The Subpolar Front of the Japan/East Sea. Part II: Inverse Method for Determining the Frontal Vertical Circulation

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ABSTRACT

An inverse method for inferring vertical velocities from high-resolution hydrographic/velocity surveys is formulated and applied to observations collected at the subpolar front of the Japan/East Sea (JES) taken during several cold-air outbreaks. The method is distinct from vertical velocity inferences based on the omega equation in that the driving mechanism for the ageostrophic flow is inferred rather than assumed and hence is particularly appropriate for application to wind- or buoyancy-forced upper-ocean currents where friction, mixing, inertial/superinertial motions, or higher-order effects can contribute along with shear/strain of the geostrophic flow to force vertical motions.

The inferred vertical circulation at the subpolar front of the JES has amplitudes $O(100$ m day$^{-1}$) compared to the $O(20$ m day$^{-1}$) vertical velocities predicted by the omega equation. Time-dependent, near-inertial motions driven by the winds and modified by the vertical vorticity of the frontal jet appear to be the primary cause of the strong vertical motions. The strongest vertical motions are associated with submesoscale, $O(5$ km), frontal downdrafts that tend to align with the slanted isopycnal surfaces of the front and advect water with low salinity and high chlorophyll fluorescence down the dense side of the front.

1. Introduction

The vertical exchange of surface waters with intram and subpycnocline waters is an important process for the subduction of water masses modified by air–sea fluxes and the transport of nutrients necessary for sustaining marine ecosystems in the photic zone. Vertical motions are enhanced in regions of high mesoscale and submesoscale variability owing to a variety of processes. Shear and strain associated with such flows disrupt the thermal wind balance and hence generate vertical circulations to restore geostrophy over subinertial time scales (Hoskins et al. 1978). Vertical circulations can also arise due to lateral variations of turbulent fluxes of momentum and buoyancy generated during the spin down of meso/submesoscale flows (Garrett and Loder 1981; Thompson 2000; Nagai et al. 2006). Forcing such flows by winds can drive Ekman pumping/suction owing to the modification of the Ekman transport by the vertical component of the relative vorticity (Stern 1965; Nüller 1969; Thomas and Lee 2005). Additionally, when the wind blows in the direction of the surface geostrophic shear, Ekman advection of denser water over light leads to localized mixing in regions of strong baroclinicity, which drives an ageostrophic secondary circulation (ASC) (Thomas and Lee 2005). Furthermore, if the mesoscale/submesoscale flow is thrown out of balance, time-dependent vertical motions can be triggered as the fluid adjusts to geostrophic equilibrium (Ou 1984; Tandon and Garrett 1994).

Regardless of the process leading to the generation of vertical motions, it is of interest to diagnose the vertical
velocity from observed meso/submesoscale flows. Presently, there are no observational techniques capable of measuring the vertical velocity \( w \) directly over areas large enough to encompass meso/submesoscale features. Consequently, \( w \) must be inferred. Vertical velocities are typically inferred through solutions of the quasigeostrophic (QG) omega equation (e.g., Pollard and Regier 1992; Rudnick 1996; Viu´dez et al. 1996; Allen and Smeed 1996; Shearman et al. 1999). If the meso/submesoscale flow is embedded at least partially in the mixed layer, where the stratification is weak and the Richardson number is low, and the currents have significant Rossby numbers, the QG approximation may not be valid. For large Rossby number flows, semigeostrophic (Naveira Garabato et al. 2001) and intermediate model (Shearman et al. 2000) versions of the omega equation have been solved to diagnose the vertical velocity from observations. Comparisons with numerical simulations show that these higher-order formulations of the omega equation yield more accurate estimates of the vertical velocity, although the presence of inertia–gravity waves and diffusive processes can lead to significant errors (Viu´dez and Dritschel 2004). These errors arise because diagnostics based on the omega equation implicitly assume that the ageostrophic velocity is dominantly driven by the shear and strain of the geostrophic flow and, hence, neglect effects due to friction, mixing of buoyancy, and/or time dependance. To account for these effects, Giordani et al. (2006) derived the “generalized omega equation” in which the \( Q \) vector, the divergence of which drives the omega equation, was modified to include contributions from turbulent fluxes of momentum and buoyancy and higher-order terms involving time dependence and advection of/or ageostrophic flows. Giordani et al. used the generalized omega equation to diagnose the vertical velocity from realistic numerical simulations of the northeastern Atlantic and found that the nontraditional forcing terms of their equation can result in significant vertical velocities. Similar to these simulations, in the real ocean it is likely that friction, mixing of buoyancy, and/or time dependence are important in the dynamics of the vertical velocity of upper-ocean meso/submesoscale flows, especially when such flows are exposed to atmospheric forcing. This implies that solutions to the traditional omega equation may poorly represent the actual vertical velocity of upper-ocean meso/submesoscale currents and motivates the use of a different method to estimate the vertical circulation. In this paper, we develop an inverse method to infer \( w \) that assesses, rather than specifies, the driving mechanism for the vertical circulation. The method is more general than omega-equation diagnostics and hence is more appropriate for use with upper-ocean datasets.

In this article, the method is applied to observations of the subpolar front of the Japan/East Sea (JES). This paper is one in a series of three detailing the dynamics of the subpolar front of the JES. The first article (C. M. Lee et al. 2008, unpublished manuscript) describes hydrographic, velocity, and shipboard meteorological measurements made at the front during periods of strong atmospheric forcing associated with cold-air outbreaks in the winter of 2000. The observations reveal thermohaline intrusions, pycnostads with weak stratification and low potential vorticity, and plumes with high chlorophyll fluorescence that evidence subduction and vertical circulation at the front. The inverse method developed and outlined in this article is used to quantify this vertical circulation. The third paper in the series (Y. Yoshikawa et al. 2008, unpublished manuscript), describes numerical experiments of an idealized representation of the subpolar front of the JES designed to study the response of the frontal vertical circulation and subduction to wind stress and heat loss and to characterize the driving forces responsible for the vertical circulation for various combinations of wind and buoyancy forcing.

The outline of the paper is as follows: first, the equation governing the ageostrophic secondary circulation that forms the theoretical basis for the inverse method is derived. Next, the inverse problem is formulated. Following this, the strategy and numerical method for the inverse calculation are outlined. In section 5, the inverse solution for the vertical circulation at the subpolar front of the JES is described. A discussion on the role of wind-driven time-dependent motions in the frontal vertical circulation and the advection of tracers by this circulation is given in section 6. The paper is concluded in section 7.

2. Governing equations for ageostrophic secondary circulations

We are interested in deriving equations governing ageostrophic secondary circulations (ASCs) valid in mixed layers where nonadiabatic, frictional, time-dependent, and high Rossby dynamics are important. The ASCs that will be considered have a characteristic horizontal length scale \( L \) longer than a kilometer and a vertical length scale \( H \) that scales with the mixed layer depth, \( O(100) \) m. Consequently the aspect ratio of the flow \( H/L \) is small, allowing for the utilization of the hydrostatic approximation. The equations governing hydrostatic motions are

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + fv = -\frac{1}{\rho_o} \frac{\partial p}{\partial x} + X, \quad (1)
\]
with the differential operators
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u = - \frac{1}{\rho_o} \frac{\partial p}{\partial y} + Y, \quad (2)
\]
where the geostrophic velocity satisfies the thermal wind balance
\[
0 = - \frac{1}{\rho_o} \frac{\partial p}{\partial y} + b, \quad (3)
\]
and
\[
\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y} + w \frac{\partial b}{\partial z} = Q, \quad (4)
\]
and
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (5)
\]
where \((u, v, w)\) are the components of the velocity, \(f\) is the Coriolis parameter (assumed to be constant), \(\rho_o\) is a reference density, \(p\) is the pressure, \((X, Y)\) are the horizontal components of frictional forces meant to represent turbulent mixing of momentum, \(b\) is the buoyancy, and \(Q\) is a generic source/sink of buoyancy caused by turbulent mixing of buoyancy.

The flow is decomposed into geostrophic and ageostrophic components
\[
u = u_g + u_{ag},
\]
and
\[
\left( L_{11} L_{12} L_{21} L_{22} \right) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \left( \begin{array}{c} F_1^g \\ F_2^g \end{array} \right) + \frac{\partial}{\partial z} \left( \begin{array}{c} Y \\ -X \end{array} \right) - \frac{\partial Q}{\partial x} \frac{\partial Q}{\partial y} \left( \begin{array}{c} \partial Q/\partial x \\ \partial Q/\partial y \end{array} \right) + \frac{D}{\partial t} \left( \partial^2 \phi/\partial z^2 \right) + \left( \begin{array}{c} R_1 \\ R_2 \end{array} \right), \quad (8)
\]
with the differential operators
\[
L_{11} = F_1^2 \frac{\partial^2}{\partial z^2} - 2 S_1^2 \frac{\partial^2}{\partial x \partial z} + N^2 \frac{\partial^2}{\partial x^2},
\]
\[
L_{12} = -\alpha \frac{\partial^2}{\partial z^2} + 2 S_2^2 \frac{\partial^2}{\partial y \partial z} + N^2 \frac{\partial^2}{\partial y \partial x},
\]
\[
L_{21} = -\alpha \frac{\partial^2}{\partial z^2} - 2 S_1^2 \frac{\partial^2}{\partial y \partial z} + N^2 \frac{\partial^2}{\partial y \partial x},
\]
\[
L_{22} = F_2^2 \frac{\partial^2}{\partial z^2} + 2 S_2^2 \frac{\partial^2}{\partial y \partial z} + N^2 \frac{\partial^2}{\partial y^2},
\]
and coefficients \(N^2, F_1^2, F_2^2, S_1^2, S_2^2, \alpha\) that are functions of the stratification and of the shear and confluence/diffuence of the geostrophic flow
\[
N^2 = \frac{\partial b}{\partial z}, \quad F_1^2 = f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad F_2^2 = f \left( \frac{\partial w}{\partial z} \right),
\]
\[
S_1^2 = \frac{\partial v}{\partial z}, \quad S_2^2 = \frac{\partial w}{\partial y}, \quad \alpha = -f \frac{\partial v}{\partial y},
\]
reflecting the tendency of the shear and strain of the geostrophic flow to disrupt the thermal wind balance and hence destroy itself (Hoskins et al. 1978). In terms of the 4 vector formalism introduced by Hoskins et al. (1978), \((F_1^g, F_2^g)\) is equal to \(-2Q\). As shown by Hoskins and Draghici (1977), Eq. (8) forced solely by term I can be converted to the semigeostrophic omega equation by taking \(-\partial/\partial x\) of the top row of (8) and adding it to \(-\partial/\partial y\) of the second row of (8).
Similar to the geostrophic forcing, vertically varying frictional forces and laterally varying buoyancy sources/sinks (terms II and III, respectively) will disrupt the thermal wind balance and hence drive ageostrophic circulations (Eliassen 1951). Also note that the equations governing Ekman dynamics are encompassed in (8) when II is balanced by the lhs terms with coefficients $F_1^2$ and $F_2^2$. Term III is important at wind-forced fronts where Ekman flow can advect dense water over light, generating convection and turbulent mixing of buoyancy localized to the front. This mixing drives (through term III) frontogenetic ageostrophic secondary circulations (Thomas and Lee 2005).

The effects of time dependence on ageostrophic flows, important for inertial and superinertial motions, enter (8) through term IV, which represents the rate of change of the potential vorticity of the geostrophic flow, such as described by Kunze (1985), as well as inertial and inertia–gravity waves propagating in geostrophic shear, $u_g\partial/\partial x + v_g\partial/\partial y$, of the thermal wind imbalance:

$$-f\frac{\partial v}{\partial z} + \frac{b}{c_1} = -f\frac{\partial u_{ag}}{\partial z},$$

$$f\frac{\partial u}{\partial z} + \frac{b}{c_1} = f\frac{\partial u_{ag}}{\partial z}.$$  

Balancing (8) with term IV captures the dynamics of inertia–gravity waves propagating in geostrophic shear, such as described by Kunze (1985), as well as inertial and symmetric instabilities that can occur when the potential vorticity of the geostrophic flow,

$$q_g = (f\mathbf{k} + \nabla \times \mathbf{u}_g) \cdot \mathbf{v}_b,$$  

(10)

takes the opposite sign of $f$ (see Hoskins 1974; Hua et al. 1997).

The last term on the rhs of (8), where

$$R_1 = -f\frac{\partial}{\partial z}(u_{ag} \cdot v_{ug}),$$

$$R_2 = f\frac{\partial}{\partial z}(u_{ag} \cdot u_{ag}),$$

(11)

is important for strong ageostrophic flows with large Rossby numbers $Ro_{ag} = U_{ag}/fL$ ($U_{ag}$ is the characteristic velocity scale of the ageostrophic flow).

If the Rossby number based on the geostrophic flow $Ro_g = U_g/fL$ ($U_g$ is the characteristic velocity scale of the geostrophic flow) is small, then the quasigeostrophic assumption may be used. In the quasigeostrophic limit, the coefficients of the operator on the lhs of (8) become $F_1^2 \rightarrow f^2, F_2^2 \rightarrow f^2, \alpha \rightarrow 0, N^2 \rightarrow N^2, S_1^2 \rightarrow 0$, and $S_2^2 \rightarrow 0$. The last two approximations involving $S_2^2$ and $S_2^2$ also assume that the Richardson number of the geostrophic flow,

$$Ri = \frac{f^2N^2}{S_1^4 + S_2^4},$$

is much greater than one, which is equivalent to restricting the potential vorticity of the geostrophic flow (10) to have the same sign as $f$ and be large in magnitude.

It should be noted that the generalized omega equation of Giordani et al. (2006) captures the same dynamics as Eq. (8).

However, the two equations differ in their choice of variable: that is, a vector streamfunction in (8) and the vertical velocity $w$ in the generalized omega equation. The vertical derivative of the vector stream-function is equal to the horizontal component of the ageostrophic velocity, a quantity that, unlike $w$, can be calculated from observations. Consequently, Eq. (8) is better suited to an inverse analysis than the generalized omega equation since observations can be used to constrain its solutions.

3. Formulation of inverse problem

Nowadays, it is commonplace in observational studies of mesoscale flows to perform high spatial resolution hydrographic and velocity surveys using a shipboard-mounted ADCP in conjunction with a towed undulating vehicle equipped with a CTD, such as a SeaSoar. Using density and velocity data from such surveys and constraining the velocity to satisfy the thermal wind balance (6) and to be horizontally nondivergent, the geostrophic flow can be inferred (Rudnick 1996). Once the geostrophic flow is inferred, an estimate of the ageostrophic velocity $(u^{ag}, v^{ag})$ can be calculated from the residual of the ADCP velocity ($u_{ADCP}, v_{ADCP}$) and the inferred geostrophic velocity,

$$u^{ag} = u_{ADCP} - u_g,$$

$$v^{ag} = v_{ADCP} - v_g.$$  

(12)

Directly estimating the ageostrophic velocity in this way is a delicate procedure that can be prone to errors if not performed carefully. In section 4 and the appendix, we describe the possible sources of these errors and strategies to minimize them.

To relate the estimate of the horizontal component of the ageostrophic velocity (12) to a vector streamfunction

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1 The forcing terms on the right-hand side of (8) are included in the “generalized” $Q$ vector of Giordani et al. (2006). The generalized $Q$ vector also includes the terms on the left-hand side of (8) that would not be there if the quasigeostrophic approximation were used.
as in (7), we proceed as follows: assuming that \((\mathbf{u}^{ag}, \mathbf{v}^{ag})\)
has been gridded in space at coordinates \((x_i, y_j, z_k)\) and then vectorized so that
\[
\mathbf{v}_{ag} = [\ldots, u^{ag}_{i,j,k}, \ldots; v^{ag}_{i,j,k}, \ldots]^T \tag{13}
\]
and defining a vector streamfunction at the grid locations
\[
\mathbf{q} = [\ldots, \phi_{i,j,k}, \ldots; \psi_{i,j,k}, \ldots]^T, \tag{14}
\]
the observations can be related to the streamfunction as
\[
\mathbf{Dq} + \mathbf{n} = \mathbf{v}_{ag}, \tag{15}
\]
where \(\mathbf{D}\) is the finite difference representation of the vertical derivative and \(\mathbf{n}\) is observational noise. By constraining the streamfunction (14) to satisfy a model based on (8),
\[
\mathbf{Lq} = \mathbf{p}, \tag{16}
\]
where \(\mathbf{L}\) is the finite difference representation of the operator on the l.h.s of (8), we can estimate \(\mathbf{p}\), the sum of the terms on the r.h.s of (8) that force the ageostrophic circulation (it should also be noted that both \(\mathbf{L}\) and \(\mathbf{p}\) also include boundary condition information). This inverse method not only assesses the driving forces for the ageostrophic flow but also yields an estimate for the ageostrophic flow observations (13) due to temporal variability since typical SeaSoar/ADCP surveys last several days. This is a longer period than the characteristic time scale of near-inertial motions, internal gravity waves, and tidal flows that could be present during a survey. This masking of temporal variability as spatial variability is problematic

Eliminating \(\mathbf{q}\) and \(\mathbf{p}\) from (18)–(20) results in the following solution for \(\mathbf{p}\):
\[
\mathbf{p} = [(\mathbf{DG})^T\mathbf{DG}]^{-1}(\mathbf{DG})^T\mathbf{v}_{ag}, \tag{21}
\]
where \(\mathbf{G} = \mathbf{L}^{-1}\).

The inverse of a matrix that represents a differential operator is also the Green’s function of that operator. This means that each column of \(\mathbf{G}\) is the solution to (16) with \(\mathbf{p}\) corresponding to a point disturbance where all the elements of \(\mathbf{p}\) are zero except for one element that has the value 1; that is,
\[
\mathbf{LG} = \mathbf{l}, \tag{22}
\]
where \(\mathbf{l}\) is the identity matrix (Wunsch 1996). With this interpretation of the matrix inverse as the Green’s function in mind, it follows from (21) that the elements of \(\mathbf{p}\) are the coefficients obtained from a least squares fit of the horizontal velocity associated with the Green’s function,
\[
\mathbf{V}_G = \mathbf{DG},
\]

Therefore, the inverse calculation involves the following steps: 1) use the hydrographic observations, inferred geostrophic velocities, and a finite differencing scheme to construct \(\mathbf{L}\); 2) calculate the Green’s function of \(\mathbf{L}\) and take its vertical derivative to obtain \(\mathbf{V}_G\); and 3) fit \(\mathbf{V}_G\) to the ageostrophic velocity inferred from observations (12) to estimate \(\mathbf{p}\), the streamfunction, and the vertical velocity. A detailed account of the numerical method and strategy used to accomplish this calculation is outlined in the next section.

4. Strategy and method for calculating inverse solution

a. Issues involving temporal variability

The inverse method relies on the spatial structure of the inferred ageostrophic velocity (13) to select the columns of \(\mathbf{V}_G\) that best match the observations. However, apparent spatial structure could arise in the gridded velocity observations (13) due to temporal variability since typical SeaSoar/ADCP surveys last several days. This is a longer period than the characteristic time scale of near-inertial motions, internal gravity waves, and tidal flows that could be present during a survey. This masking of temporal variability as spatial variability is problematic
for the inverse calculation and requires a strategy to minimize its effects. The strategy employed on the winter 2000 dataset from the subpolar front of JES is described below.

The SeaSoar/ADCP surveys at the subpolar front of the JES consisted of 5–6 north–south, cross-frontal sections separated by ~20 km in the zonal direction (Fig. 1). The SeaSoar profiled repeatedly from the ocean surface to roughly 350-m depth while being towed at 8 kt. This configuration provided ~3 km along-track profile separation. Horizontal velocities were measured continuously using a 150-kHz ADCP installed on the ship, yielding velocity profiles with a vertical resolution of 8 m and a vertical extent of ~350 m. The velocity measurements were averaged every 3 min, resulting in an along-track resolution of ~0.75 km. The velocity and density fields used in the inverse calculation were objectively mapped using the method of Le Traon (1990).

Each field minus a quadratic fit was mapped at each depth using 5% noise and an anisotropic cosine modulated Gaussian covariance function with $e$-folding lengths of 20 and 10 km in the $x$ and $y$ directions and wave-lengths for the cosine modulation of 25 and 20 km in $x$ and $y$ directions, respectively.

FIG. 1. Objective map of the potential density at $z = -18$ m (contours) and the ship track (thick gray) for survey 2 of the subpolar front of the JES taken on 26–29 Jan 2000. Section numbers are to the left of the ship track. The black line denotes the truncated domain used for the inverse calculation along section 2. Contour interval for the density is 0.1 kg m$^{-3}$.
For the analysis of the vertical circulation at the subpolar front, $T$ was chosen to be 1.3 h or 0.07 inertial periods. This length of time was selected so as to minimize aliasing by inertial oscillations, the dominant time-dependent motions in the region (Takematsu et al. 1999). The distance traveled by the ship in this time is $L_s \approx 20$ km. This distance is sufficient to cover submesoscale features in the hydrography that evidence vertical circulation at the subpolar front. These features are associated with plumes of low salinity water with high chlorophyll fluorescence extending from the surface mixed layer to the pycnocline that appear to have been subducted on the dense side of the front. An example of such a plume is shown in Fig. 2. Plumes such as this were observed on the majority of sections and were characterized by a horizontal scale of 5–10 km, a distance smaller than $L_s$. Thus, the strategy for the inverse calculation is to perform the analysis section by section, using data collected within $6 \times 1.3$ h or $6 \times 20$ km from the frontal crossing. In using this strategy, however, it is necessary that the inverse solution calculated on each section be treated independently because significant temporal variation in the ageostrophic flow is likely from section to section, given the $\sim 0.6$ inertial period transit time from one frontal crossing to the next.

b. Two-dimensional methodology

For a fully three-dimensional ageostrophic flow, the value of the vector streamfunction $(\phi, \psi)$ along a meridional section, at a zonal location $x = x_o$, will be influenced by rhs forcing terms of (8) nonlocal to that section. That is, the solution vector of the inverse problem, $\mathbf{p}$, will in general have nonzero elements for grid locations at $x \neq x_o$. From (23) it can be seen that, depending on the number of elements in $\mathbf{p}$ relative to the number of data points in $\mathbf{v}_{ag}$, (23) changes from an under- to an overdetermined system of equations. For the strategy described above, where only data taken on each section is used to constrain the solution to minimize temporal aliasing, the inverse problem involves solving an underdetermined system of equations, since the elements of $\mathbf{v}_{ag}$ only include observations taken along $x = x_o$, whereas those of $\mathbf{p}$ include contributions from locations where $x = x_o$ and $x \neq x_o$. If the ageostrophic flow were approximately two dimensional, that is, invariant in the zonal direction, however, then $\mathbf{p}$ would be at most a weak function of $x$ and velocity data along a single section could provide enough information to make (23) a square or overdetermined system of equations. To perform the inverse calculation, we will therefore make the assumption that the ageostrophic flow is approximately two dimensional while performing tests to determine the validity of the two-dimensional assumption.

The governing equation for ageostrophic flow that has no zonal variation $(\partial / \partial x = 0)$ is

$$
\begin{pmatrix}
L_{11}^{2D} & L_{12}^{2D} \\
L_{21}^{2D} & L_{22}^{2D}
\end{pmatrix}
\begin{pmatrix}
\phi \\
\psi
\end{pmatrix} = I + II + III + IV + V,
$$

(24)

where terms I–V are given in (8) and the two-dimensional differential operators are

![Fig. 2. Evidence of frontal downdrafts in the tracer fields: (a) salinity and (b) chlorophyll fluorescence for section 5 of survey 2, reveal 5–10-km-wide plumes with high chlorophyll fluorescence and low salinity that extend $\sim 50$ m beneath the mixed layer on the dense side of the front. Isopycnals are contoured in intervals of 0.1 kg m$^{-3}$.](http://journals.ametsoc.org/jpo/article-pdf/40/1/3/4730842/2009jpo4018_1.pdf)
\[ L_{11}^{2D} = f_{1} \frac{\partial^{2}}{\partial x^{2}}; \quad L_{12}^{2D} = -a \frac{\partial^{2}}{\partial z^{2}} \]
\[ L_{21}^{2D} = -a \frac{\partial^{2}}{\partial z^{2}} - 2s_{1}^{2} \frac{\partial^{2}}{\partial y \partial z}; \]
\[ L_{22}^{2D} = f_{2} \frac{\partial^{2}}{\partial z^{2}} + 2s_{1}^{2} \frac{\partial^{2}}{\partial y \partial z} + N^{2} \frac{\partial^{2}}{\partial y^{2}}. \]

Equation (24) is similar to the modified semigeostrophic omega equation and along-front vorticity balance derived by Nagai et al. (2006) with the difference that in their equations \( F_{1} = f^{2} \) and \( S_{1}^{2} = 0 \). The contributions from \( S_{1}^{2} \) and \( \partial u / \partial x \) in \( F_{1} \) are retained in (24) for completeness, although solutions to (24) calculated with and without these contributions show only minor differences.

If we denote the finite difference representation of the operator on the lhs of (24) as \( L^{2D} \), then the two-dimensional methodology for the inverse calculation entails solving on each section the system of equations
\[ \mathbf{V}^{2D}_{G} \mathbf{p} + \mathbf{n} = \mathbf{v}_{\text{ag}}, \quad (25) \]
where \( \mathbf{V}^{2D}_{G} = \mathbf{DG}^{2D} \) and
\[ L^{2D} \mathbf{G}^{2D} = \mathbf{I}. \quad (26) \]

1) NUMERICAL METHOD

The matrix \( L^{2D} \) is derived using the objectively mapped hydrography, geostrophic flow field, and second-order accurate finite difference discretizations. For example, the term \( N^{2} \partial^{2} \psi / \partial y^{2} \) is discretized as
\[ 2N_{j,k} \left[ \frac{\psi_{j+1,k}^{(y+1)} - \psi_{j,k}^{(y+1)}}{(y_{j+1} - y_{j})^{(y+1)}} - \frac{\psi_{j,k}^{(y-1)}}{(y_{j} - y_{j-1})^{(y-1)}} + \frac{\psi_{j-1,k}^{(y-1)}}{(y_{j} - y_{j-1})^{(y-1)}} \right]. \]

where \( N_{j,k} \) is the square of the buoyancy frequency at each vertical and meridional grid along a particular section. Also contained in \( L^{2D} \) are the boundary conditions applied at the edges of the domain over which the Green’s function is calculated (referred to as the computational domain). The particular boundary condition used for the calculation follows Rudnick (1996): that is, zero vertical velocity on all the edges of the computational domain. In terms of the components of the vector streamfunction, the \( w = 0 \) boundary condition becomes \( \phi = \psi = 0 \) and \( \phi = \partial \psi / \partial y = 0 \) at the top/bottom and meridional boundaries of the computational domain, respectively.

The next step in the inverse calculation is to construct the Green’s function \( \mathbf{G}^{2D} \). This was accomplished by solving (26) column by column using a generalized minimum residual iterative method, an efficient algorithm for large sparse matrices such as \( \mathbf{L}^{2D} \) (Press et al. 1993, 83–86). Once the columns of the Green’s function were calculated, \( \mathbf{p} \) was obtained via the least squares fit, \( \mathbf{p} = (\mathbf{DG}^{2D})^{-1}(\mathbf{DG}^{2D})^{T} \mathbf{v}_{\text{ag}} \). From \( \mathbf{p} \), the vector streamfunction was calculated as \( \mathbf{q} = \mathbf{pG}^{2D} \), from which the vertical velocity at each grid on the section was estimated as
\[ w_{j,k} = \frac{\psi_{j+1,k} - \psi_{j,k}}{y_{j+1} - y_{j}}. \]

2) VALIDITY OF THE TWO-DIMENSIONAL SOLUTION

Inspection of (8) reveals that lateral variability in the ageostrophic flow field can be driven by horizontal vari-
The Burger number of submesoscale geostrophic flows is typically $O(1)$ (Thomas et al. 2008). Through geostrophic forcing these flows can drive ageostrophic motions with Burger numbers similar to those of the geostrophic currents that drive them. Therefore, the use of the two-dimensional version of (8) for such flows comes into question. To check the validity of the two-dimensional method for this case, solutions to the semigeostrophic omega equation were calculated. The vertical velocity obtained by solving the three- and two-dimensional versions of the omega equations, that is, (16) and $L^{2D}q = p$, with $p$ equal to the geostrophic forcing calculated from the objectively mapped buoyancy and geostrophic velocity fields were compared. Scatterplots of the two- and three-dimensional solutions ($w_{2D}^3$ and $w_{3D}^3$, respectively) are shown in Fig. 3 for each section.

To quantify how representative the two-dimensional solution is of $w_{3D}^3$, a linear regression between the two solutions was performed. That is, $w_{3D}^3$ was fit to $w_{2D}^3$ following the form

$$w_{3D}^3 = \sigma_w a + bw_{2D}^3,$$

(27)

where $a$ and $b$ are nondimensional coefficients that measure the offset and scaling between the two solutions and $\sigma_w$ is the standard deviation of $w_{2D}^3$ on a given section. If $w_{3D}^3$ is well represented by $w_{2D}^3$, then $b \approx 1$ and $|a| < 1$. As can be seen in Fig. 3, there are several sections where $b = 1 \pm 0.1$ and $|a| < 0.25$ (specifically sections 1–4 of survey 2; 1, 5, and 6 of survey 3; and 2 and 4 of survey 4). Based on this metric, we judge that, on these sections, the two-dimensional solutions to (24) are a good approximation to the fully three-dimensional solutions to (8) at the subpolar front for ageostrophic flows driven by geostrophic forcing.

5. **Vertical circulation and forcing terms of the ageostrophic flow at the subpolar front of the JES**

   **a. Inferred vertical velocity**

Cross sections of the vertical velocity inferred using the inverse method for all sections occupied during three of the surveys of the subpolar front of the JES are plotted in Fig. 4. As evident in the figure, submesoscale laterally banded structures in $w$ of $O(5–10 \text{ km})$ in width are prominent. As described in the appendix, analyses aimed at assessing random and systematic errors in the inverse method arising from noise in the data, aliasing, and boundary conditions indicate that features in the vertical velocity field are robust for $z > -150 \text{ m}$. For depths greater than $150 \text{ m}$, the arbitrary $w = 0$ bottom boundary condition can influence the solution and lead to significant errors. The magnitude of $w$ is large, with amplitudes of $O[1–2 \text{ mm s}^{-1} (\sim 100–200 \text{ m day}^{-1})]$. Comparing the inverse solution to the solution to the omega equation, $w_{3D}^3$, shown in Fig. 5, it is clear that the amplitude and structure of the inferred vertical velocity is very different from what one obtains using the omega equation; namely, the magnitude of $w_{3D}^3$ is nearly an order of magnitude weaker than $w$ and the patterns of upwelling and downwelling in $w_{3D}^3$ rarely resemble those of $w$. This result qualitatively indicates that the vertical circulation at the front is not primarily driven by geostrophic forcing. A more thorough and quantitative analysis assessing what physical processes drive the ageostrophic flow at the front will be presented in section 5c.

   **b. Slantwise nature of ageostrophic flow**

Although it is not obvious from the sections of vertical velocity in Fig. 4, frontal downdrafts and updrafts are often correlated with meridional ageostrophic flow, indicating that the frontal circulations are slantwise. A clear example of this slantwise circulation is shown in Fig. 6. The ageostrophic flow of the strong frontal updraft observed within the core of the front on this section tends to be oriented parallel to isopycnals, whereas flow in the downdraft tends to be parallel to surfaces of constant geostrophic absolute momentum,

$$M_g = u_{g} - fy,$$

(28)

the latter of which are tilted owing to the strong vertical shear of the frontal jet. In the absence of buoyancy mixing, fluid parcels conserve buoyancy and travel along isopycnals if the flow is steady. Similarly, without friction or zonal pressure gradients, fluid parcels conserve absolute momentum and flow parallel to surfaces of constant $M_g$ (again for steady conditions). Therefore, the departure of the ageostrophic velocity vector from being parallel to isopycnals or $M_g$ surfaces indicates, for steady flows, the presence of friction, diabatic processes, or zonal pressure gradients; for time-dependent motions, it indicates the existence of inertia–gravity waves or symmetric instability (Hoskins 1974).

   **c. Driving forces for the ageostrophic flow**

Apart from the vertical velocity, the inverse method yields, in the $2MN$ element vector $p$, estimates for the rhs forcing terms of (8). The first (last) $MN$ elements of $p$ correspond to the sum of all the rhs forcing terms in the upper (lower) equation of (8) and is referred to as $\text{RHS}_1$ ($\text{RHS}_2$). We would like to know what physical process—geostrophic forcing, friction, mixing of buoyancy, time-dependence, and/or nonlinearities in the ageostrophic
FIG. 3. Scatterplots of the vertical velocity for the three-dimensional $w_3D$ and two-dimensional $w_{2D}$ solutions to the semigeostrophic omega equation for each of the sections. A line of slope one and the coefficients of the linear regression between $w_3D$ and $w_{2D}$, $a$ and $b$, see (27), are included in each panel.
flow [i.e., the terms I–V in (8)]—is associated with (RHS\textsubscript{1}, RHS\textsubscript{2}). As neither microstructure measurements nor repeated observations of the ageostrophic flow capable of sufficiently sampling inertial/superinertial motions were made during the cruise, terms II–V cannot be estimated directly. The geostrophic forcing (9) can be calculated from the observations; therefore, it is the only forcing term that we can state with certainty does or does not resemble (RHS\textsubscript{1}, RHS\textsubscript{2}) in structure and magnitude.

1) GEOSTROPHIC FORCING

The degree to which the spatial structure of the rhs of (8) resembles the geostrophic forcing is quantified by calculating the correlations between RHS\textsubscript{1} and $F_{g1}^x$, $r$(RHS\textsubscript{1}, $F_{g1}^x$) and between RHS\textsubscript{2} and $F_{g2}^x$, $r$(RHS\textsubscript{2}, $F_{g2}^x$). To assess whether the magnitude of the geostrophic forcing is strong enough to drive the inferred ageostrophic flow, the ratios of the standard deviation of $F_{g1}^x$ and RHS\textsubscript{1}, $\sigma(F_{g1}^x)/\sigma($RHS\textsubscript{1}$)$, as well as $F_{g2}^x$ and RHS\textsubscript{2}, $\sigma(F_{g2}^x)/\sigma($RHS\textsubscript{2}$)$, were calculated. The results for the correlations and standard deviation ratios for each section are listed in Table 1. For the majority of the sections, the correlation between the RHS and the geostrophic forcing is not significantly different than zero or is negative (the physical model for secondary circulations driven by geostrophic forcing requires that the correlation coefficient be positive). For those sections where the correlation is significant and positive, the standard deviation ratios have values ranging from 0.05 to 0.21, indicating that the magnitude of the geostrophic forcing is about an order of magnitude smaller than the inferred forcing. Since the geostrophic forcing is the driving force for the omega equation, these
results reaffirm the conclusion made in section 5a that the inferred vertical circulation at the front is not well represented by solutions to the omega equation.

2) FRICTION AND MIXING OF BUOYANCY

Although no direct measurements of turbulent mixing of momentum and buoyancy were made, the meteorological observations of wind stress \((\tau_w^x, \tau_w^y)\) taken from the ship can be used to derive scalings for terms II and III of (8), assuming that these terms are primarily associated with wind-driven frictional forces and mixing of buoyancy caused by the Ekman advection of dense water over light.

A wind-driven frictional force goes as \((X, Y) \sim (\tau_w^x, \tau_w^y)/\rho_o\delta_e\), presuming that the stress is distributed over the thickness of the Ekman layer \(\delta_e\). Consequently, an appropriate scaling for term II of (8) is

\[
\left[ f \frac{\partial Y}{\partial z} \right] = \frac{f|\tau_w^y|}{\rho_o\delta_e} \tag{29}
\]

and

\[
\left[ f \frac{\partial X}{\partial z} \right] = \frac{f|\tau_w^x|}{\rho_o\delta_e^2}, \tag{30}
\]

where

\[
\delta_e = \frac{0.4}{f} \left( \frac{\sqrt{\left(\tau_w^x\right)^2 + \left(\tau_w^y\right)^2}}{\rho_o} \right)^{1/2}
\]
is the turbulent Ekman layer depth (Wimbush and Munk 1970). The scalings (29) and (30) were calculated using the time-mean wind stress for each section and then normalized by the standard deviation of RHS$_1$ and RHS$_2$, respectively. The results are tabulated in Table 2 and range from 0.05 to 0.35. These values are typically larger than the normalized geostrophic forcing listed in Table 1, suggesting that wind-driven frictional forces contribute a greater amount to the generation of ageostrophic motions at the front than frontogenetic/frontolytic shear and strain in the geostrophic flow.

For all sections the wind stress had a component toward the east, which would drive a southward Ekman transport. The combination of southward Ekman flow and a negative meridional buoyancy gradient yields the right conditions for Ekman-driven convection, that is, gravitational instability triggered by Ekman advection of denser water over light. The nonhydrostatic simulations of Thomas and Lee (2005) demonstrate how the strength of the buoyancy flux of Ekman-driven convection $F^B_c$ is scaled by a wind-driven buoyancy flux

$$F^B_c \sim F^B_{\text{wind}} = M_e \cdot V_h \cdot b,$$

where $M_e$ is the Ekman transport. Divergence of the convective buoyancy flux induces turbulent mixing of buoyancy, $Q_c = -\frac{\partial F^B_c}{\partial z}$, the lateral variations of which drive ASCs via term III of (8). At the subpolar front the wind-driven buoyancy flux can take on values nearly an order of magnitude larger than the buoyancy loss to the atmosphere by air–sea fluxes (Thomas and Lee 2005). In addition, unlike the buoyancy loss due to air–sea fluxes, the wind-driven buoyancy flux is focused at the front where the lateral buoyancy gradient is largest and hence creates spatial variations in mixing of buoyancy, which are critical to driving ASCs through term III. Consequently, an upper bound on term III can be estimated using the wind-driven buoyancy flux and is as follows:

$$\frac{\partial Q}{\partial y} = \frac{\tau^z}{\rho_o f H_c L_f} \left| \frac{\partial b}{\partial y} \right|_{\text{max}},$$

where $H_c = 75$ m is the nominal depth of the mixed layer and hence an estimate for the vertical extent of the

Fig. 6. Example of the slantwise nature of the ageostrophic flow near the front: the ageostrophic flow $(u_{ag}, w)$ (vectors with dots indicating their base), isopycnal surfaces (gray solid contours), and surfaces of constant geostrophic absolute momentum $M_g$ (gray dashed contours), for section 2 of survey 2.
Ekman-driven convection, $L_r = 5$ km is a scaling for the width of the front and the lateral distance over which mixing of buoyancy occurs. $|\partial \theta / \partial z|_{\text{max}}$ is the magnitude of the maximum buoyancy gradient at the center of the front at $z = -8$ m, and the wind stress is time averaged over the cross-front section as in (29) and (30). As can be seen in Table 2, (32) normalized by the standard deviation RHS$_2$ ranges from 0.01 to 0.18 and is generally smaller than both the geostrophic forcing and frictional terms, implying that mixing of buoyancy plays a relatively minor role in driving ASCs at the front.

### 3) TIME DEPENDENCE

In the previous two sections, it was shown that the estimates and scalings for the contributions from geostrophic forcing, friction, and mixing of buoyancy to the generation of ageostrophic motions are all a fraction of the magnitude of the inferred rhs forcing term of (8). This suggests that the remaining terms on the rhs of (8), namely time variability in the ageostrophic shear and/or nonlinear advection by the ageostrophic flow, contribute significantly to the dynamics of the frontal ASC.

The rate of change of the thermal wind imbalance following the geostrophic flow can drive ageostrophic flows through term IV. To fully evaluate this term, information is required on the local rate change and spatial variation in the direction of the geostrophic flow of the ageostrophic vertical shear. Owing to the infrequent sampling of the ageostrophic flow during the cruise, such information is not available. However, there are indications from the structure of the rhs forcing terms and the ageostrophic velocity field that time dependence is a critical driving term of (8) at the front.

For many of the sections, the rhs forcing term is characterized by a vertically banded structure oriented along isopycnals that is intensified in regions of strong stratification. This structure is exemplified in Figs. 7a,b for survey section 2 of survey 2. Banding can be seen in both the RHS$_1$ and RHS$_2$ fields. For example, near $y = -10$ km and within the pycnocline ($-200 < z < -75$), there is an alternation in the sign of RHS$_1$ with depth. Also in this region, hodographs of the ageostrophic velocity show a clockwise turning of the velocity vector with depth. The degree to which the velocity vector turns clockwise (CW) or counterclockwise (CCW) with depth can be quantified by calculating rotary spectra of the ageostrophic velocity profiles. The total energy in the CW and CCW components of the flow can be calculated as

$$E_{\text{CW}} = \int_0^{m_{\text{max}}} \Phi_{\text{CW}}(m) \, dm$$  \hfill (33)$$

and

$$E_{\text{CCW}} = \int_0^{m_{\text{max}}} \Phi_{\text{CCW}}(m) \, dm,$$  \hfill (34)

where $m$ is the vertical wavenumber, $\Phi_{\text{CW}}(m)$ and $\Phi_{\text{CCW}}(m)$ are the CW and CCW spectra, and $m_{\text{max}}$ is the

---

**Table 1.** Comparison between the rhs forcing term of (8) and the geostrophic forcing, where $r(RHS_2, F_2)$ and $r(RHS_1, F_1)$ denote the correlations between RHS$_2$ and $F_2$ and between RHS$_1$ and $F_1$, respectively. Also tabulated are the ratios of the standard deviations of $F_2$ to RHS$_2$ and $F_1$ to RHS$_1$, denoted by $\sigma(F_2) / \sigma(RHS_2)$ and $\sigma(F_1) / \sigma(RHS_1)$, respectively.

<table>
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<th>$r(RHS_2, F_2)$</th>
<th>$\sigma(F_2) / \sigma(RHS_2)$</th>
<th>$r(RHS_1, F_1)$</th>
<th>$\sigma(F_1) / \sigma(RHS_1)$</th>
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<td>0.01*</td>
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<td>0.21*</td>
<td>0.21</td>
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<td>0.20*</td>
<td>0.12</td>
</tr>
<tr>
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<td>0.10</td>
<td>0.14*</td>
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</table>

* Significant correlation value.
Nyquist wavenumber. The cross-front structures of $E_{CW}$ and $E_{CCW}$ reveal that there is more energy in the CW component of the flow to the south of the front, coincident with the banded structure in RHS (Fig. 7d).

Two physical phenomena may explain the CW spiraling of ageostrophic flow: Ekman shear or downward propagation of an internal wave (Leaman and Sanford 1975). If friction were the cause of the spiraling, then the vertical length scale associated with the spiral (∼50 m) would scale with the Ekman depth, requiring a vertical viscosity of around 0.1 m² s⁻¹. A vertical viscosity of this magnitude in the pycnocline (where the spiraling is enhanced) is implausible. Therefore, it is more likely that the turning of the velocity vector is a consequence of downward propagating internal waves rather than frictional effects. If this spiraling was associated with an internal wave, then the wave’s frequency should follow the internal gravity wave (hydrostatic) dispersion relation $\omega = \sqrt{f^2 + N^2(\lambda_z/\lambda_h)^2}$. Given the pycnocline stratification, $N \sim 80f$ and an estimate for the vertical/horizontal wavelength taken from the structure of the rhs forcing ($\lambda_z \sim 50$ m, $\lambda_h \sim 15$ km), the dispersion relation yields a wave frequency that is slightly superinertial, $\omega \sim 1.04f$. It has been observed that mixed layer energy generated by time-variable wind forcing can be transferred into the interior of the ocean by downward propagating, slightly superinertial gravity waves (D’Asaro et al. 1995). Given the fact that the subpolar front is forced by strong and variable winds, it is likely that we are observing a similar phenomenon, although on a smaller horizontal scale (∼15 km versus 50–100 km).

Assuming that the turning of the ageostrophic velocity vector is due to internal waves, then the cross-front contrast in the relative strengths of CW versus CCW energy shown in Fig. 7d can be interpreted as evidence of an asymmetry in the amount of energy propagating up versus down on either side of the front. This asymmetry...
was not an isolated occurrence specific to this particular frontal crossing. Averaging $E_{CW}$ and $E_{CCW}$ over all the sections for meridional locations south of the front yields 0.0032 and 0.0022 m$^2$ s$^{-2}$ in the clockwise and counterclockwise components of the flow, respectively. On the north side of the front, there is on average 0.0031 and 0.0029 m$^2$ s$^{-2}$ in the clockwise and counterclockwise components of the flow, suggesting that there is less energy propagating upward south of the front. The region of enhanced CW versus CCW energy is coincident with an area of anticyclonic vorticity (e.g., Fig. 7d). The increase in the energy of downwelling propagating waves in regions of negative vorticity was also observed at the subpolar front of the JES during a summertime cruise (Shcherbina et al. 2003), indicating that time-dependent ageostrophic motions influenced by the frontal vorticity field may be a common feature to the front during all seasons. In addition, eddy-resolving numerical simulations reveal an enhancement of downwelling propagating near-inertial energy in regions of anticyclonic vorticity, consistent with these findings (Lee and Niiler 1998; Zhai et al. 2007).

4) **HIGHER-ORDER EFFECTS**

For strong ageostrophic flows, higher-order effects can contribute to the dynamics of ASCs. In the framework of Eq. (8), these effects are contained within term $\mathbf{V}$ [i.e., (11)] and involve advection of the ageostrophic momentum by the ageostrophic flow. Although term $\mathbf{V}$ cannot be accurately evaluated from the observations because of errors arising from temporal variability, the Rossby number $\text{Ro}_{ag} = U_{ag}/fL$, a measure of this term’s relative importance as a driving mechanism, can be estimated. The Rossby number was calculated using the standard deviation of the ageostrophic velocity on a given section, that is, $U_{ag} = \sqrt{\sigma_{uag}^2 + \sigma_{vag}^2}$ and assuming that the length scale of the ageostrophic flow was the nominal width of the front, $L = 5$ km. For all of the sections, $\text{Ro}_{ag} = O(0.1)$, suggesting that higher-order effects play a relatively minor, but not insignificant, role in the dynamics of the ageostrophic flow at the subpolar front.

6. Discussion

a. **Inertial pumping due to nonlinear Ekman effects**

As described in section 3, there are indications that the vertical velocities inferred at the front are associated with near-inertial motions. In this section, we show how time-dependent Ekman flow modified by the vertical vorticity of the front can induce inertial oscillations and inertial pumping, which could be a significant source of vertical motions at the front.

For large Rossby number flows, the steady Ekman transport is modified by the vorticity of the fluid and Ekman pumping can result, even for a spatially uniform wind field (Stern 1965; Niiler 1969; Thomas and Lee 2005). If the wind forcing is allowed to vary in time, the Ekman pumping/suction that results oscillates at frequencies close to the inertial period. This is best illustrated by solving the vertically integrated momentum equations in the Ekman layer for a wind-forced two-dimensional (invariant in the x direction) geostrophic zonal jet with vertical vorticity $\xi_g = -\partial u_g/\partial y$,

$$\frac{\partial M^x_E}{\partial t} - (f + \xi_g)M^y_E = \frac{\tau^x_v}{\rho_o} \tag{35}$$

$$\frac{\partial M^y_E}{\partial t} + fM^x_E = \frac{\tau^y_v}{\rho_o},$$

where $(M^x_E, M^y_E)$ and $(\tau^x_v, \tau^y_v)$ are the components of the Ekman transport and wind stress, respectively. In deriving (35), it has been assumed that the vertical variation of $u_g$ through the Ekman layer is minimal and that the Ekman flow is weaker than the geostrophic flow so that terms involving advection of the Ekman flow by the Ekman flow can be neglected. The solution for $M^x_E$, assuming that at some time $t = t_0, M^x_E = 0$, is

$$M^x_E = \int_{t_0}^t \tau^x_v(s) F_1(t - s) \, ds + \int_{t_0}^t \tau^y_v(s) F_2(t - s) \, ds \tag{36}$$

and involves the convolution of the wind stress and functions

$$F_1 = \frac{1}{\sqrt{1 + \xi_g/f}} \sin f_{eff}t \tag{37}$$

and

$$F_2 = \cos f_{eff}t \tag{38}$$

that oscillate at the effective Coriolis frequency $f_{eff} = \sqrt{f(f + \xi_g)}$. For this two-dimensional flow, Ekman pumping/suction $w_c = \partial M^y_E/\partial y$ can be generated from spatial variations in the vertical vorticity that modify both amplitude and phase of $M^y_E$.

To determine if time-variable nonlinear Ekman transport can drive vertical motions of similar magnitude to those inferred from the inverse method, $M^x_E$ and $w_c$ were calculated using observed winds and the vertical vorticity distribution near the front. An example solution derived using the surface vorticity of the geostrophic flow for section 2 of survey 2 is shown in Fig. 8.
The Ekman transport has been set to zero at 0000 UTC 23 January. This instance corresponds to a period of weak winds that preceded the onset of a strong cold-air outbreak with wind stress magnitudes exceeding 0.4 N m$^{-2}$. If the Ekman transport at 0000 UTC 23 January were known, it could have been added as an initial condition. However, as this information was unavailable, it was not included in the solution. Consequently, the solution shown in Fig. 8 should be interpreted as the portion of the Ekman flow driven by the cold-air outbreak that took place after 23 January, not the total wind-driven flow, which could include currents set in motion from wind events that took place prior to 23 January. As evident in the figure, the Ekman transport and pumping tend to oscillate near the effective inertial frequency. Because of the high Rossby number of the frontal jet, the period of the oscillations differs significantly from an inertial period. In particular, in the anticyclonic portion of the frontal jet ($-16 \text{ km} < y < -4 \text{ km}$) the effective inertial period, $2\pi f_{\text{eff}}$, exceeds a day. In this region the Ekman transport and pumping is amplified as well, yielding meridional Ekman flows (estimated as $M_{\psi}^e \delta_e$, where $\delta_e$ is the time-mean thickness of the Ekman layer) of $O(5-15 \text{ cm s}^{-1})$ and Ekman pumping/suction with a magnitude of 1–3 mm s$^{-1}$ for the time period when the observations on section 2 of survey 2 were collected (indicated by the shaded gray bar in Fig. 8c). It is precisely this region where the prominent overturning cell with the clockwise spiraling ageostrophic flow was observed (e.g., Figs. 6 and 7). Within the overturning cell the observed amplitude of the horizontal ageostrophic flow was 10–15 cm s$^{-1}$ and the inferred vertical velocities reached 2 mm s$^{-1}$. These values are wholly consistent with the magnitude of the horizontal and vertical velocities derived from (36). Based on this qualitative comparison and that geostrophic forcing and mixing of
buoyancy cannot account for the inferred vertical velocity, we conclude that time-dependent nonlinear Ekman dynamics contributes significantly to the generation of strong vertical motions at the subpolar front.

b. Vertical advection of tracers

The vertical transport of tracers by submesoscale secondary circulations is of considerable interest to questions involving biogeochemical processes in the upper ocean as well as the subduction of water masses recently modified by atmospheric forcing. At the subpolar front the spatial structure of salinity (which can be considered a passive tracer at the front) and chlorophyll fluorescence (a proxy for phytoplankton biomass) presented evidence of such vertical transport. An example of this can be seen along section 2 of survey 2, Fig. 9. On this section, a strong slantwise frontal downdraft is aligned with a streamer of water with low salinity as well as with a plume of high chlorophyll fluorescence that extends down from the surface. The correlation of the ageostrophic flow, salinity, and chlorophyll fluorescence suggests a causal relation between the inferred vertical velocity and the apparent vertical displacements in the tracer fields. These observations attest to the potential impact of frontal scale vertical motions in the generation of submesoscale bio–optical features and thermohaline intrusions (such as the ones seen at $y = 22$ km and $y = -17$ km in Figs. 2 and 9, respectively), both of which are common features of the subpolar front of the Japan/East Sea (e.g., the majority of the sections of the SeaSoar survey contained such features) and ocean fronts in general (e.g., Barth et al. 2001; Fedorov 1983). The vertical extent of the plumes of water with low salinity and high chlorophyll fluorescence suggest that fluid parcels near the front experience vertical displacements of $O(50\text{ m})$. The issue of whether such displacements are consistent with the vertical velocities inferred from the inverse method is discussed below.

In the previous section, it was argued that the frontal vertical circulation is primarily associated with near-inertial motions driven by time-dependent nonlinear Ekman dynamics. An oscillatory vertical velocity of magnitude $w_\omega$ oscillating at a frequency $f_{\text{eff}}$ will induce a maximum vertical displacement of $2w_\omega/f_{\text{eff}}$ in one cycle. The strongest frontal downdrafts are found on the dense side of the front where $f_{\text{eff}} = (1 - 1.5)f$ and $w_\omega \sim 2 \text{ mm s}^{-1}$. Given these values, the maximum vertical excursion experienced by fluid parcels should not be much greater than 40 m, a distance similar to the vertical extent of the thermohaline intrusions and bio-optical features observed at the front. This suggests that the large magnitude of the inferred vertical velocities and the interpretation that the frontal vertical circulation is associated with near-inertial motions are both plausible.

The inferred vertical velocity could conceivably be used to estimate vertical advective tracer fluxes by correlating tracer and vertical velocity fields from each section. However, given the apparent time variability in the ageostrophic flow, it is likely that such a calculation would be prone to errors. This is because, although the time-dependent vertical motions lead to large vertical displacements in the tracer fields, these displacements...
are purely reversible to leading order and hence do not lead to a net vertical tracer flux or net subduction. Therefore, without enough cross-front sections sampled at various phases of the oscillatory flow, the reversible portion of the vertical tracer advection may not be adequately averaged out, resulting in errors in the flux calculation. It may also be the case that the oscillatory motions are of sufficient amplitude that they result in a net displacement of fluid parcels via a Stokes drift, the quantification of which will be the topic of future research. However, to estimate from observations this Stokes drift and the net tracer flux that it induces, one would again need a sufficient number of temporal realizations of the flow to minimize the errors in the calculation. Given these potential pitfalls and the infrequent sampling of the cross-front sections, a calculation to quantify the flux of the various tracers observed at the front was not attempted.

7. Conclusions

An inverse method was derived for diagnosing vertical velocities from quasi-synoptic, high-resolution surveys of upper-ocean mesoscale/submesoscale flows. The method consists of calculating the Green’s function to an equation governing the vector streamfunction of ASCs and fitting the horizontal component of the velocity of the Green’s function to observationally derived estimates of the ageostrophic flow. In doing so, the method both yields an estimate for the vertical velocity and assesses the driving force for the vertical circulation. The equation that forms the theoretical basis of the method is quite general, with solutions that can describe ageostrophic motions such as nonlinear Ekman flow, mixing-driven ASCs, solutions to the omega equation, and inertia–gravity waves in a geostrophic flow. Owing to its semigeostrophic nature, the equation accounts for higher-order corrections to the quasigeostrophic omega equation, similar to the equations that form the basis of the vertical velocity diagnostics of Shearman et al. (2000), Naveira Garabato et al. (2001), and Pallàs Sanz and Viúdez (2005). Unlike these diagnostics, however, the inverse method is designed to be able to infer vertical velocities driven by mixing, friction, time-dependent inertial/superinertial motions, or nonlinearities in the ageostrophic flow, making it advantageous for application to datasets of upper-ocean flows forced strongly by the atmosphere, such as the high-resolution surveys of the subpolar front of the Japan/East Sea taken during the winter of 2000 in periods of cold-air outbreaks.

At the subpolar front, the inferred vertical velocity was characterized by submesoscale (5–10 km) features with magnitudes of 1–2 mm s⁻¹ (100–200 m day⁻¹). The vertical circulation took on a slantwise nature with strong frontal downdrafts that tended to parallel the sloped isopycnal surfaces of the front. Downdrafts were located on the dense side of the front and were aligned with plumes of water with anomalously low salinity and elevated levels of chlorophyll fluorescence, suggesting that the frontal vertical circulation contributed to the formation of the submesoscale bio-optical features and thermohaline intrusions observed at the front. The strength of the vertical velocity inferred from the inverse method was nearly an order of magnitude larger than that predicted by the omega equation, a consequence of the weakness of the geostrophic forcing relative to the total forcing of the ASC. As evidenced by the clockwise spiraling of the ageostrophic velocity vector with depth, near-inertial waves of relatively small horizontal scale (~15 km) may have contributed to the lack of applicability of the omega-equation solution. A simple model for time-dependent nonlinear Ekman transport indicates that, given the high Rossby number of the frontal jet and the strength and variability of the wind forcing, nonlinear Ekman transport and inertial pumping are capable of generating inertial oscillations and vertical motions of the amplitude observed and inferred at the front. In addition, scalings for frictional forces and turbulent mixing of buoyancy by Ekman-driven convection, based on shipboard wind measurements, suggest that wind-driven friction is important in the dynamics of the ASC at the subpolar front, whereas mixing of buoyancy is not.

Intensification of the frontal vertical circulation by atmospheric forcing has also been observed in numerical simulations of an idealized representation of the subpolar front forced by winds and/or cooling (Y. Yoshikawa et al. 2008, unpublished manuscript). Comparing experiments with and without surface forcing, Y. Yoshikawa et al. (2008, unpublished manuscript) found that cooling and winds lead to the formation of submesoscale frontal downdrafts, 2–3 km in width, having vertical velocities $O(1 \text{ mm s}^{-1})$ that were nearly an order of magnitude stronger than the $O(20 \text{ km})$ wide downdrafts present in the unforced run. Using the numerical solutions, Y. Yoshikawa et al. were able to diagnose the various rhs forcing terms of (8). In the wind-forced experiments,
geostrophic forcing was weaker than both friction and the time-dependent term IV, the latter of which took on a vertically banded structure. The enhancement of the vertical velocity in the wind-forced experiments was partially attributable to the presence of near-inertial motions. The similarities between the solutions of the wind-forced experiments of Y. Yoshikawa et al. and the observed ageostrophic flow at the subpolar front of the JES and its inferred forcing and vertical velocity [viz., the secondary role of geostrophic forcing in driving the frontal ASC, the vertically banded structure of the rhs of (8), and the importance of time-dependent motions] bolster our conclusion that near-inertial motions driven by the winds and modified by the vertical vorticity of the frontal jet appear to be the primary cause of the strong vertical motions at the front. The wind-forced simulations of Y. Yoshikawa et al. (2008, unpublished manuscript) show that, while the instantaneous frontal vertical circulation is strongly influenced by time-dependent processes, geostrophic forcing plays an important role in driving net subduction and the vertical transport of tracers. The vertical circulation in simulations with weak atmospheric forcing is dominated by meander-driven subduction, suggesting that during seasons when winds are weak over the JES the vertical circulation at the subpolar front is primarily ascribable to geostrophic forcing.

A multitude of high-resolution hydrographic and velocity observations as well as state-of-the-art numerical simulations reveal that submesoscale high Rossby number currents are ubiquitous to the upper ocean, especially in frontal zones (Thomas et al. 2008). When forced by winds, such submesoscale features will likely induce time-dependent nonlinear Ekman flows with corresponding inertial oscillations, Ekman pumping, and inertial pumping. The small horizontal spatial scales of the inertial motions generated in this way could trigger downward propagating near-inertial waves that could efficiently transfer inertial energy from the mixed layer to the ocean interior. Indeed, at the subpolar front of the JES there was evidence for enhanced downward-propagating near-inertial energy associated with the submesoscale structure of the vorticity of the frontal jet. It is possible that the interaction of wind-forced inertial motions with the submesoscale surface vorticity field could affect the global budget of inertial energy in the mixed layer and ocean interior, with implications for interior diapycnal mixing—this is a topic that should be explored in the future.

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APPENDIX

Errors in the Inverse Solution

Sources of error in the inverse solution can be both systematic and random. In this appendix, an estimate for the magnitude of random errors will be calculated by probing the sensitivity of the inverse solution to noise in the observed fields. In addition, a discussion of the possible sources of systematic errors will be described.

a. Sensitivity of inverse solution to noise

Following Rudnick (1996) and Naveira Garabato et al. (2001), the sensitivity of the inferred vertical velocity to noise was investigated by adding noise to the density and velocity fields and repeating the inverse calculation. The noise had amplitude 5% of the standard deviation of $b$, $u_{ADCP}$, and $v_{ADCP}$. The horizontal structure of the noise was prescribed such that the two-dimensional correlation function of the noise field was isotropic, Gaussian, and had a correlation scale of 10 km. For $\zeta > -125$ m the noise was uniform in the vertical; beneath this level the noise was set to zero. Noise with this spatial structure was meant to represent internal gravity waves that could not be resolved by the ~20 km zonal spacing of the survey.

The sensitivity test was performed on the section at $x = -2$ km of survey 4 using 100 realizations of noise and averaging the results. The spatial structure and sense of the vertical velocity and forcing terms $p$ were nearly identical to $w$ and $p$ calculated using the uncorrupted data. The magnitude of the error was at worst 3% for the vertical velocity and 2% for $p$, indicating that the features of both fields are robust.

b. Systematic errors

1) ERRORS IN THE ESTIMATION OF AGEOSTROPHIC FLOW

The inverse method relies on the estimate of the horizontal ageostrophic velocity (12) to infer the vertical
circulation; therefore, errors in the magnitude and spatial structure of (12) can propagate into the inverse solution. Systematic errors in (12) can arise owing to aliasing of high-frequency variability to lower frequencies in the sampling of velocity and density. This is not so much of an issue for quantities that are derived from along-track, meridional variability, such as \( u_g \), \( u_{\text{ADCP}} \), and \( v_{\text{ADCP}} \). Because, as described in section 4, the data collected in the along-track direction is sampled quickly enough to resolve near-inertial variability. However, zonal density gradients and the meridional geostrophic velocity could be prone to aliasing errors since they are calculated from data taken on separate sections. Hence, it is likely that aliasing errors would be most manifest in the estimate of the meridional component of the ageostrophic flow. Assuming that the ageostrophic flow is nearly isotropic\(^{A1}\) (i.e., \( |u_{\text{ag}}| \sim |v_{\text{ag}}| \)) and that the estimate for \( u_{\text{ag}} \) is accurate,\(^{A2}\) then errors in \( v_{\text{ag}} \) can be assessed by comparing the relative magnitudes of the zonal and meridional components of the ageostrophic flow. In Table A1, the standard deviations of \( u_{\text{ag}} \) and \( v_{\text{ag}} \), \( \sigma(u_{\text{ag}}) \) and \( \sigma(v_{\text{ag}}) \), are tabulated for each section as a metric for the magnitudes of the horizontal components of the ageostrophic flow. The ratio \( \sigma(v_{\text{ag}})/\sigma(u_{\text{ag}}) \) takes values between 0.5 and 1.5 for the majority of the sections suggesting that errors in the estimate of \( |v_{\text{ag}}| \) are no greater than 50% on these sections.

Comparison of the spatial structure of \( u_{\text{ADCP}} \) and \( v_{\text{ag}} \) on these sections reveals that the structure of \( v_{\text{ag}} \) is dominantly determined by \( u_{\text{ADCP}} \), not \( u_g \). The spatial structure of \( u_{\text{ADCP}} \) on each section is unlikely to be affected by aliasing since the velocity data used in the inverse calculation was collected within \( \sim 0.14 \) of an inertial period (see section 4). We therefore surmise that the spatial structure of \( v_{\text{ag}} \) is accurately estimated and conclude that errors in the inferred vertical velocity owing to errors in \( v_{\text{ag}} \) are less than 50% for most of the sections.

\(^{A1}\) The observations suggest that the ageostrophic flow is primarily attributable to near-inertial motions that are isotropic (see sections 3 and 6).

\(^{A2}\) The accuracy of \( u_{\text{ag}} \) is limited by errors in the estimate for \( u_g \). A measure of the accuracy of \( u_g \) is the closeness to which the thermal wind relation models the observed vertical shear in the zonal flow. Following Rudnick and Luyten (1996), this measure was quantified by calculating the regression coefficient \( A = -(\partial \theta/\partial x)(\partial \theta/\partial y)/(\partial \theta/\partial y)^2 \) (the brackets denote an average over volume for a given survey, excluding grid points where the ratio of error to signal variance exceeded 0.3), which would be equal to one if the zonal flow satisfied the thermal wind relation exactly. The regression coefficients for surveys 2, 3, and 4 are 0.93, 0.79, and 0.74, respectively. These relatively high values of \( A \) suggest that the zonal geostrophic flow was captured accurately in the surveys.

The spatial structure of the ageostrophic circulation and, hence, the inferred vertical circulation can be affected by the along-track resolution of the observations. As stated in section 4, the along-track separation between hydrographic and velocity measurements was \( \sim 3 \) and \( \sim 0.75 \) km, respectively. To test the sensitivity of the inverse solution to the resolution of the observations, the measurements were subsampled and the inverse calculation repeated. Reducing the resolution of the velocity measurements to 3 km resulted in only a slight (less than 10%) modification of the inferred vertical velocity. However, decreasing the resolution of both hydrographic and velocity observations to 7.5 km was found to reduce the magnitude of \( w \) by \( \sim 50\% \). Clearly the best observational strategy for the inverse method is to collect measurements at as high a spatial resolution as possible while sampling rapidly so as to avoid aliasing of inertial motions.

2) BOUNDARY CONDITIONS

Setting the vertical velocity at the bottom and side boundaries to a constant value of zero is an arbitrary choice that could lead to errors in the solution. If we consider that the vertical velocity obtained from the inverse calculation \( w \) is equal to the actual vertical velocity plus an error \( w = w_{\text{act}} + w_{\text{err}} \), then at the boundary \( w = 0 \) the inverse solution will have errors as large as \( w_{\text{act}} \). The errors attributed to the \( w = 0 \) boundary condition are not confined to the boundary but are felt over a finite distance into the interior of the domain. To assess how large this area of error is, (8) was solved with the

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rhs forcing terms set to zero and a nonzero vertical velocity meant to represent $w_{err}$ specified at the lateral and bottom boundaries separately. As demonstrated in Fig. A1, the area of error depends on the spatial structure of the vertical velocity specified on the boundary. When the vertical velocity on the lateral boundaries extends deeper into the fluid, the area of error reaches farther into the interior of the domain (cf. Figs. A1a and A1b). Specifying a vertical velocity on the bottom boundary with a larger horizontal scale yields an area of error that extends higher up into the water column (cf. Figs. A1c,d).

Given the 5–10-km horizontal and ~100-m vertical length scales of the inferred $w$, this sensitivity test indicates that errors due to the boundary conditions should be less than 10% outside of a region 10 km from the lateral boundaries and above $z = -150$ m, indicating that the solutions for the vertical circulation in the proximity of the front and above and within the pycnocline are robust.

REFERENCES
