg \beta^2 \mu^{-1}$. To test Eq. (13), we compare it with the time-mean $K_{eff}$ diagnosed from the simulation in Fig. 5 for the three periods corresponding to Fig. 3. (See appendix D of NZ for the method of computation for $K_{eff}$ from hyperdiffusion.) For all periods $K_{TP} = 2.3 \times 10^9$ m$^2$ s$^{-1}$ and $\mu = 6$ are assumed, whereas the time-mean $\partial Q / \partial y$ obtained from each period of the simulation is used to evaluate Eq. (13). The two curves agree remarkably well in all three periods. In particular, the very small values of $K_{eff}$ associated with the maximal PV gradients and jet axes are reproduced accurately.

4. Simulation with parameterized PV mixing and forcing

Although the parameterized $\Delta \Sigma$ and $K_{eff}$, Eqs. (3) and (13), reasonably describe these variables observed in the numerical simulation, whether they capture the dynamics of jet formation remains to be seen. To this end, we substitute Eqs. (3) and (13) into Eq. (1)

$$\frac{\partial}{\partial t} (\bar{u} + A) = -K_{TP} \left\{ 1 + \mu \left[ \left( \frac{\partial Q}{\partial y} \right)^2 \right] ^{1/2} \right\} \frac{\partial Q}{\partial y}$$

$$+ F_0 \left( \frac{\partial Q}{\partial y} \right)^{-1}$$

(14)

and consider Eq. (14) on a periodic domain of length L. Here, we treat $K_{TP}$, $F_0$, and $\mu$ as constants representative of beta-plane turbulence. Since these constants involve internal parameters, such as eddy kinetic energy and eddy length scale, that are not predicted, Eq. (14) is not a full closure of the mean flow; however, it highlights the role of PV gradient in regulating the nonconservative driving. The following nondimensionalization is applied to Eq. (14):
The third relationship in Eq. (15) uses Eq. (2), which transforms Eq. (14) into

\[ \frac{\partial Q}{\partial y} \beta^{-1} = G = 1 - \frac{\partial^2}{\partial y^2} (\pi + A), \quad t K_{TP} L^{-2} \to t, \]

\[ F_0 (K_{TP} \beta^2)^{-1} \to F_0. \quad (15) \]

The third relationship in Eq. (15) uses Eq. (2), which transforms Eq. (14) into

\[ \frac{\partial}{\partial t} (\pi + A) = -G(1 + \mu G^2)^{-1} + F_0 G^{-1}, \quad 0 \leq y \leq 1. \quad (16) \]

The nondimensionalized \( F_0 \) measures the magnitude of forcing relative to stirring in homogeneous turbulence. Below, it is assumed that \( F_0 = 0.12 \) and \( \mu = 6 \). The choice closely reflects the observed (dimensional) value of \( F_0 \) for the third \( \frac{1}{5} \) period of the numerical simulation in section 2 and \( K_{TP} = 2.3 \times 10^5 \text{ m}^2 \text{ s}^{-1} \). We solve Eq. (16) numerically on 512 equally spaced grids with a periodic boundary condition, using a first-order explicit time differencing and a second-order center space differencing with a time interval of \( \Delta t = 5 \times 10^{-6} \). The procedure of solution is to (i) initialize \( \pi + A \), (ii) compute \( G \) with Eq. (15), (iii) evaluate the right-hand side of Eq. (16), and (iv) predict \( \pi + A \) for the next time step.

In the first example, \( \pi + A \) is initialized as \( 0.01 + 0.000 25 \cos(12\pi y + 15) \), \( 0.1 \cos(2\pi y + 15) \). Figure 6 summarizes the subsequent evolution of \( \pi + A \) for \( 0 \leq t \leq 2 \). Initially \( \pi + A \) is primarily characterized by a weak sinusoidal perturbation with wavenumber 6 (Fig. 6a), but by \( t = 0.05 \) pointed jets emerge at three locations and start to grow in amplitude (Figs. 6b,c). By \( t = 0.5 \) the jets attain substantial amplitudes and nearly equilibrate (Fig. 6d). Note that despite the formation of jets, \( \pi + A \) is everywhere decreasing (the vertical axes of Figs. 6a–d are shifted so the curves remain centered), which means that the dissipation of wave activity through the diffusive flux of PV exceeds its gain through the source term. The deceleration, however, is not uniform, which causes the jet profile to form. The time–latitude cross section of \( \pi + A \) minus its domain average shows that the emerging three jets are remarkably stable (Fig. 6e).

Figure 7 depicts the corresponding structure of PV. Initially, PV is nearly linear in latitude, but small stepwise variation develops at the same locations where the pointed jets are formed (Figs. 7a,b). The step size increases with time whereas the slope between the steps becomes less steep than the background beta (Fig. 7c).
By $t = 0.5$ a well-defined piecewise linear PV profile develops (Fig. 7d). In the time–latitude diagram of Fig. 7e the growth of discontinuities is captured by converging PV contours into the lines of discontinuity.

The above example produces three stable jets from the beginning, but more complex behaviors ensue from a slightly different initial condition. The following example, $\tau + A$ is initialized as $0.01 + 0.00025 \cos(12\pi y + 15)[1 + 0.1 \cos(2\pi y + 15)]$. Figures 8 and 9 describe the subsequent solution, to be compared with Figs. 6 and 7. In this case, five jets and PV steps form early (Figs. 8b and 9b), but two of them weaken and disappear. The remaining three jets are similar to those in Fig. 6 (Figs. 8c,d and 9c,d), but their emergence is significantly delayed compared to Fig. 6e. The disappearance of the PV steps at $y = 0.45$ and $y = 0.8$ is accompanied by diverging PV contours in Fig. 9e.

As the above comparison suggests, the numerical solution of Eq. (16) is generally very sensitive to the initial condition; for example, with the above choice of $F_0$ and $\mu$, no jet is formed if the amplitude of sinusoidal variation in the initial condition is too weak. The solution is also sensitive to the model resolution. Figure 10 shows the same time–latitude diagrams as in Figs. 8e and 9e but for the calculation with half the grid resolution (256 grids; the temporal resolution and the initial condition are identical). Comparing the two runs, we see that even though they produce a similar three-jet profile in the end, the low-resolution run reaches that profile much earlier than the high-resolution run. In particular, the demise of the PV step at $y = 0.45$ occurs almost a unit time earlier (Fig. 10b vs 9e). A weak sixth PV step also forms briefly at the beginning of the low-resolution run (Fig. 10b).

Despite these sensitivities, the solutions are robust in that all PV staircases (if they form) have similar slopes, and the associated multiple jets are qualitatively similar to the numerical simulation of beta-plane turbulence. The exact locations of the PV steps in the final state, however, appear to be unpredictable. In the next section we will perform a more detailed analysis of Eq. (16) to better understand the behavior of the numerical model.

5. Anatomy of staircase formation

The numerical results in the previous section reveal that the finite-difference solutions to Eq. (16) support formation of discontinuities in PV from a smooth initial condition. The origin of the discontinuities is best understood in terms of the PV gradient equation obtained by differentiating Eq. (16) twice in $y$. 

![Fig. 9. As in Fig. 8, but for PV. The conventions are as in Fig. 7.](http://journals.ametsoc.org/doi/abs/10.1175/1520-0477(2010)067-0271.1?journalCode=jas)
\[
\frac{\partial G}{\partial t} = \frac{\partial^2}{\partial y^2} \left[ G(1 + \mu G^2)^{-1} - F_0 G^{-1} \right] \\
= \frac{\partial}{\partial y} \left\{ (1 - \mu G^2)(1 + \mu G^2)^{-2} + F_0 G^{-2} \right\} \frac{\partial G}{\partial y}.
\]  

Equation (17) represents diffusion for \( G \), but it becomes backward diffusion if the expression in the square brackets is negative or

\[ F_0 < (\mu G^2 - 1)G^2(1 + \mu G^2)^{-2}. \]  

It is well known that backward diffusion is ill posed, and Eq. (17) becomes unstable (see appendix). Since the linear growth rate for backward diffusive instability is infinite at a vanishing scale, Eq. (17) predicts that PV gradient becomes spiky instantly in regions where Eq. (18) is met, creating discontinuities in PV.

Figure 11 depicts subspaces in the \((\mu, F_0)\) plane that satisfy Eq. (18) for \( G = 1 \) and \( G = 1.35 \). Because \( F_0 > 0 \), Eq. (18) requires that \( \mu > G^{-2} \). For an unperturbed flow \((G = 1, \mu > 1)\), which means \( L_1 > L_0 \) [Eq. (12)]. Thus, spontaneous generation of a PV staircase occurs only if the flow is sufficiently anisotropic. In addition, \( F_0 \) must be sufficiently small (mixing must be sufficiently fast) for a given \( \mu \) and \( G \). In Fig. 11, the plus sign marked “b” (point B) corresponds to the choice of \((\mu, F_0)\) in section 4. Generally \( G \) varies with \( y \) and \( t \); therefore, it is possible that only a part of the domain satisfies Eq. (18). For example, in the solution shown in Fig. 6, the maximum value of \( G \) in the initial condition is 1.35 because of the sinusoidal perturbation in \( \tilde{u} + A \) (the unperturbed value is 1). It is clear that point B satisfies Eq. (18) for \( G = 1.35 \) but not for \( G = 1 \). Therefore, one expects that spikes in the PV gradient form only in regions where \( G \) is close to its maximum value (i.e., it is a finite-amplitude instability).

The top row of Fig. 12 describes the early evolution of \( G \) in the experiment shown in Figs. 6 and 7. Although
initially $G$ has six maxima and minima, their amplitudes are not uniform, and Eq. (18) is satisfied only in the vicinity of the three greatest maxima. As a result, $G$ develops spikes in these regions through backward diffusive instability (Figs. 12b–d; growth rates are kept finite by a finite grid size $\Delta y$). Closer examination reveals multiple spikes within each of the three regions in Fig. 12b, but they are quickly consolidated into a single spike. Since outside these regions Eq. (17) is diffusive, $G$ is flattened between the spikes (Fig. 12e). Also shown are the corresponding effective diffusivity $(1 + \mu G^2)^{-1}$ (Figs. 12f–j), the diffusive flux of PV (Figs. 12k–o, dashed curve), and the source term (Figs. 12k–o, solid curve). Because of the spikes in $G$, these quantities develop narrow gaps (barriers) in which they take extremely small values but otherwise they exhibit considerable structure. The diffusive flux is greater than the source term in the early stage, consistent with the deceleration of the flow (Figs. 12k–n). Once the jets mature, the two terms nearly balance and become flat (Fig. 12o).

Thus, the locations at which discontinuities form initially are sensitive to the distribution of $G$ as long as $(\mu, F_0)$ is near the stability boundary, which explains why the initial flow geometries in Figs. 8 and 9 are different from those in Figs. 6 and 7. How do we interpret the numerical simulation in section 2 in light of Fig. 11? The three plus signs in Fig. 11 correspond to the first, third, and last 1/5 periods of the numerical simulation. The points descend from point A to point C as forcing diminishes relative to mixing. The simulation starts well above the stability boundary for $G = 1$ but becomes closer to it as mixing increases. It is conceivable that at some point of time some parts of the domain will reach the boundary because of a finite-amplitude perturbation to $G$, where PV steps begin to form.

We have also initialized the model with $\mu$, $F_0$, and $G$ that satisfy Eq. (18) everywhere. In this case, $G$ spawns numerous spikes from a very weak initial perturbation (it is no longer a finite-amplitude instability), leading to many small PV steps; however, later they collapse one after another until a few robust steps and jets emerge (Fig. 13). Witelski et al. (2001) report a very similar solution for a model of shear-band formation in a granular medium, which possesses a similar mathematical structure to ours despite different motivation.

One might ask why the finite-difference solution can even integrate stably if Eq. (17) becomes ill posed. It is related to its nonlinear coefficient in that even when Eq. (17) is ill posed, the same is not true for the corresponding PV equation:

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial y} \left[ G(1 + \mu G^2)^{-1} - F_0 G^{-1} \right]$$

$$= \frac{\partial}{\partial y} \left[ (1 + \mu G^2)^{-1} - F_0 G^{-2} \right] \frac{\partial Q}{\partial y}. \quad (19)$$

The square brackets in Eq. (19) are positive for $G^2 > (F_0^{-1} - \mu)^{-1}$, in which case Eq. (19) is forward diffusive and thus well posed (examination reveals that $G$ remains in this range everywhere throughout the integrations in section 4). Thus, despite the ill-posedness of Eq. (17), PV is bounded and its monotonicity is preserved. As long as the discrete analog of Eq. (19) is enforced, the solution does not blow up even when step discontinuities form.

To understand the asymptotic behavior of the discrete model, it is useful to consider the magnitudes of the two terms on the right-hand side of Eq. (16) as functions of $G$, which are shown in Fig. 14. A steady state ensues where the two curves meet, which occurs at two positive values of $G$: $G_1^* = (F_0^{-1} - \mu)^{-1/2}$ and $G_2^* = \infty$. The first value requires $F_0 < \mu^{-1}$, which our choice ($F_0 = 0.12$, $\mu = 6, G_1 = 0.655$) meets. A steady solution must consist of linear segments of PV with the gradient $G_1^*$, separated by “jumps” where $G = G_2^* = \infty$. Discontinuities are inevitable, because a continuous linear function with the slope $G_1^*$ cannot satisfy the periodicity of PV; namely, $Q(y + 1, t) = Q(y, t) + 1$, since $G_1 < 1$. The locations or magnitudes of the jumps, however, cannot be determined a priori (the steady state is nonunique).
It is easy to verify that $G_1$ and $G_2$ are both attracting solutions of $G$; the sign of the square brackets in Eq. (17) is positive at $G = G_1$, whereas it is negative but vanishes as $G \to G_2$. Thus, a perturbation to $G_1$ tends to be diffused, whereas a large $G$ becomes even larger by backward diffusion, which results in a very robust binary distribution of $G$, consistent with Fig. 12c. The slope corresponding to $G_1$ is also indicated in Figs. 7a–d, 9a–d, and 13f–j. The piecewise linear PV gives rise to piecewise parabolic jets (Figs. 6, 8, and 13).

Although $G = G_2 = \infty$ is a steady-state solution to Eq. (16), it is never achieved exactly in the discrete model, because the PV gradient is limited by the finite grid size or by an implicit numerical diffusion [$G_2 \approx \Delta Q/(\Delta y)^{-1}$, where $\Delta Q$ is the amount of PV jump]. A small negative imbalance remains on the right-hand side.
of Eq. (16) at the PV steps, which causes $\pi + A$ to decay there. The decay rate is larger for smaller $\Delta Q(D_y)^{-1}$, because the imbalance increases as $G$ decreases (Fig. 14). Therefore, weaker jets with smaller $\Delta Q$ decay faster than stronger jets and eventually disappear. Unless the strengths of the jets are equal, as in Fig. 6, the number of the jets decreases as observed in Figs. 9 and 13. The decay rate is also affected by $D_y$; a larger $D_y$ allows the jet to decay faster, which is consistent with the faster demise of weaker jets/PV steps in Fig. 11 compared to Figs. 8e and 9e. In principle, jets continue to decrease until the last one remains, but it can take a long time; in most simulations with 512 grids, it is common to observe two jets by $t = 50$, but a single jet is seldom reached.

Because of the ill-posedness of the PDE, the limit $D_y \to 0$ may not give convergence to a physically relevant solution, but if one draws an analogy between a finite $D_y$ and the Kolmogorov scale at which molecular viscosity takes over enstrophy cascade, then the discrete solution permits a physical (albeit uncomfortable) interpretation that the evolution of the jets depends critically on the viscosity at the barriers, which determines their “leakiness” because stirring vanishes there (cf. Nakamura 2008).

6. Summary and discussion

We have investigated the role of nonconservative processes in the formation of a PV staircase and reinforcement of multiple jets in barotropic beta-plane turbulence. Numerical integration of vorticity equation on the beta plane with a steady small-scale forcing produces robust zonal jets, and they are reinforced by the concentrated PV gradients that form at the axes of the prograde jets. Based on the formalism developed in the companion paper (NZ), the diffusive flux of PV and the source of wave activity due to forcing are diagnosed as the primary agents of the nonconservative arrangement of PV and the mean flow (the sum of the zonal-mean zonal flow and finite-amplitude wave activity). It is found that the two quantities are both strongly anticorrelated with PV gradients, suggesting that PV gradients control the nonconservative driving of the flow.

On the basis of the above observation and in reference to the small-amplitude theory, the source term is parameterized as a uniform enstrophy forcing $F_0$ divided by the PV gradient, whereas the PV flux is modeled with the effective diffusivity $K_{\text{eff}}$ formula proposed by FN. The parameterized $K_{\text{eff}}$ is a decreasing function of the square of the PV gradient, emphasizing that PV regulates its own mixing: strong PV gradients act as barriers whereas weak gradients accelerate mixing.

There are two constants in the parameterization that need to be specified: $F_0$ and $\mu$. The former represents the strength of forcing relative to mixing in a homogeneous background, and the latter measures anisotropy associated with beta. Since these “internal” parameters are specified, the parameterization does not represent a complete turbulence closure. Rather, it highlights the role of PV gradient in forming an extreme inhomogeneity under a prescribed flow regime.
The numerical solution of the nonlinear PDE with the parameterized mixing and forcing predicts formation of a "PV staircase" (piecewise linear PV) and multiple jets when $F_0$, $m$, and the PV gradient satisfy Eq. (18). The flow has to be sufficiently anisotropic and mixing must be strong ($m > 1$ and $F_0/C^2_m > 1$) for jets to form from a nearly uniform condition ($G \approx 1$). It is indicated that the numerical simulation starts outside, but near, the boundary of Eq. (18) in ($m$, $F_0$) space, and as soon as it crosses the boundary, the PV staircase begins to form.

The generation of discontinuous PV in the parameterized model is associated with the ill-posedness of the underlying PDE, which spawns spikes in PV gradient through backward diffusive instability. Although nonlinearity allows the discrete model to integrate stably, the solution is sensitive to the initial condition and resolution. The sensitivity to the initial condition may be acceptable given that the parent problem (turbulence) itself is highly dependent on the initial condition. Dependence on the resolution begs the question of convergence, but this is a common problem for any transport modeling in the presence of near discontinuity (e.g., Nakamura 2008; Mizuta and Yoshimura 2009). Since formation of discontinuities in finite-difference models of ill-posed nonlinear PDEs is widely studied in other fields such as image processing (Kichenassamy 1997, and references therein) and granular matter dynamics (Witelski et al. 2001), we believe that the relevance of the present model to geophysical fluids is at least worth consideration.

There are, of course, plenty of discrepancies between the solution of the 1D model and the numerical simulation of beta-plane turbulence. The PV steps in the 1D model certainly exaggerate inhomogeneity. The concentrated PV gradients in the simulation are narrow but do not collapse to the grid size and they form gradually, whereas in the 1D model the spikes in PV gradients form very quickly. It appears that the nonlinearity in the parameterized effective diffusivity, though crucial for spontaneous step formation, may be too strong. Another discrepancy is that the barriers in the 1D model do not move once they form (Figs. 7, 9, and 13). The created jets either persist or disappear without merging, although mergers occur commonly during the direct numerical simulation (Fig. 1a). The lack of barrier movement is due to the symmetry of diffusivity and source terms about the barrier (Fig. 12); for the barrier to move, mixing and/or forcing must be asymmetric across the barrier.

Some modifications to Eqs. (3) and (13) are necessary to improve the transient solution of the parameterized model, although the transient behavior may be inherently difficult to model, because it depends on the small difference between two large terms that are both parameterized. [On the other hand, the steady-state slope of the PV staircase that arises from the balance of the two terms is robust: $(F_0^{-1} - \mu)^{-1/2}$.] In addition, the current form of parameterization does not address why the numerical simulation occupies a region in ($m$, $F_0$) space that is close to the stability boundary. Further work is necessary to constrain the internal parameters to better confine a dynamically available subspace in ($m$, $F_0$).

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APPENDIX

Necessary Condition for Backward Diffusive Instability

Consider the diffusion equation for $\chi$:

$$\frac{\partial \chi}{\partial t} = \frac{\partial}{\partial y}(K \frac{\partial \chi}{\partial y}),$$  \hspace{1cm} (A1)

where both $\chi$ and $K$ are assumed periodic in $y$. Multiply Eq. (A1) by $\chi$ to obtain...
\[
\frac{\partial}{\partial t}(\chi^2/2) = \frac{\partial}{\partial y}(\chi K \partial \chi/\partial y) - K \left(\frac{\partial \chi}{\partial y}\right)^2. \tag{A2}
\]

Taking the domain average (denoted by the angle bracket), the flux divergence term on the right-hand side vanishes because of the boundary condition:

\[
\frac{\partial}{\partial t}(\chi^2/2) = -\langle K(\partial \chi/\partial y)^2 \rangle. \tag{A3}
\]

If \(K\) is positive everywhere in the domain (diffusive), the right-hand side of Eq. (A3) is negative, so the average variance of \(\chi\) decays with time. If \(K\) is everywhere negative (backward diffusive), then the variance will grow. Generally, the necessary condition for backward diffusive instability is that \(K\) is negative somewhere in the domain. This condition is often sufficient as well, because the fastest-growing modes are the ones with the shortest scales, so even a small region of negative \(K\) can accommodate them.

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