Regime of Validity of Soundproof Atmospheric Flow Models

RUPERT KLEIN

Mathematik und Informatik, Freie Universität Berlin, Berlin, Germany

ULRICH ACHATZ

Meteorologie, Goethe-Universität, Frankfurt, Germany

DIDIER BRESCH

LAMA, CNRS, Université de Savoie, Chambéry, France

OMAR M. KNIO

Department of Mechanical Engineering, The Johns Hopkins University, Baltimore, Maryland

PIOTR K. SMOLARKIEWICZ

National Center for Atmospheric Research, Boulder, Colorado*

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ABSTRACT

Ogura and Phillips derived the original anelastic model through systematic formal asymptotics using the flow Mach number as the expansion parameter. To arrive at a reduced model that would simultaneously represent internal gravity waves and the effects of advection on the same time scale, they had to adopt a distinguished limit requiring that the dimensionless stability of the background state be on the order of the Mach number squared. For typical flow Mach numbers of $M \sim 1/30$, this amounts to total variations of potential temperature across the troposphere of less than one Kelvin (i.e., to unrealistically weak stratification). Various generalizations of the original anelastic model have been proposed to remedy this issue. Later, Durran proposed the pseudoincompressible model following the same goals, but via a somewhat different route of argumentation. The present paper provides a scale analysis showing that the regime of validity of two of these extended models covers stratification strengths on the order of $(h_{sc}/\theta)d\theta/dz < M^{2/3}$, which corresponds to realistic variations of potential temperature θ across the pressure scale height h_{sc} of $\Delta\theta|_0^{h_{sc}} < 30\,\mathrm{K}$.

Specifically, it is shown that (i) for $(h_{sc}/\theta)d\theta/dz < M^{\mu}$ with $0 < \mu < 2$, the atmosphere features three asymptotically distinct time scales, namely, those of advection, internal gravity waves, and sound waves; (ii) within this range of stratifications, the structures and frequencies of the linearized internal wave modes of the compressible, anelastic, and pseudoincompressible models agree up to the order of M^{μ}; and (iii) if $\mu < 2/3$, the accumulated phase differences of internal waves remain asymptotically small even over the long advective time scale. The argument is completed by observing that the three models agree with respect to the advective nonlinearities and that all other nonlinear terms are of higher order in M.

1. Introduction

Ogura and Phillips (1962) derived the original anelastic model through systematic formal asymptotics using the flow Mach number as the expansion parameter. Their goal was to derive a set of model equations that would simultaneously represent internal gravity waves and the effects of advection while suppressing any sound modes.

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Corresponding author address: Rupert Klein, FB Mathematik und Informatik, Freie Universität Berlin, Arnimallee 6, 14195 Berlin, Germany.
E-mail: rupert.klein@math.fu-berlin.de

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Such a model would not only cast the notion that compressibility plays only a subordinate role in the majority of atmospheric flow phenomena in systematic mathematical terms, but it would also lend itself to numerical integration without the necessity of handling the fast yet unimportant acoustic modes through computationally cumbersome numerical means. Such ‘‘soundproof’’ models may also be considered as conceptually important building blocks in a model hierarchy for analyses of fundamental processes of weather and climate (Held 2005).

By design, the soundproof models should be able to address deep atmospheres vertically covering a typical pressure scale height $h_{sc} \sim 10$ km or more, and nonhydrostatic flow regimes corresponding to horizontal scales down to 10 km or less (cf. Bannon 1996). Thus, the characteristic vertical and horizontal length scales for the design regime of these models are comparable to h_{sc}. As can be seen in Table 1, the characteristic acoustic time scale t_{ac} is small, on the order of $O(\varepsilon)$, relative to the advection time t_{adv}, whereas the time scale for internal waves, t_{int} (i.e., the inverse of the Brunt–Väisälä frequency N), is on the order of $O\{\varepsilon[(h_{sc}/\theta)d\theta/dz]^{-1/2}\}$, where θ is the potential temperature. As a consequence, if we follow Ogura and Phillips (1962) and construct an asymptotic single time scale model that resolves the advection time scale and includes internal waves at the same time, we would have to adopt a weak stratification so that $(h_{sc}/\theta)d\theta/dz = O(\varepsilon^2)$.

For typical flow Mach numbers of $M \sim 1/30$ such stratifications amount to total variations of potential temperature across the troposphere of less than one Kelvin (i.e., to unrealistically weak stratification). Various generalizations of Ogura and Phillips’ anelastic model have been proposed to remedy this issue (e.g., Dutton and Fichtl 1969; Lipps and Hemler 1982; cf. Bannon 1996). Durran (1989) proposed the pseudoincompressible model following the same goals but via a somewhat different route of argumentation.

According to Table 1, however, any such stronger stratification with

$$\frac{h_{sc}}{\theta}\frac{d\theta}{dz} = O(\varepsilon^{\mu}), \quad \text{where} \quad 0 < \mu < 2, \tag{1}$$

will induce a three–time scale asymptotic limit so that

$$t_{ac} \ll t_{int} \ll t_{adv} \quad \text{with} \quad t_{ac} = O(\varepsilon t_{adv}), \quad t_{int} = O(\varepsilon^{1-\mu/2} t_{adv}). \tag{2}$$

Soundproof models derived for such a regime of stratifications will thus constitute asymptotic two-scale models in time, retaining a scale separation between the internal and advection time scales. In deriving their models, Dutton and Fichtl (1969), Lipps and Hemler (1982), Durran (1989), and Bannon (1996) provide a range of physical arguments for their validity. However, the two–time scale nature of the resulting soundproof models for stratifications within the regime from (1) is not addressed. Neither have we found the internal wave–Lagrangian time scale separation addressed in more recent scaling or asymptotic analyses of Davies et al. (2003) and Almgren et al. (2006). At the same time, numerical experience indicates that soundproof models work well on a much broader range of scales and problems than would be anticipated based on theoretical arguments (cf. Prusa et al. 2008, and references therein).

TABLE 1. Characteristic inverse time scales.

	Dimensional	Dimensionless
Advection:	$\dfrac{u_{ref}}{h_{sc}}$	1
Internal waves:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{gh_{sc}}}{u_{ref}}\sqrt{\dfrac{h_{sc}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon}\sqrt{\dfrac{h_{sc}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$
Sound:	$\dfrac{\sqrt{gh_{sc}}}{h_{sc}}$	$\dfrac{\sqrt{gh_{sc}}}{u_{ref}} = \dfrac{1}{\varepsilon}$

The presence of multiple scales in the soundproof models is, nevertheless, an issue because both the spatial structures and frequencies of internal waves featured by the soundproof models only approximate those represented by the full compressible flow equations. As a consequence, there are two necessary conditions for the validity of the soundproof models over the targeted advective time scales:

(a) the spatial structures of corresponding internal wave eigenmodes of the soundproof and compressible systems should be asymptotically close as $\varepsilon \to 0$, and
(b) the accumulation of phase differences between such soundproof and compressible internal waves should remain asymptotically small at least over the advective time scale.

Motivated by these considerations, we consider in this paper atmospheres with stratifications in the regime from (1) and

(i) compare the internal wave eigenmode structures of the compressible Euler equations and selected soundproof models;
(ii) assess the approximation errors due to ‘‘soundproofing’’ for both the spatial eigenmodes and the associated frequencies in terms of the Mach number; and

(iii) demonstrate, as our main result, that internal wave solutions of the soundproof and compressible models remain asymptotically close for $t = O(t_{\mathrm{adv}})$ for sufficiently weak stratification. Specifically, for both Lipps and Hemler's and Durran's soundproof models, the corresponding bound on the stratification is

$$\frac{h_{\mathrm{sc}}}{\theta}\frac{d\theta}{dz} = O(\varepsilon^{\mu}) \quad \text{with} \quad \mu > \frac{2}{3}. \tag{3}$$

This corresponds to realistic stratifications with $\Delta\theta|_0^{h_{\mathrm{sc}}} = 30$–$50\,\mathrm{K}$ over 10–15 km.

The rest of the paper is organized as follows. In section 2 we summarize the model equations to be studied. In section 3 we introduce a new set of variables that explicitly reveal the multiscale nature of fully compressible flows within the regime of stratifications from (1). In section 4, using formal asymptotic analysis and vertical mode decompositions, we compare the vertical internal wave eigenmodes and eigenfrequencies for the pseudoincompressible and the Lipps and Hemler anelastic models with those of the compressible equations and show that they are asymptotically close as long as $(h_{\mathrm{sc}}/\theta)\, d\theta/dz = O(\varepsilon^{\mu})$ for any $\mu > 0$. In that section we also assess the time it takes compressible and soundproof internal waves to accumulate leading-order deviations of their phases because of these differences in the dispersion relations, and this will lead to the abovementioned principal result in (3). In section 5 we draw conclusions and provide an outlook for future work.

2. Compressible and soundproof model equations

The exposition in this section of the three sets of model equations to be analyzed subsequently closely follows Klein (2009). Here, we restrict our considerations to flows under gravity, but without Coriolis effects and nonresolved-scale closures, and present consistent dimensionless forms of the compressible Euler equations and of two soundproof models.

a. Compressible Euler equations

$$\rho_t + \nabla \circ (\rho \mathbf{v}) = 0, \tag{4a}$$

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + P \nabla \pi = -\rho \mathbf{k}, \tag{4b}$$

$$P_t + \nabla \cdot (P \mathbf{v}) = 0, \tag{4c}$$

where $(\rho, \mathbf{v})$ are the density and flow velocity, $P = p^{1/\gamma} = \rho\theta$ is a modified thermodynamic pressure variable, θ is potential temperature, and $\pi = p^{\kappa}/\kappa$, where $\kappa = (\gamma - 1)/\gamma$, and $\gamma = c_p/c_v$ is the ratio of the specific heat capacities. Let an asterisk, for the moment, denote dimensional variables; then the dimensionless quantities appearing in (4) are defined as

$$t = \frac{t^* c_{\mathrm{ref}}}{h_{\mathrm{sc}}}, \quad \mathbf{x} = \frac{\mathbf{x}^*}{h_{\mathrm{sc}}}, \quad \rho = \frac{\rho^*}{\rho_{\mathrm{ref}}}, \quad p = \frac{p^*}{p_{\mathrm{ref}}},$$
$$\mathbf{v} = \frac{\mathbf{v}^*}{c_{\mathrm{ref}}}, \quad \rho\theta = p^{1/\gamma}, \tag{5}$$

where $c_{\mathrm{ref}} = \sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}$ and $h_{\mathrm{sc}} = p_{\mathrm{ref}}/\rho_{\mathrm{ref}} g$, and where p_{ref}, ρ_{ref}, and g respectively denote the sea level pressure, the corresponding density at a temperature of, say, 300 K, and the acceleration of gravity.

b. Pseudoincompressible model

If we refrain, in contrast to Durran (1989), from subtracting the background hydrostatic balance from the vertical momentum equation, then the pseudoincompressible model is obtained from (4) by simply dropping the pressure time derivative and assuming P to match a prescribed background distribution $P \equiv \overline{P}(z)$. Thus, we find

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{6a}$$

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + \overline{P} \nabla \pi = -\rho \mathbf{k}, \tag{6b}$$

$$\nabla \cdot (\overline{P} \mathbf{v}) = 0. \tag{6c}$$

c. Anelastic model

Bannon (1996) discusses various versions of anelastic models that differ from the pseudoincompressible one in that they adopt the mass conservation law to impose the sound-removing velocity divergence constraint instead of the pressure equation. The generic anelastic model proposed by Bannon (also Lipps and Hemler 1982), to be analyzed below, is obtained from (4) by dropping the density time derivative, assuming the density to be equal to some prescribed background distribution, $\rho \equiv \overline{\rho}(z)$, and by slightly modifying the pressure gradient and gravity terms. With these modifications, we obtain

$$\nabla \cdot (\overline{\rho} \mathbf{v}) = 0, \tag{7a}$$

$$(\overline{\rho} \mathbf{v})_t + \nabla \cdot (\overline{\rho} \mathbf{v} \circ \mathbf{v}) + \overline{\rho} \nabla \hat{\pi} = \overline{\rho} \frac{\theta - \overline{\theta}}{\overline{\theta}} \mathbf{k}, \tag{7b}$$

$$(\overline{\rho}\theta)_t + \nabla \cdot (\overline{\rho}\theta \mathbf{v}) = 0. \tag{7c}$$

In all three cases, $\overline{\theta}(z)$ is the mean background potential temperature distribution that defines the background pressure variable, $\overline{P}(z)$, and the background density, $\overline{\rho}(z)$, via $d\overline{p}/dz = -\overline{\rho}g$, $\overline{p}(0) = 1$, $\overline{\rho}\overline{\theta} = \overline{P}$, and $\overline{P} \equiv \overline{p}^{1/\gamma}$. For later reference we note the exact solution,

$$\overline{p}(z) = \overline{P}(z)^{\gamma} = [\kappa\overline{\pi}(z)]^{1/\kappa}, \quad \overline{\rho}(z) = \overline{P}(z)/\overline{\theta}(z), \quad \text{where}$$
$$\overline{\pi}(z) = \frac{1}{\kappa} - \int_0^z \frac{1}{\overline{\theta}(\zeta)}\,d\zeta. \tag{8}$$

We also note that in the anelastic model (7) the pressure-related quantity $\hat{\pi}$ is defined as

$$\hat{\pi} = \frac{p - \overline{p}}{\overline{\rho}}; \tag{9}$$

that is, it is a density-scaled perturbation of the pressure p but not of the Exner pressure π.

3. Scaled variables

To arrive at a system of equations that lends itself to our subsequent scale analysis, we first rewrite (4) in a nonconservative (advective) perturbational form, with the primary unknowns

$$\theta' = \theta - \overline{\theta}(z), \quad \mathbf{v}, \quad \pi' = \pi - \overline{\pi}(z). \tag{10}$$

Here θ' is the potential temperature perturbation away from a static background distribution, $\mathbf{v}$ is the velocity with vertical component w, and π' is the perturbation Exner pressure. The hydrostatic background variables satisfy

$$d\overline{\pi}/dz = 1/\overline{\theta} \quad \text{with} \quad \overline{\pi}(0) = 1/\kappa. \tag{11}$$

This yields the equivalent advective form of the compressible Euler equations:

$$\theta'_t + \mathbf{v}\cdot\nabla\theta' + w\frac{d\overline{\theta}}{dz} = 0, \tag{12a}$$

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + (\overline{\theta} + \theta')\nabla\pi' = \frac{\theta'}{\overline{\theta}}\mathbf{k}, \tag{12b}$$

$$\pi'_t + \mathbf{v}\cdot\nabla\pi' + w\frac{d\overline{\pi}}{dz} + \gamma\kappa(\overline{\pi} + \pi')\nabla\cdot\mathbf{v} = 0. \tag{12c}$$

Now the following transformation of variables will explicitly reveal the asymptotic scalings to be discussed in the sequel. First we introduce a time coordinate nondimensionalized by the characteristic advection time

$$\tau = \varepsilon t \tag{13}$$

and then we let

$$\theta(t, \mathbf{x}, z; \varepsilon) = 1 + \varepsilon^{\mu}\overline{\Theta}(z) + \varepsilon^{\mu+\nu}\tilde{\theta}(\tau, \mathbf{x}, z; \varepsilon)$$
$$(\nu = 1 - \mu/2), \tag{14a}$$

$$\pi(t, \mathbf{x}, z; \varepsilon) = \overline{\pi}(z) + \varepsilon\tilde{\pi}(\tau, \mathbf{x}, z; \varepsilon), \tag{14b}$$

$$\mathbf{v}(t, \mathbf{x}, z; \varepsilon) = \varepsilon\tilde{\mathbf{v}}(\tau, \mathbf{x}, z; \varepsilon). \tag{14c}$$

The velocity $\mathbf{v}$ was nondimensionalized by $\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}$, which is comparable to the sound speed; thereupon the scaling in (14c) implies low Mach number flow when $\varepsilon \ll 1$. The representation of the background potential temperature stratification,

$$\overline{\theta}(z) = 1 + \varepsilon^{\mu}\overline{\Theta}(z), \tag{15}$$

follows from the stratification regime in (1). The exponent ν determines the scaling of the dynamic potential temperature perturbations. Its specific value as given in (14a) implies the correct scaling for internal gravity waves, as we will see shortly. Furthermore, $\overline{\pi}(z)$ denotes the background Exner pressure distribution given the stratification from (1). We assume a pressure perturbation amplitude on the order of the Mach number, $O(\varepsilon)$, so as to not preclude leading-order acoustic modes at this stage.

For compressible flows, the new variables $\tilde{\theta}, \tilde{\pi}, \tilde{\mathbf{v}}$ satisfy

$$\tilde{\theta}_{\tau} + \frac{1}{\varepsilon^{\nu}}\tilde{w}\frac{d\overline{\Theta}}{dz} = -\tilde{\mathbf{v}}\cdot\nabla\tilde{\theta}, \tag{16a}$$

$$\tilde{\mathbf{v}}_{\tau} - \frac{1}{\varepsilon^{\nu}}\frac{\tilde{\theta}}{\overline{\theta}}\mathbf{k} + \frac{1}{\varepsilon}(1 + \varepsilon^{\mu}\overline{\Theta})\nabla\tilde{\pi} = -\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi}, \tag{16b}$$

$$\tilde{\pi}_{\tau} + \frac{1}{\varepsilon}\left(\gamma\kappa\overline{\pi}\nabla\cdot\tilde{\mathbf{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = -\tilde{\mathbf{v}}\cdot\nabla\tilde{\pi} - \gamma\kappa\tilde{\pi}\nabla\cdot\tilde{\mathbf{v}}. \tag{16c}$$

These equations are obtained from a straightforward *equivalent* transformation of the compressible flow equations in (4) without any asymptotic simplifications.

Besides the tendencies of temporal change, there are three groups of terms in (16): the terms multiplied by $\varepsilon^{-\nu}$ induce internal waves, the terms multiplied by ε^{-1} represent the acoustic modes, and the terms on the right-hand side cover all nonlinearities. In fact, all terms on the left-hand sides are linear in the unknowns. Notice that all terms on the right are nonsingular as $\varepsilon \to 0$; that is, they are $O(\varepsilon^{\alpha})$ with $\alpha \geq 0$. This clean Mach number scaling of acoustic, internal wave, and nonlinear (advective) terms justifies in hindsight the choice $\nu = 1 - \mu/2$ introduced earlier.

In the new variables the pseudoincompressible model reads

$$\tilde{\theta}_\tau + \frac{1}{\varepsilon^\nu}\tilde{w}\frac{d\overline{\Theta}}{dz} = -\tilde{\mathbf{v}}\cdot\nabla\tilde{\theta}, \tag{17a}$$

$$\tilde{\mathbf{v}}_\tau - \frac{1}{\varepsilon^\nu}\frac{\tilde{\theta}}{\overline{\theta}}\mathbf{k} + \frac{1}{\varepsilon}(1+\varepsilon^\mu\overline{\Theta})\nabla\tilde{\pi} = -\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi}, \tag{17b}$$

$$\left(\gamma\kappa\overline{\pi}\nabla\cdot\tilde{\mathbf{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = 0, \tag{17c}$$

whereas the anelastic model becomes

$$\tilde{\theta}_\tau + \frac{1}{\varepsilon^\nu}\tilde{w}\frac{d\overline{\Theta}}{dz} = -\tilde{\mathbf{v}}\cdot\nabla\tilde{\theta}, \tag{18a}$$

$$\tilde{\mathbf{v}}_\tau - \frac{1}{\varepsilon^\nu}\frac{\tilde{\theta}}{\overline{\theta}}\mathbf{k} + \frac{1}{\varepsilon}\nabla\tilde{\pi} = -\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}}, \tag{18b}$$

$$-\varepsilon^\mu\frac{\gamma\kappa\overline{\pi}}{\overline{\theta}}\tilde{w}\frac{d\overline{\Theta}}{dz} + \left(\gamma\kappa\overline{\pi}\nabla\cdot\tilde{\mathbf{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = 0. \tag{18c}$$

Notice that in order to obtain the scaling of the pressure term in (18b), we have accounted for the fact that $\hat{\pi}$ in the unscaled anelastic model from (7) already denotes a deviation from the background pressure according to (9), whereas we had retained the full dimensionless Exner pressure in writing down the compressible and pseudoincompressible models. Thus, we have replaced (14b) with $\hat{\pi} = \varepsilon\tilde{\pi}(\tau, \mathbf{x}, z; \varepsilon)$ in deriving (18b).

We observe that the potential temperature transport equations are in agreement between all three models. This was to be expected since in the present adiabatic setting this equation reduces to a simple advection equation. The momentum equations of the compressible and pseudoincompressible models are in complete agreement, whereas the anelastic model's momentum equation lacks the respective last terms on the left and right from (16b) or (17b) that combine to yield $\varepsilon^{\mu-1}(\overline{\Theta} + \varepsilon^\nu\tilde{\theta})\nabla\tilde{\pi}$.[1] This reduces baroclinic vorticity production in the anelastic model in essence to the effects of horizontal gradients of buoyancy (cf. Smolarkiewicz and Dörnbrack 2008).

The only difference between the compressible Euler equations from (16) and the pseudoincompressible model is found in the Exner pressure evolution equation (16c), which becomes the pseudoincompressible divergence constraint in (17c). The anelastic divergence constraint in (18c) again differs from the pseudoincompressible one through an additional term involving the background potential temperature stratification.

[1] Notice that $\mu - 1 + \nu = 2(1 - \nu) - 1 + \nu = 1 - \nu$.

TABLE 2. Switching parameters in Eq. (19).

Model	A	B	C
Compressible	1	1	0
Pseudoincompressible	0	1	0
Anelastic	0	0	1

4. Internal gravity waves

a. Gravity wave scaling

The compressible flow equations from (16) feature three distinct time scales: for sound propagation, $\tau = O(\varepsilon)$; for internal waves, $\tau = O(\varepsilon^\nu)$; and for advection, $\tau = O(1)$. In this section we consider solutions that do not feature sound waves but evolve on time scales comparable to the internal wave time scale. The only "sound term" $O(\varepsilon^{-1})$ in the momentum equation is the one involving the pressure gradient. This term will reduce to $O(\varepsilon^{-\nu})$ and thus induce changes on the internal wave time scale only, provided that the pressure perturbations satisfy $\tilde{\pi} = \varepsilon^{1-\nu}\pi^*$ with $\pi^* = O(1)$. By introducing this additional rescaling of the pressure fluctuations and by adopting an internal wave time coordinate $\vartheta = \varepsilon^{-\nu}\tau$, the compressible, pseudoincompressible, and anelastic systems can be represented as

$$\tilde{\theta}_\vartheta + \tilde{w}\frac{d\overline{\Theta}}{dz} = -\varepsilon^\nu\tilde{\mathbf{v}}\cdot\nabla\tilde{\theta}, \tag{19a}$$

$$\tilde{\mathbf{v}}_\vartheta - \frac{\tilde{\theta}}{\overline{\theta}}\mathbf{k} + (1 + B\varepsilon^\mu\overline{\Theta})\nabla\pi^* = -\varepsilon^\nu\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}} - B\varepsilon^{\mu+\nu}\tilde{\theta}\nabla\pi^*, \tag{19b}$$

$$A\varepsilon^\mu\pi^*_\vartheta - C\varepsilon^\mu\frac{\gamma\kappa\overline{\pi}}{\overline{\theta}}\tilde{w}\frac{d\overline{\Theta}}{dz} + \left(\gamma\kappa\overline{\pi}\nabla\cdot\tilde{\mathbf{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = -A\varepsilon^{\mu+\nu}(\tilde{\mathbf{v}}\cdot\nabla\pi^* + \gamma\kappa\pi^*\nabla\cdot\tilde{\mathbf{v}}), \tag{19c}$$

with the choices of switching parameters summarized in Table 2.

We observe that in the gravity wave scaling all differences between the compressible model on the one hand and both of the soundproof models on the other hand are $O(\varepsilon^\mu)$ or smaller (i.e., at least on the order of the stratification strength). At leading order in ε, all models agree from a formal scaling perspective, although switching off the pressure tendency by letting $A = 0$ fundamentally changes the mathematical type of the equations from strictly hyperbolic to mixed hyperbolic–elliptic. We will demonstrate below through formal asymptotics that this, nevertheless, affects the internal gravity wave solutions only weakly. Between the pseudoincompressible and anelastic systems there is no such singular switch, however, so that their solutions will differ only by $O(\varepsilon^{\mu})$ at least on internal wave time scales with $\vartheta = O(1)$.

b. *The constraint on the stratification*

The leading perturbation terms in (19) involve terms $O(\varepsilon^{\mu})$ in the linearized part on the left and terms $O(\varepsilon^{\nu})$ in the nonlinear part of the equations on the right. This suggests that for $\mu < \nu$ (i.e., for $\varepsilon^{\mu} \gg \varepsilon^{\nu}$), the linearized internal wave eigenmodes and eigenvalues of the three systems differ by $O(\varepsilon^{\mu})$ only, and the nonlinearities represent even higher-order effects. In this setting, we may expect solutions of the three models that start from comparable internal wave initial data to remain close with differences $O(\varepsilon^{\mu})$ over the internal wave time scale with $\vartheta = O(1)$. However, we are really interested in flow evolutions over advective time scales with $\tau = \varepsilon^{\nu}\vartheta = O(1)$. Over such longer time scales, the expected differences in the internal wave eigenfrequencies $O(\varepsilon^{\mu})$ will accumulate to phase shifts on the order of $\varepsilon^{\mu}\vartheta = O(\tau\varepsilon^{\mu-\nu}) = O(\varepsilon^{\mu-\nu})$. As a consequence, the linearized internal wave solutions of the three models should remain asymptotically close even over advective time scales provided that

$$\varepsilon^{\mu-\nu} = \varepsilon^{(3/2)\mu-1} = o(1) \quad \text{as} \quad \varepsilon \to 0 \quad \text{or} \quad \mu > \frac{2}{3}. \tag{20}$$

This constitutes our main result: For any stratifications weaker than $d\overline{\theta}/dz = O(\varepsilon^{2/3})$, the internal wave dynamics of the compressible, pseudoincompressible, and anelastic models should remain asymptotically close in terms of the flow Mach number *over advective time scales*. This is a considerable improvement over the Ogura and Phillips' original condition for the validity of their anelastic model, which requires that $d\overline{\theta}/dz = O(\varepsilon^2)$. For $\varepsilon \sim 1/30$ Ogura and Phillips' estimate amounts to potential temperature variations on the order of $\Delta\theta|_0^{h_{sc}} \sim 0.33\,\mathrm{K}$ over the pressure scale height, whereas our new estimate implies validity of the soundproof models even if

$$\Delta\theta|_0^{h_{sc}} \sim \theta_{ref} h_{sc}\frac{1}{\theta^*}\frac{d\theta^*}{dz^*} = T_{ref}\frac{1}{\theta}\frac{d\theta}{dz} \sim 300\,\mathrm{K} \times (1/30)^{2/3} \sim 30\,\mathrm{K}, \tag{21}$$

where the asterisk denotes dimensional quantities.

Another indication that at the threshold of $\mu = 2/3$, the dynamics change nontrivially arises as follows. When $\mu = 2/3$ we have $\nu = 1 - \mu/2 = \mu$, so that the leading nonlinearities on the rhs of (19), which are $O(\varepsilon^{\nu})$, become comparable to the perturbation terms of the linearized system on the lhs of (19), which are $O(\varepsilon^{\mu})$. Thus, for $\mu \leq 2/3$ any perturbation analysis of internal waves in compressible flows that go beyond the leading-order solution must necessarily account for nonlinear effects.

Note that there is no noticeable transition or change in the structure of the linear eigenmodes and eigenvalues considered in the next section as μ decreases below the threshold of $\mu = 2/3$. The importance of this threshold is associated entirely with the more subtle effects just explained.

The present estimates rely on the linearized equations. However, since all three models considered feature the same leading nonlinearities represented by the nonlinear advection of potential temperature and velocity in (19a) and (19b) [see the terms $O(\varepsilon^{\nu})$], we expect asymptotic agreement of the solutions over advective time scales as long as the fast linearized dynamics do not already lead to leading-order deviations between the model results (i.e., as long as $\mu > 2/3$). A mathematically rigorous proof of the validity of the fully nonlinear pseudoincompressible, and possibly the anelastic, models over advective time scales is a work in progress.

c. *Vertical mode decomposition and the Sturm–Liouville eigenvalue problem*

Here we summarize the analysis of internal wave vertical eigenmodes for the three flow models. For simplicity, we assume rigid-wall top and bottom boundaries at $z = 0$ and $z = H = O(1)$, respectively, and seek horizontally traveling waves described by

$$(\tilde{\theta}, \tilde{\mathbf{u}}, \tilde{w}, \pi^*)(\vartheta, \mathbf{x}, z) = (\breve{\theta}, \breve{\mathbf{u}}, \breve{w}, \breve{\pi})(z)\exp[i(\omega\vartheta - \boldsymbol{\lambda}\cdot\mathbf{x})]. \tag{22}$$

Inserting this ansatz into (19), neglecting the nonlinearities, and eliminating $\breve{\theta}$, $\breve{\mathbf{u}}$, and $\breve{\pi}$, we obtain a Sturm–Liouville-type second-order differential equation for a suitable vertical velocity structure function $W(z)$,

$$-\frac{d}{dz}\left(\frac{1}{\lambda^2 - A\varepsilon^{\mu}/\Lambda\overline{c}^2}\phi_{BC}\frac{dW}{dz}\right) + \phi_{BC}W = \Lambda(N^2\phi_{BC})W, \tag{23}$$

with boundary conditions

$$W(0) = W(H) = 0. \tag{24}$$

Here we have used the following abbreviations:

$$\phi_{\mathrm{BC}} = \frac{\overline{\theta}^C}{\overline{\theta}^B}\overline{P}, \quad \overline{c}^2 = \frac{\gamma\overline{p}}{\overline{\rho}}, \quad N^2 = \frac{1}{\overline{\overline{\theta}}}\frac{d\overline{\Theta}}{dz}, \tag{25}$$

and

$$\Lambda = \frac{1}{\omega^2},$$
$$W = \begin{cases} \overline{P}\breve{w} & \text{compressible or pseudoincompressible} \\ \overline{\rho}\breve{w} & \text{anelastic} \end{cases}. \tag{26}$$

See the appendix for details of the derivation, and note that $\overline{\theta}^B$, $\overline{\theta}^C$ are to be read as $\overline{\theta}$ to the power B and C, respectively.

For $A = 0$ (i.e., for either the anelastic or the pseudoincompressible model), and for any fixed horizontal wavenumber vector $\boldsymbol{\lambda}$, (23) and (24) represent a classical Sturm–Liouville eigenvalue problem, about which the following facts are well known (Zettl 2005):

- (i) There is a sequence of eigenvalues and associated eigenfunctions, $(\Lambda_k^0, W_k^0)_{\kappa=0}^{\infty}$, with $0 < \Lambda_0^0 < \Lambda_1^0 \ldots$, and $\Lambda_k^0 \to \infty$ as $k \to \infty$.
- (ii) The $(W_k^0)_{\kappa=0}^{\infty}$ form an orthonormal basis of a Hilbert space of functions $f{:}[0, H] \to \mathbb{R}$ with scalar product $\langle U, V\rangle = \int_0^H U(N^2\phi_{\mathrm{BC}})V\,dz$. Note that the scalar product and, thus, the Hilbert space are independent of the horizontal wavenumber $\boldsymbol{\lambda}$.
- (iii) The vertical mode number k equals the number of zeroes of the associated eigenmodes on the open interval $0 < z < H$ (i.e., excluding the boundary points). Thus, $k = 0$ represents the leading, vertically nonoscillatory mode.

We conclude that the two soundproof models considered here feature well-defined internal wave modes, with one such hierarchy of eigenvalues and vertical structures for each wavenumber vector $\boldsymbol{\lambda}$. The only differences in the linearized eigenmodes between the pseudoincompressible and the present anelastic model consist of the scaling factor of $\overline{\theta} = \overline{P}/\overline{\rho}$ in the definition of the structure function $W(z)$ in (26) and the slightly different way in which the background potential temperature distribution enters the Sturm–Liouville equation. Specifically,

$$\phi_{\mathrm{BC}} = \begin{cases} 1/\overline{\theta}\overline{P} & \text{pseudoincompressible} \\ \overline{\theta}/\overline{P} & \text{anelastic} \end{cases}. \tag{27}$$

Notice that the compressible and pseudoincompressible models share the definition of W as well as that of ϕ_{BC}.

d. Asymptotics for the compressible internal wave modes

The eigenvalue–eigenfunction problem for the linearized compressible equations [i.e., (23) and (24) with $A = 1$] is *nonlinear* in the eigenvalue Λ. Here we construct first-order accurate approximations to the *weakly* compressible eigenvalues and eigenfunctions, for which $\lambda^2 \gg \varepsilon^\mu/\Lambda\overline{c}^2$, so that the compressibility term in the denominator of the first term in (23) remains a small perturbation, and we may expand the solution as

$$(\Lambda_k^\varepsilon, W_k^\varepsilon) = (\Lambda_k^0, W_k^0) + \varepsilon^\mu(\Lambda_k^1, W_k^1) + O(\varepsilon^{2\mu}), \tag{28}$$

where the W_k^0 are taken to be the eigenfunctions corresponding to the pseudoincompressible model.

Notice that there is a set of eigenvalues with $\Lambda = 1/\omega^2 = O(\varepsilon^\mu)$ that correspond to the system's high-frequency acoustic modes. Those will not be considered further in this paper.

The perturbation structure functions $W_k^1(z)$ are then expanded in terms of the leading-order eigenfunction basis, $(W_j^0)_{j=0}^{\infty}$, so that

$$W_k^1 = \sum_j \psi_{k,j} W_j^0. \tag{29}$$

Inserting (28) in (23) we first find that the leading-order terms $O(1)$ cancel identically because (Λ_k^0, W_k^0) already solve the eigenvalue problem for $A = 0$ and $B = 1$. At $O(\varepsilon^\mu)$ we have, letting $\phi_{\mathrm{BC}} \equiv \phi$ for simplicity of notation,

$$-\frac{d}{dz}\left(\frac{\phi}{\lambda^2}\frac{dW_k^1}{dz}\right) + \phi W_k^1 = \Lambda_k^0(N^2\phi)W_k^1 + \Lambda_k^1(N^2\phi)W_k^0 + \frac{d}{dz}\left(F\frac{dW_k^0}{dz}\right), \tag{30}$$

where

$$F = \frac{\phi}{\Lambda_k^0\overline{c}^2\lambda^4}. \tag{31}$$

Multiplying by W_k^0, integrating from $z = 0$ to $z = H$, and using the orthonormality from item (ii) above as well as the fact that W_k^0 is the leading-order eigenfunction with eigenvalue Λ_k^0, we find that the left-hand side and the first term on the right cancel each other, whereas the remaining two terms yield

$$\Lambda_k^1 = \int_0^H F\left[\frac{dW_k^0}{dz}\right]^2 dz. \tag{32}$$

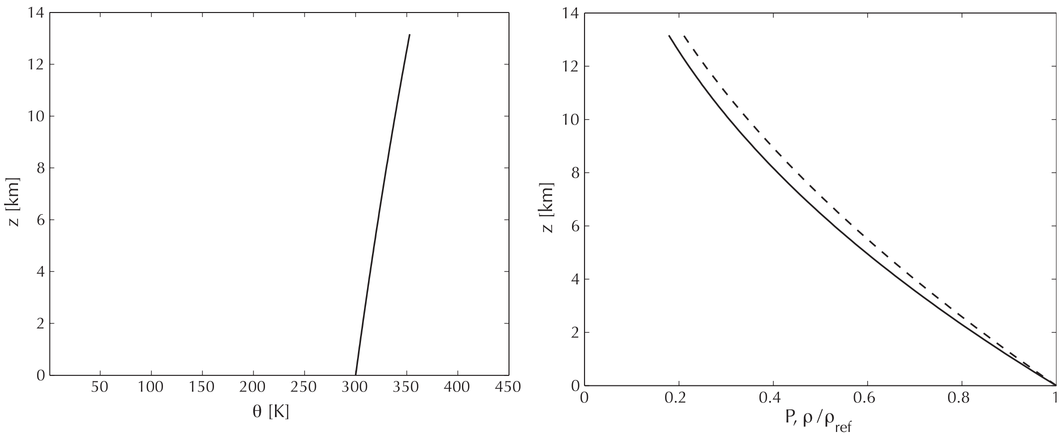

FIG. 1. (left) Sample potential temperature distribution, and (right) the resulting dimensionless vertical distributions of $\overline{P} = \overline{\rho}\overline{\theta}$ (dashed line) and $\overline{\rho}$ (solid line).

Similarly we find, after multiplication with W_j^0 for $j \neq k$ and integration,

$$\psi_{k,j}^1 = -\frac{1}{\Lambda_j^0 - \Lambda_k^0}\int_0^H F\frac{dW_k^0}{dz}\frac{dW_j^0}{dz}\,dz. \tag{33}$$

Because of normalization of the eigenfunctions, it turns out that $\psi_{k,k}^1 = O(\varepsilon^\mu)$ and thus contributes a higher-order correction only.

This determines the first-order perturbations in terms of ε^μ from (28) and (29). For a forthcoming companion paper, two of the authors are currently working on a rigorous proof that the remainders are actually $O(\varepsilon^{2\mu})$ as indicated in (28). A *remark* is in order. If $\Lambda_j(p, q, r)$ is a simple eigenvalue of a Sturm–Liouville operator $L(p, q, r)$ on [0, 1]—that is, if there exists a unique eigenfunction W_j such that $L(p, q, r)W_j = -(pW_j')' + qW_j = r\Lambda_j W_j$ with $W_j(0) = W_j(1) = 0$—then $\Lambda_j(p, q, r)$ depends analytically on the functions p and q in a neighborhood of the coefficients. The derivative of the eigenvalue Λ_j and eigenvector W_j are given by the expressions in (32) and (33) (see Kato 1995; Kong and Zettl 1996).

e. Examples

Here we evaluate the leading- and first-order results for a background potential temperature distribution

$$\overline{\theta}(z) = (1 - 0.1z)^{-1}\ (0 \le z \le 1.5). \tag{34}$$

With $h_{sc} = p_{ref}/\rho_{ref}g \sim 8.8$ km and $T_{ref} = 300$ K, the potential temperature distribution from Fig. 1 results, showing a vertical variation of about 40 K over ~13 km. The maximum relative deviation between $\overline{P} = \overline{p}^{1/\gamma}$ and $\overline{\rho}$ amounts to 15% in this example.

For the present hydrostatic background and horizontal wavenumbers $\lambda = 0.5, 2.0, 8.0$, corresponding to horizontal wavelengths of 110.6, 27.6, and 6.9 km, respectively, the eigenvalues for the compressible and soundproof systems deviate from each other by less than two percent. Figure 2 shows the leading-order relative difference between the Sturm–Liouville eigenvalues for the pseudoincompressible and anelastic models on the one hand and the first-order approximations of the eigenvalues for the compressible model on the other hand. The approximate eigenvalues for the compressible case have been computed here from the first iterate of a Picard iteration in terms of Λ in (23)—that is, from the perturbed regular Sturm–Liouville equation

$$-\frac{d}{dz}\left(\frac{1}{\lambda^2 - A\varepsilon^\mu/\Lambda_k^0\overline{c}^2}\phi\frac{dW}{dz}\right) + \phi W = \Lambda^1(N^2\phi)W. \tag{35}$$

The resulting Λ_k^1 equals the compressible eigenvalue of mode number k up to errors $O(\varepsilon^{2\mu})$ as shown rigorously by two of the authors in a forthcoming paper. The $\Lambda_j^1(k)$ for $j \neq k$ resulting from (35) have no physical meaning.

We observe that the relative deviation of the eigenvalues between the soundproof and compressible cases is surprisingly small in practice. According to our previous analysis, we would expect deviations of the same order of magnitude as the relative vertical variation of the potential temperature, which in the present case is $\varepsilon^\mu \sim 0.1$. Yet, the maximum relative deviation between the eigenvalues is less than 0.02 (in modulus) for the cases documented in Fig. 2 for mode number $k = 0$, and it decreases rapidly for larger k. The situation is very similar for other horizontal wavenumbers (not shown).

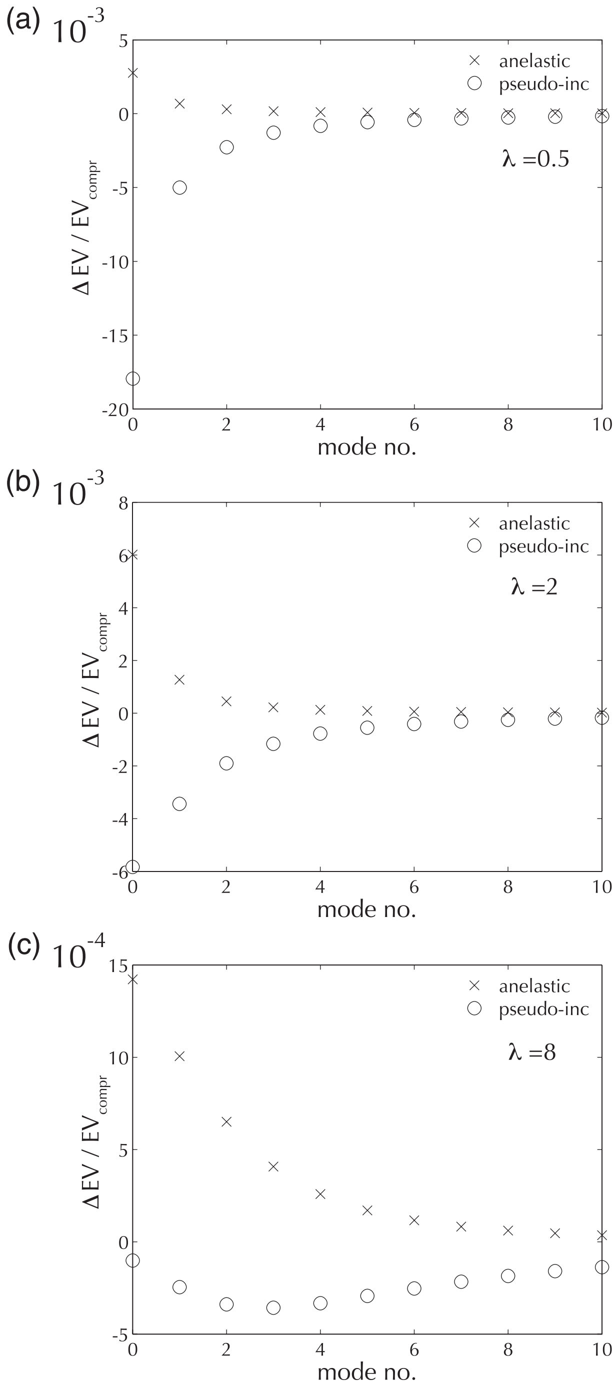

FIG. 2. Comparison between the soundproof and first-order accurate approximations to the compressible internal wave eigenvalues for the background state from Fig. 1 and horizontal wavenumbers $|\boldsymbol{\lambda}|$ = (a) 0.5, (b) 2.0, and (c) 8.0. The approximate compressible eigenvalues and eigenmodes are defined through the perturbed Sturm–Liouville problem in (35). The graph shows relative differences of the eigenvalues $(\Lambda_j^0 - \Lambda_j^1)/\Lambda_j^0$.

The deviations in the vertical structure functions are similarly small as demonstrated in an exemplary fashion by the differences in the vertical velocity structure functions, $\breve{w}_{10}^1 - \breve{w}_{10}^0$ for $|\boldsymbol{\lambda}| = 0.5$ and $\breve{w}_0^1 - \breve{w}_0^0$ for $|\boldsymbol{\lambda}| = 8.0$ in Fig. 3. We have observed similarly small differences in the structure functions for a range of vertical mode numbers. Note, however, that we have assumed $H = 1.5$, so that deep internal modes with characteristic scales much larger than the pressure scale height are excluded. A systematic study of such deep modes as well as much larger horizontal scales is left for future work.

f. The long-wave limit

Considering (23), one may wonder whether compressibility will play less of a subordinate role for large-scale internal gravity waves with $|\boldsymbol{\lambda}| \ll 1$, as in this case the two terms in the denominator, $\lambda^2 - \varepsilon^\mu/\Lambda\overline{c}^2$, could become comparable. That this is not the case becomes clear after multiplication of the entire equation (23) by λ^2 and considering the rescaled eigenvalue $\Lambda^*(\lambda) = \boldsymbol{\lambda}^2\Lambda$. The Sturm–Liouville equation for this variable then reads

$$-\frac{d}{dz}\left[\frac{1}{1-\varepsilon^\mu/\Lambda^*(\lambda)\overline{c}^2}\phi\frac{dW}{dz}\right] + \lambda\phi W = \Lambda^*(\lambda)(N^2\phi)W. \tag{36}$$

As λ^2 vanishes, the equation approaches a well-defined limit in which second term on the left vanishes asymptotically, and the term $\varepsilon^\mu/\Lambda^*\overline{c}^2$ remains a small perturbation in the denominator of the second-derivative term. As a consequence, the long-wave limiting behavior of the original eigenvalues will be

$$\lambda^2\Lambda_k \rightarrow \Lambda_k^*(0) \quad \text{as} \quad |\lambda| \rightarrow 0, \tag{37}$$

where $\Lambda_k^*(0)$ is an eigenvalue of the limit problem

$$-\frac{d}{dz}\left[\frac{1}{1-\varepsilon^\mu/\Lambda^*(\lambda)\overline{c}^2}\phi\frac{dW}{dz}\right] = \Lambda^*(\lambda)(N^2\phi)W \tag{38}$$

with the same rigid-wall boundary conditions. Of course, to correctly capture the behavior of internal wave modes at large horizontal scales we will have to include the Coriolis effect. This is left for future work.

5. Conclusions

In this paper we have addressed the formal asymptotics of weakly compressible atmospheric flows involving three asymptotically different time scales for sound, internal waves, and advection. Both the pseudoincompressible and a particular anelastic model yield

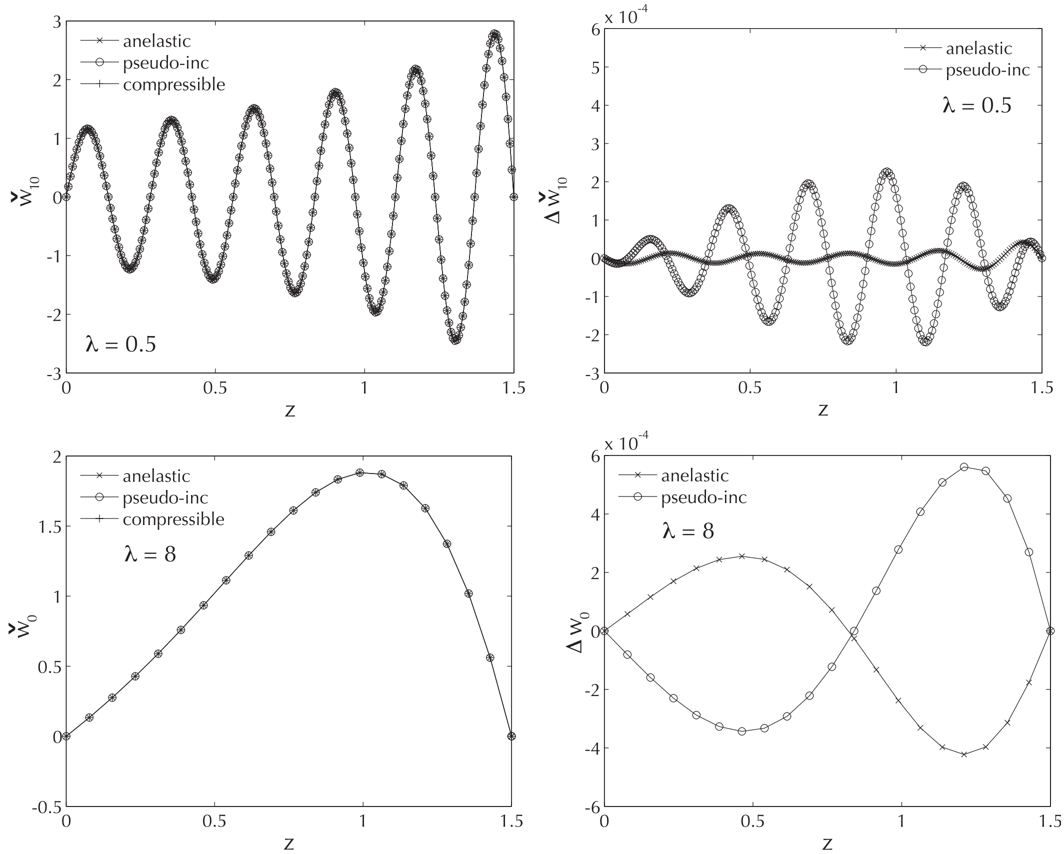

FIG. 3. (left) Vertical velocity structure functions and (right) deviations between the soundproof and compressible modes for the same case as in Fig. 2 and (top) mode number $k = 10$, horizontal wavenumber $|\boldsymbol{\lambda}| = 0.5$ and (bottom) mode number $k = 0$, horizontal wavenumber $|\boldsymbol{\lambda}| = 8$.

very good approximations to the linearized internal wave dynamics in a compressible flow for realistic background stratifications and on length scales comparable to the pressure and density scale heights. These soundproof models should be applicable for stratification strengths $(h_{sc}/\overline{\theta})(d\overline{\theta}/dz) < O(\varepsilon^{2/3})$, where ε is the flow Mach number. This constraint guarantees that the soundproof and compressible internal waves evolve asymptotically closely even over advective time scales. For typical flow Mach numbers $\varepsilon \sim 1/30$, this amounts to vertical variations of the mean potential temperature over the pressure scale height of $\Delta\overline{\theta} \sim 30\,\mathrm{K}$. Considering that $h_{sc} \sim$ 8.8 km for $T_{ref} = 300$ K and that typical tropospheric heights are about 10–15 km, the estimate for the validity of the soundproof models yields realistic potential temperature variations of $\delta\overline{\theta} \sim 30 - 50$ K across the troposphere. We have thus provided an explicit estimate for the regime of validity of the considered soundproof models that considerably extends Ogura and Phillips' original estimate, which required $(h_{sc}/\overline{\theta})(d\overline{\theta}/dz) = O(\varepsilon^2)$ and implied unrealistically weak background stratifications.

A number of important open questions remain to be addressed, such as (i) Could either of the soundproof models be also justified even for $(h_{sc}/\overline{\theta})(d\overline{\theta}/dz) = O(1)$, and if so, what are the pertinent flow regimes when linear as well as nonlinear effects are taken into account? (ii) Is there a mathematically rigorous justification of the present formal asymptotic results? (iii) How does inclusion of Coriolis effects influence the regime of validity of these soundproof models, especially with regard to horizontal scales comparable to synoptic or even planetary distances, and vertical extensions much larger than the pressure scale height? See also the discussions in Davies et al. (2003) and Almgren et al. (2006) in this context.

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APPENDIX

Derivation of the Sturm–Liouville Eq. (23)

Consider the linearized (19), that is, (19) with the rhs terms set to 0. Let $1 + B\varepsilon^\mu\overline{\Theta} \equiv \overline{\theta}^B$. Then we have

$$\tilde{\theta}_\vartheta + \tilde{w}\frac{d\overline{\Theta}}{dz} = 0, \tag{A1a}$$

$$\tilde{\mathbf{u}}_\vartheta + \overline{\theta}^B\nabla\pi^* = 0, \tag{A1b}$$

$$\tilde{w}_\vartheta - \frac{\tilde{\theta}}{\overline{\overline{\theta}}} + \overline{\theta}^B\pi^*_z = 0, \tag{A1c}$$

$$A\varepsilon^\mu\pi^*_\vartheta - C\varepsilon^\mu\frac{\gamma\kappa\overline{\pi}}{\overline{\theta}}\tilde{w}\frac{d\overline{\Theta}}{dz} + \left(\gamma\kappa\overline{\pi}\nabla\cdot\tilde{\mathbf{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = 0. \tag{A1d}$$

Introduce the vertical mode expansion from (22). Then the first two equations in (A1) yield

$$i\omega\breve{\theta} + \breve{w}\frac{d\overline{\Theta}}{dz} = 0, \tag{A2a}$$

$$i\omega\breve{\mathbf{u}} - \overline{\theta}^B i\lambda\breve{\pi} = 0, \tag{A2b}$$

$$i\omega\breve{w} - \frac{\breve{\theta}}{\overline{\overline{\theta}}} + \overline{\theta}^B\frac{d\breve{\pi}}{dz} = 0, \tag{A2c}$$

$$A\varepsilon^\mu i\omega\breve{\pi} - C\varepsilon^\mu\frac{\gamma\kappa\overline{\pi}}{\overline{\theta}}\breve{w}\frac{d\overline{\Theta}}{dz} + \gamma\kappa\overline{\pi}\left(-i\boldsymbol{\lambda}\cdot\breve{\mathbf{u}} + \frac{d\breve{w}}{dz}\right) + \breve{w}\frac{d\overline{\pi}}{dz} = 0. \tag{A2d}$$

Eliminate $\breve{\mathbf{u}}$ from the fourth equation in (A2) using the second equation in (A2) to obtain

$$i\left(A\varepsilon^\mu\omega - \gamma\kappa\overline{\pi}\overline{\theta}^B\frac{\lambda^2}{\omega}\right)\breve{\pi} + \left[\left(\frac{d\overline{\pi}}{dz} - \varepsilon^\mu C\frac{\gamma\kappa\overline{\pi}}{\overline{\theta}}\frac{d\overline{\Theta}}{dz}\right)\breve{w} + \gamma\kappa\overline{\pi}\frac{d\breve{w}}{dz}\right] = 0. \tag{A3}$$

Use (A2a) and (A2b) to obtain

$$i\frac{d\breve{\pi}}{dz} = \frac{\omega}{\overline{\theta}^B}\left(1 - \frac{N^2}{\omega^2}\right)\breve{w} \quad \text{where} \quad N^2 \equiv \frac{1}{\overline{\overline{\theta}}}\frac{d\overline{\Theta}}{dz}, \tag{A4}$$

and then to solve (A3) for $\breve{\pi}$, take the z derivative and eliminate $id\breve{\pi}/dz$ using (A4). This yields, after division by ω,

$$\frac{1}{\overline{\theta}^B}\left(1 - \frac{N^2}{\omega^2}\right)\breve{w} - \frac{d}{dz}\left[\frac{1}{\lambda^2 - A\varepsilon^\mu\omega^2/\overline{c}^2}\frac{1}{\overline{\theta}^B} \times\left(\frac{\breve{w}}{\gamma\kappa\overline{\pi}}\frac{d\overline{\pi}}{dz} - \varepsilon^\mu C\frac{\breve{w}}{\overline{\theta}}\frac{d\overline{\Theta}}{dz} + \frac{d\breve{w}}{dz}\right)\right] = 0, \tag{A5}$$

where we have used

$$\overline{c}^2 = \frac{\gamma\overline{p}}{\overline{\rho}} = \frac{\gamma\overline{p}\overline{\theta}}{\overline{p}^{1/\gamma}} = \gamma\kappa\overline{\pi}\overline{\theta}, \tag{A6}$$

the definition of $\overline{\pi}$ in (8), and the fact that, according to (19) and Table 2, we have $\overline{\theta}^B \equiv \overline{\theta}$ whenever $A \neq 0$. Realize that

$$\left(\frac{\breve{w}}{\gamma\kappa\overline{\pi}}\frac{d\overline{\pi}}{dz} - \varepsilon^\mu C\frac{\breve{w}}{\overline{\theta}}\frac{d\overline{\Theta}}{dz} + \frac{d\breve{w}}{dz}\right) = \frac{\overline{\theta}^C}{\overline{P}}\frac{d}{dz}\left(\frac{\overline{P}}{\overline{\theta}^C}\breve{w}\right), \tag{A7}$$

which, given $\overline{P} = (\kappa\overline{\pi})^{1/\gamma\kappa}$ from (8), is obvious for $C = 0$ and follows from $\overline{\theta}(z) = 1 + \varepsilon^\mu\overline{\Theta}(z)$ for $C = 1$. Letting $\overline{\rho}_C = \overline{P}/\overline{\theta}^C$, $W = \overline{\rho}_C\breve{w}$, $\phi_{\rm BC} = \overline{\theta}^C/\overline{\theta}^B\overline{P}$, and $\Lambda = 1/\omega^2$, we collect (A5)–(A7) to obtain the Sturm–Liouville equation from (23):

$$-\frac{d}{dz}\left(\frac{\phi_{\rm BC}}{\lambda^2 - A\varepsilon^\mu/\Lambda\overline{c}^2}\frac{dW}{dz}\right) + \phi_{\rm BC}W = \Lambda(\phi_{\rm BC}N^2)W, \tag{A8}$$

where

$$\phi_{\rm BC} = \frac{\overline{\theta}^C}{\overline{\theta}^B\overline{P}} = \begin{cases} \dfrac{\overline{\theta}}{\overline{P}} & \text{anelastic} \\ \dfrac{1}{\overline{\theta}\,\overline{P}} & \text{compressible and pseudoincompressible} \end{cases}. \tag{A9}$$

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