Stochastic Parameterization Schemes for Use in Realistic Climate Models

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(Manuscript received 22 March 2010, in final form 25 August 2010)

ABSTRACT

Stochastic parameterizations of fast-evolving, subgrid-scale processes are increasingly being used in a range of models from conceptual models to general circulation models. However, stochastic terms are generally included in an ad hoc fashion. In this study, a systematic method—"Hasselmann’s method"—of stochastic parameterization is developed through the direct application of rigorously justified limit theorems that predict the effective slow dynamics in systems with coupled slow and fast variables. The multiple Hasselmann models form a hierarchy of models ordered by the time scales over which they are expected to provide good approximations to the slowly evolving variables. Adaptable, efficient algorithms for integrating these reduced models are developed that require minimal changes to the unreduced model.

Hasselmann’s method is tested on an O(10 000)-dimensional (planetary and synoptic scale) quasigeostrophic model of atmospheric low-frequency variability. Low-dimensional deterministic and stochastic models in the planetary-scale modes alone are derived, which accurately generate the statistics of the corresponding modes of the unreduced model, including the statistical signatures of jet regime behavior. It is shown that deterministic nonlinearity through slow forcing averaged with respect to the fast modes distribution dominates over multiplicative noise in generating the regime behavior.

1. Introduction

Parameterizations of the effects of unresolved/"weather" processes on resolved/"climate" processes are generally highly simplified representations of reality that are limited to the modeling of specific processes within specific models (i.e., they are developed in an ad hoc fashion). Mathematically rigorous dimension-reduction theorems allow the systematic development of effective dynamics for the slow variables in coupled slow–fast systems. Although many climate–weather systems satisfy key assumptions underlying such theorems, they have not often been invoked in developing parameterization schemes because of computational barriers to their implementation. In this paper, we develop efficient numerical schemes (as part of what we here call Hasselmann’s method; cf. Hasselmann 1976) by which theorems valid for large time scale separation between slow and fast processes may be directly applied in the parameterization of climate–weather/slow–fast interactions in realistic climate models.

These theorems predict that the best parameterizations are stochastic, verifying the intuition that guided the development of stochastic climate modeling in Mitchell (1966) and Hasselmann (1976). Motivated by the predominant red-noise structure of many climate variables (e.g., Pandolfo 1993), Klaus Hasselmann argued that the slowly evolving ocean (climate) integrates...
the effectively stochastic contributions from the weather to produce a red-noise response to white-noise forcing (in analogy to the classic case of the Brownian motion of a grain of pollen induced by random bombardments of neighboring water molecules). Based on this physical reasoning, Hasselmann constructed a stochastic differential equation (SDE) in the climate variable alone, which represented deterministic nonlinear interactions between scales through averaged forcing and state-dependent (i.e., multiplicative) noise forcing. Since Hasselmann’s seminal work, stochastic parameterizations have been utilized in a range of models from conceptual models of ocean–atmosphere processes (e.g., Penland and Sardeshmukh 1995) to general circulation models (GCMs) (e.g., Berner et al. 2008). However, the full potential of stochastic modeling has not been realized because systematic parameterization methods have not been fully developed for use in complex climate models. Two methods based on rigorously justified reduction theorems in the limit of large-scale separation are available for application to climate problems: the Majda–Timofeyev–Vanden-Eijnden (MTV) method and Hasselmann’s method.

Following Kurtz (1973), the MTV method was developed in Majda et al. (2001, 2003, 2006). All deterministic and stochastic corrections to the truncated equations are predicted by the theory, and the effective coefficients of the reduced equation are explicitly derived. In Franzke et al. (2005) and Franzke and Majda (2006), the MTV method was applied to quasigeostrophic (QG) models of extratropical atmospheric low-frequency variability (LFV). Atmospheric LFV is a phenomenon manifest at the planetary scale, characterized by a maximum in power at around wavenumbers 2, 3, and 4 in the wintertime extratropical troposphere, corresponding to time scales of weeks to months (e.g., Pandolfo 1993). Reduction/parameterization methods are particularly relevant for understanding LFV because LFV appears to be determined by only a few dynamically important modes (e.g., D’Andrea and Vautard 2001; Kondrashov et al. 2006). The MTV models reproduced many of the salient statistical features of the full model, but tuning of the coefficients was required to eliminate climate drift in the reduced models. A drawback of the MTV method as applied in these papers is the limitation that the climate–weather system be multilinear (although this assumption can be relaxed as it is not inherent to the theory).

At the heart of Hasselmann’s method is deterministic averaging, which was first used in early celestial mechanics applications but not rigorously justified until much later (Bogolyubov and Mitropolskii 1961). Khasminskii (1966) applied deterministic averaging to a system driven by a fast stochastic process independent of the slow variable and justified the emergence of a stochastic correction to the deterministic averaged solution. By comparison, Hasselmann (1976) proposed that the effective dynamics of a deterministic, coupled slow–fast system is determined by a multiplicative noise correction to the averaged forcing. This latter approximation has only been recently rigorously justified in Kifer (2003), but under more restrictive conditions than proposed by Hasselmann. In Arnold et al. (2003), Hasselmann’s method was formalized and expanded to include a hierarchy of deterministic and stochastic reduced equations and applied to an idealized atmosphere–ocean model.

In this study, Hasselmann’s method is adapted for use in complex climate models and applied to a QG model of atmospheric LFV. This paper is organized as follows. The limit theorems underlying the Hasselmann equations are detailed in section 2. The quasigeostrophic model of LFV [the Kravtsov et al. (2005, hereafter KRG05) model] to which the reduction methods will be applied is described in section 3. The reduced Hasselmann equations of the KRG05 model are derived in section 4. A statistical comparison of the reduced and unreduced models of LFV is detailed in section 5. Conclusions are presented in section 6.

2. Limit theorems underlying Hasselmann’s method

Hasselmann’s method is based on rigorously justified limit theorems that predict the effective dynamics of the (multidimensional) slow variable in a system with coupled slow and fast scales. These theorems are briefly considered below; for further details, see Arnold et al. (2003), Vanden-Eijnden (2003), Weinan et al. (2005), and Monahan and Culina (2010, manuscript submitted to J. Climate).

The starting point of Hasselmann’s method is the coupled system of ODEs:

\[
\begin{align*}
\dot{x}_f &= f(x, y), \quad x_0 \in \mathbb{R}^d, \quad \text{(slow, climate ODE),} \\
\dot{y}_f &= g(x, y), \quad y_0 \in \mathbb{R}^m, \quad \text{(fast, weather ODE),}
\end{align*}
\]  

(1)

where \(0 < \epsilon \ll 1\) is the ratio of the \(y\) to \(x\) time scales and \(f\) and \(g\) are \(O(1)\). On the \(O(1)\) climate time scale, only the statistics and not the particular path of the weather variable \(y\) are important to the evolution of the climate variable \(x\). That is, the effective climate is driven by the statistics of the weather equation

\[
\dot{y}_s = g(x, y_s), \quad y_0 \in \mathbb{R}^m,
\]  

(3)

where the superscript \(s\) denotes that the climate variable is held fixed and time has been fine-grained as \(s = \epsilon t\).
This equation approximates the weather dynamics on the weather time scale over which the climate variable does not change significantly.

Assuming ergodicity of the weather dynamics, the weather variable can be averaged out of the climate variable tendency:

$$\lim_{t \to \infty} \frac{1}{T} \int_0^T f(x, y_t) \, dt = \int f(x, y) \mu_y(dy) = \bar{f}(x), \quad (4)$$

where $\mu_y(dy)$ is the invariant distribution of $y$ conditioned on $x$ [derived from Eq. (3)]. The averaged function $\bar{f}$ drives an effective climate ODE:

$$\dot{x}_t = \bar{f}(x), \quad x_0 \in \mathbb{R}^d, \quad (5)$$

the solution of which is denoted the (A) approximation. On the $O(1)$ time scale, the (A) approximation $X$ and the climate variable $x$ are close. That is, the (A) approximation can be used for “short-term climate forecasts.”

However, the (A) approximation cannot break down on longer time scales, which may be of interest for climate variability. In particular, the (A) approximation may not accurately capture the distribution of the climate variable. Although both the unreduced and reduced models are deterministic, higher-order corrections to the averaged climate are stochastic. Specifically, the best approximations of the climate component of a deterministic, chaotic climate–weather system are given by diffusion processes (i.e., solutions of SDEs).

Under the assumption of a sufficiently mixing weather variable (which implies that the weather variable decorrelates sufficiently rapidly), it was justified in Khasminskii (1966) through application of the central limit theorem (CLT) that the $\sqrt{c}$-scaled error in the (A) approximation of the full model climate variable is (approximately) a Gauss–Markov (or Ornstein–Uhlenbeck) process (here denoted as $\xi$). Specifically,

$$\frac{x_t^\varepsilon - x_t}{\sqrt{c}} \overset{D}{\to} \xi_t, \quad (6)$$

$$x_t^\varepsilon \overset{D}{=} x_t + \sqrt{c} \xi_t, \quad (7)$$

where $\overset{D}{\to}$ denotes approximate equality in distribution. This approximation, consisting of a stochastic correction to the averaged solution, is denoted the (L) approximation.

The Gauss–Markov process is driven by the averaged solution, but the averaged solution is independent of $\xi_t$; specifically,

$$\dot{x}_t = \bar{f}(x_t), \quad x_0 \in \mathbb{R}^d, \quad (8)$$

$$\xi_t = \nabla_x \bar{f}(x_t) \xi_t + \sigma(x_t) W_t, \quad \xi_0 = 0, \quad (9)$$

where $W_t$ is white noise, $\nabla_x \bar{f}$ is the Jacobian of $\bar{f}$ with respect to the slow variable $x$, and $\sigma$ is the nonnegative matrix square root of

$$\sigma(x) \sigma(x)^T = 2 \int_0^\infty \mathbb{E} \{ f(x, y_t^\varepsilon - \bar{f}(x)), f(x, y_0^\varepsilon - \bar{f}(x)) \} \, ds, \quad (10)$$

where the expectation (covariance) is found with respect to the joint distribution of $y_t^\varepsilon$ and $y_0^\varepsilon$. Equation (9) and all other SDEs are interpreted in the Ito sense, unless otherwise indicated. On the $O(1)$ time scale, on which the (L) approximation has been theoretically validated, the (A) approximation resembles a low-pass filtered version of the (L) approximation.

The (L) approximation is an improvement over the (A) approximation on short time scales but is not guaranteed to capture the long-term dynamics. An approximation valid on the long, $O(e^{-1})$ time scale, denoted the (N) approximation, is formally derived for the coupled slow–fast system at the level of the Fokker–Planck equation (FPE) in Just et al. (2001):

$$\dot{z}_t = \bar{f}(z_t) + \sqrt{c} \sigma(z_t)^T W_t, \quad z_0 \in \mathbb{R}^d, \quad (11)$$

which is interpreted in the Stratonovich sense. In Ito form, Eq. (11) becomes

$$\dot{z}_t = \bar{f}(z_t) + c B(z_t) + \sqrt{c} \sigma(z_t)^T W_t, \quad z_0 \in \mathbb{R}^d, \quad (12)$$

where the noise-induced drift is

$$B(x) = \int_0^\infty \mathbb{E} \{ (\nabla_x f(x, y_t^\varepsilon) - \nabla_x \bar{f}(x))(f(x, y_0^\varepsilon) - \bar{f}(x)) \} \, dt, \quad (13)$$

and $\bar{f}$ and $\sigma$ are defined above. In contrast to the (L) approximation, the averaged climate trajectory is not resolved in the (N) approximation, as the averaged climate and noise terms are now mutually interacting within one SDE. The (deterministic) noise-induced drift forcing arises because ($\delta$-correlated) white noise is used to approximate fast-evolving processes with finite lag-correlation time scales (Penland 2003).

The original Hasselmann approximation (Hasselmann 1976), denoted the (N) approximation, is obtained by suppressing the noise-induced drift term $B$ in the Ito-interpreted Eq. (12):

$$\dot{z}_t = \bar{f}(z_t) + \sqrt{c} \sigma(z_t)^T W_t, \quad z_0 \in \mathbb{R}^d. \quad (14)$$

The (N) approximation is strictly valid only on short time scales (Kifer 2003). However, the numerical integration
of the (N) model is made much simpler by the exclusion of $B$; in particular, the (N) model forcing has no derivatives.

The (N) model may be further simplified by suppressing the state dependence of the diffusion matrix $\alpha$. This approximation [denoted (W)] is driven by climate–weather feedback through the averaged climate forcing and by an additional “one-way feedback” of the fast onto the slow through the effect of additive Gaussian white noise on the climate forcing. Although not a theoretically justified approximation, the (W) approximation is intuitively at an intermediate level between the (N) and (L) approximations.

In theory, the (N) approximation is not an improvement over the (L) approximation on short time scales. Neither of these approximations is strictly valid on long time scales, but the (N) model alone can potentially generate a non-Gaussian climate distribution in the case that the (A) approximation is Gaussian distributed. For example, the (A) approximation of a damped double-well system does not simulate well-hopping in such a system with small periodic forcing coupled to the (fast) Lorenz system (Just et al. 2001). The (L) approximation cannot simulate jumps between wells because it is determined as a Gaussian stochastic correction superposed on the (A) model solution, which is confined to one well. On the other hand, the (N) model allows for the interaction of the slow averaged forcing and stochastic correction, permitting the possibility of well-hopping. The (W) approximation also permits the possibility of well-hopping, but the exit time scale from a well, for example, may be misestimated because of the neglect of the modulation of noise strength by the climate state.

The hierarchy of Hasselmann approximations is summarized in Table 1. Note that analytic expressions exist for Hasselmann’s models at all levels of approximation, although closed-form analytic expressions for the various integrals will not be available in general.

### Table 1. Hasselmann’s reduced equations.

<table>
<thead>
<tr>
<th>Approximation</th>
<th>Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>$\dot{x} = f = \int f(x,y) \mu_x(y) , dy$</td>
</tr>
<tr>
<td>(L)</td>
<td>$z = x + \sqrt{\epsilon} \xi(x)$</td>
</tr>
<tr>
<td>(W)</td>
<td>$\dot{z} = \bar{f}(z) + \sqrt{\epsilon \alpha} W$</td>
</tr>
<tr>
<td>(N)</td>
<td>$\dot{z} = \bar{f}(z) + \sqrt{\epsilon \alpha} W$</td>
</tr>
<tr>
<td>(N+)</td>
<td>$\dot{z} = \bar{f}(z) + \epsilon D + \sqrt{\epsilon \alpha} W$</td>
</tr>
</tbody>
</table>

3. **KRG05 model of LFV**

The KRG05 model is a classic two-layer QG model (Pedlosky 1987) of the form

$$\frac{\partial q_k}{\partial t} = -J(\psi_1, q_k) + \mathcal{S} - \mathcal{D}, \quad (15)$$

where $\psi_1, \psi_2$ are respectively the upper- and lower-layer streamfunctions, $q = L(\psi_1, \psi_2)$ is the QG potential vorticity, and $\mathcal{S}$ and $\mathcal{D}$ are respectively forcing and dissipation terms described in KRG05. The model is discretized on a rectangular $128 \times 41$ grid, requiring that $128 \times 41 \times 2$ equations be numerically solved.

The model simulates an idealized midlatitude jet. Depending on the strength of the bottom friction, the model has been shown to exhibit a range of dynamical behavior, including irregular meridional displacement of the jet between two quasi-stationary states (Fig. 1). A leading stationary mode of variability captures much of this displacement and is similar to its counterpart in the real atmosphere, the Arctic Oscillation (AO).

In this paper, Hasselmann’s method is applied to the KRG05 model at a value of the bottom drag (corresponding to a spin-down time scale of 6.7 days) for which there is pronounced multiple regime behavior and associated skewness of the stationary mode distribution. The model with these chosen parameter settings will be referred to as the (full or unreduced) KRG05 model.

**FIG. 1.** Time series of the jet-center position: $K^{-1} = (a) 7.7, (b) 7.1, (c) 6.7, (d) 6.3,$ and (e) 5.9 days. Thick solid lines mark the climatological location of the jet in the high- and low-latitude states. Figure from KRG05.
4. The Hasselmann equations of the KRG05 model

The algorithms (cf. appendix A) for integrating the Hasselmann equations require minimal changes to the original KRG05 model algorithm. Furthermore, these algorithms can be readily adapted to apply Hasselmann’s method to other complex climate models. Although the equations need not be in such a simple form to apply these algorithms, the KRG05 equations can be written as

\[
\frac{\partial \psi_1}{\partial t} = F_1(\psi_1, \psi_2), \quad \text{and} \quad \frac{\partial \psi_2}{\partial t} = F_2(\psi_1, \psi_2).
\]

(16)

(17)

Letting \( p = (\psi_1, \psi_2) \), Eq. (17) may be rewritten as

\[
\frac{\partial p}{\partial t} = F(p) = [F_1(p), F_2(p)].
\]

(18)

There are two primary steps in the systematic parameterization of the fast/weather processes: first, the equations are projected onto a suitable basis, on which the separation into slow and fast variables is defined; second, the reduction theorems are applied to derive a system in the slow variable alone.

a. Partitioning into the slow–fast system

Potentially, there are several bases, such as empirical orthogonal functions (EOFs; Preisendorfer 1988) and principal interaction patterns (PIPs; Kwasniok 1996), by which a system may be efficiently partitioned. Each choice of basis has its strengths and weaknesses. For example, a few leading EOFs may capture a large fraction of the variance, but EOFs do not generally correspond to dynamical modes (e.g., Monahan et al. 2009), nor are the leading dynamical modes necessarily contained in the space spanned by the leading EOFs (e.g., Crommelin and Majda 2004). However, the leading EOFs (i.e., the stationary mode EOF and the propagating wave-4, -5, and -6 EOFs) of the KRG05 model are close to eigenvectors of the full system linearized about the climatology (KRG05). To facilitate comparison between the reduced and unreduced models, EOFs are used here as the basis for reduction.

The set of EOFs \( \{e_i(x, y)\} \) of the streamfunction \( p(x, y, t) \) satisfies

\[
p(x, y, t) = \bar{p}(x, y) + \sum \alpha_i(t) e_i(x, y),
\]

(19)

where \( \bar{p} \) denotes the time-mean streamfunction and

\[
\alpha_i(t) = \langle e_i, p \rangle.
\]

(20)

where \( \langle \cdot, \cdot \rangle \) denotes an inner product and the \( \alpha_i(t) \) are the principal component (PC) time series. The \( L_2 \) norm of the streamfunction field \( \| \psi \|_2^2 = \int \psi^2 \, dA \), where \( dA \) is an infinitesimal area of the channel, is used here for mode reduction of the KRG05 model instead of the total energy norm. The total energy norm ensures the stability of the nonlinear operator and that the total energy associated with a conservative equation remains conserved by projection onto any truncated set of EOFs (Selten 1995; Ehrendorfer 2000; Franzke et al. 2005). In the system under consideration, the total energy norm is not necessarily superior to other norms at conservation of total energy because of the presence of nonconservative terms driving the forced-dissipative dynamics in the OQ equations.

Using the dot product to represent the above defined inner product, the projection of the governing equation onto the set of EOFs \( \{e_i\} \) yields a system of independent ODEs for the PC time series \( \alpha_i(t) \):

\[
\frac{d\alpha_i}{dt} = e_i \cdot [F(\bar{p} + \sum \alpha_j e_j)], \quad \alpha \in \mathbb{R}^{d+m}.
\]

(21)

The EOFs/PCs are divided into slow-evolving and fast-evolving modes based on their autocorrelation time scales, such that \( \sum \alpha_i e_i = \sum_{i \in S_s} a_i e_i + \sum_{j \in S_f} b_j e_j \); \( S_s \cap S_f = \emptyset \), where \( S_s \) and \( S_f \) are the index sets of the slow and fast variables, respectively (Majda et al. 2001; Berner 2005). As in Berner (2005), the autocorrelation time scale of a PC \( \alpha_i \) is defined as

\[
\tau_{\alpha_i} = \frac{1}{\alpha_i^2} \int_0^\infty \| \alpha_i(t) \alpha_i(0) \| \, dt.
\]

(22)

Applied to the KRG05 model, there is an obvious division into slow and fast modes (Fig. 2). In particular, the propagating wave-4 mode (determined by the projection of the full system onto the first two EOFs) and the stationary mode, which together describe nearly two-thirds of the system’s variance, comprise the set of slow climate modes while the rest of the modes comprise the fast weather modes. This also represents the division into planetary- and synoptic-scale modes.

ODE (21) can be rewritten to reflect the partitioning into slow (a) and fast (b) variables as

\[
\frac{da_i}{dt} = e_i \cdot [F(\bar{p} + \sum a_k e_k + \sum b_k e_k)], \quad a_0 \in \mathbb{R}^{d=3},
\]

(23)

\[
\frac{db_j}{dt} = e_j \cdot [F(\bar{p} + \sum a_k e_k + \sum b_k e_k)], \quad b_0 \in \mathbb{R}^{m=128 \times 41 \times 2 \times 3},
\]

(24)
where the time component of the slow modes is given by 
\[ a = \text{(wave-4 PC1, wave-4 PC2, stationary PC)} \].

Although there is an obvious division into slow and fast modes, Hasselmann approximations of Eq. (23) with \( a = \text{(stationary PC)} \) (i.e., with the set of climate modes restricted to the stationary mode alone and wave 4 included among the weather modes) will also be considered. As the time scale separation between the stationary and wave-4 modes is approximately unity, \( \epsilon = 1 \) in this case.

System (23) and (24) is of the form

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y), \quad x_0 \in \mathbb{R}^d, \quad \text{and} \\
\frac{dy}{dt} &= g(x, y), \quad y_0 \in \mathbb{R}^m,
\end{align*}
\]

where it is known only that \( f/g \) is \( O(\epsilon) \). However, the effective Hasselmann approximations can be derived directly from this system, without knowing the orders of \( f \) and \( g \) and without any rescaling of time. To see this, assume that \( f \) is \( O(1) \) and \( g \) is \( O(\epsilon^{-1}) \). Recall that the “weather equation” is found on fine-grained time \( s = t/\epsilon \). If time \( t \) is not fine-grained in determining the weather statistics, then the lag covariance is

\[
\begin{align*}
\sigma(x)\sigma(x)^T &= 2 \int_0^\infty \mathbb{E} \{ f[x, y_s - \tilde{f}(x)], f[x, y_0 - \tilde{f}(x)] \} \, dt \\
&= 2\epsilon \int_0^\infty \mathbb{E} \{ f[x, y_s - \tilde{f}(x)], f[x, y_0 - \tilde{f}(x)] \} \, ds \\
&= \epsilon \sigma(x)\sigma(x)^T,
\end{align*}
\]

where \( \sigma(x) \) is determined by Eq. (10). Thus, the (N) equation (for example) is

\[
z_i' = \tilde{f}(z_i') + \sigma(z_i')W_i, \tag{27}
\]

which differs from Eq. (14) only by the absence of the (explicit) \( \epsilon \)-scaling of the noise term.

If \( f \) is \( O(\epsilon) \) or \( O(\epsilon^{-1}) \) in Eq. (25), then the same effective Eq. (27) is derived using \( f \) and \( g \) and without rescaling time. Similarly, \( \epsilon \) is dropped from the other Hasselmann stochastic approximations. The Hasselmann approximations of the KRG05 model will be derived in this fashion, without the explicit \( \epsilon \)-scaling of the correction terms.

b. Integrating the Hasselmann equations

As the Hasselmann stochastic differential equations involve expectations and their time integrals, methods for their efficient approximation must be developed. In this section, we apply methods for efficient averaging developed in Fatkullin and Vanden-Eijnden (2004) and Weinan et al. (2005). We first introduce notation that is used to describe the numerical integration of the Hasselmann equations. A time step of the effective climate equation, or a macrotime step, is denoted \( \Delta t \), and a time step of the weather equation, or a microtime step, is denoted \( \delta t \). For each macrotime step, the number of integration microtime steps is denoted \( N \). Of these \( N \) microtime steps, \( N_1 \) are treated as “spinup” to bring the weather trajectory onto its \( (a\text{-dependent}) \) weather attractor. An ensemble of \( R \) weather realizations is initialized at the beginning of the climate integration by choosing \( R \) well separated points from a long weather integration (with fixed climate variable).

1) (A) APPROXIMATION

The averaged equation in the climate variable alone can be written succinctly as

\[
\begin{align*}
\frac{da_i}{dt} &= \mathbb{E} \left[ \mathbb{E} \left[ F \left( \mathbb{E} \left[ f + \sum a_k e_k + \sum b_k e_k \right] \right) \right] \right] \, d\mu_a(\mathbf{b}), \\
&= a(0) \in \mathbb{R}^d. 
\end{align*}
\]

The distribution \( \mu_a \) cannot be expressed in an explicit functional form for all but the simplest systems (of which the KRG05 model is not one). However, by the (assumed) ergodicity of the weather equation

\[
\frac{db^a_i}{ds} = \mathbb{E} \left[ F \left( \mathbb{E} \left[ f + \sum a_k e_k + \sum b_k e_k \right] \right) \right], \quad b^a(0) \in \mathbb{R}^m,
\]

the effective tendency forcing for a particular PC value can be computed as the time average over realizations of the fast equation:
\[
F_i = \lim_{T \to \infty} \frac{1}{T} \int_0^T e_i \cdot \left\{ F \left[ \bar{\mathbf{p}} + \sum a_k e_k + \sum b_k^e(s) e_k \right] \right\} ds.
\]

The averaged equation becomes

\[
\frac{da}{dt} = F_i(a), \quad a(0) \in \mathbb{R}^d.
\]

Thus, the numerical integration of the effective system is divided into an integration of the effective climate equation and an integration of the weather equation for fixed climate.

For each given fixed climate value, it would take a large number of weather iterations to obtain a converging solution of the averaged forcing function (30). However, as justified in Fatkullin and Vanden-Eijnden (2004) and Weinan et al. (2005) for sufficiently large-scale separation and small macrotime step, only a small sampling of the weather attractor for each fixed climate value is needed to obtain an approximation of the averaged forcing sufficiently good that the climate solution of the full system is well approximated by the averaged model solution. This is because averaging is occurring over successive macrotime steps, over which the climate variable does not change significantly, necessitating only a small amount of averaging for each fixed climate variable. With the reinitialization after each macrotime step of

\[
b^{(t)}_{r,i} = b^{(t-\Delta t)}_{r,i},
\]

which ensures that the distance between successive weather attractors on which the weather solutions settle is small, \(O(\varepsilon^{-1})\) efficiency gains are possible. Fatkullin and Vanden-Eijnden (2004), for example, averaged a toy climate model with a large-scale separation of \(\varepsilon = 2^{-7}\) to obtain an accurate approximation of the climate statistics with an efficiency gain of a factor of \(2^5\).

It might seem necessary to explicitly resolve all the fast modes in order to carry out the numerical integration of the averaged model, given that the fast values are to be averaged out of the slow equation. Since there are more than 10,000 fast PCs in the KRG05 model, the cost of explicitly resolving the fast modes would be enormous. However, the averaged slow tendency forcing can be found using only the (discretized) full model PDE and the slow mode ODE. To see this, note that the full model tendency forcing function \(F\) and its argument are the same in Eqs. (29) and (30). The function \(F \left[ \bar{\mathbf{p}} + \sum a_k e_k + \sum b_k^e(s) e_k \right] \) is simply determined from integration of the full model with the slow modes \((a_i)\) fixed, without explicitly resolving the fast modes. Iterates of \(F\), not the fast modes, are saved, to be projected onto the slow modes and time averaged.

The alterations to the original model code to implement such a scheme are minimal. The scheme can be made more versatile because the explicit full model tendency forcing \(F\) is not needed (see appendix A). This versatility is of practical significance, as \(F\) is not explicitly defined in the KRG05 algorithm, or in models of similar or greater complexity for which it may be impossible to explicitly derive \(F\).

2) (N) APPROXIMATION

Few additions to the (A) algorithm are needed to obtain the (N) algorithm. The saved tendency forcings, which are used to compute the averaged tendency, are also used to compute the diffusion coefficient \(\mathbf{\alpha}\), according to Eq. (27).

However, the (N) model cannot be efficiently integrated in cases in which the weather does not mix sufficiently quickly and the scale separation is not large without simplifying the formula for \(\mathbf{\alpha}\). In contrast to the computation of expectations (including the lag covariance), the time integral of the lag covariance must be well approximated at each macrotime step and for each ensemble member. With a rapidly mixing weather component, \(\mathbf{\alpha}\) can be efficiently computed without simplification because the lag covariance tends to zero sufficiently fast. In the KRG05 model, the weather may mix sufficiently rapidly to ensure the validity of the stochastic approximations, but its decorrelation time scale (and hence the integral time scale for determining \(\mathbf{\alpha}\)) is on the order of days. This algorithmic challenge is addressed by noting that the integrated lag covariance for large lag (approximating \(\mathbf{\alpha} \alpha^T\)) is well approximated by the integrated lag covariance for small lag multiplied by an extrapolation factor.

Figure 3 shows time series of the integrated lag covariance, indexed by the upper limit of the integral, for the KRG05 model with \(a = \) (stationary PC). This series appears to converge for a lag time scale of 2 days. It would be prohibitively costly to integrate the weather equation for two days at each climate realization, as the macrotime step is on the order of hours. However, the upscaled (by a factor of 36) time series of lag covariances integrated up to only 1 h is strongly correlated with the series integrated up to 2 days. Thus, the computation of \(\mathbf{\alpha}\) in the (N) model only requires the integration of the weather equation at each climate realization up to a lag of 1 h and the use of an appropriate scaling factor. Using the matrix norm of the integrated lag covariance, similar results hold in the case that we take \(a = \) (wave-4 PC1, wave-4 PC2, stationary PC).

As the (W) model is not rigorously justified, with \(\mathbf{\alpha}\) defined only as being independent of climate, multiple additive noise models may be defined (for a multidimensional
climate). A simple additive noise model, to be called the
(W1) model, may be defined with \( s = g I \), where \( g \) is
a scalar and \( I \) is the identity matrix. It is natural to define
\( g \) as the square root of the norm of the time-integrated
lag covariance for large lag. An alternative formulation
of the diffusion matrix is to find the average \( h/C_1i \)
over the range of climate realizations as \( s W_2 \equiv (h/s)T \)
where \( s W_2 \) is the diffusion matrix of the (W2) model.

5. Results of Hasselmann’s method applied to
KRG05 model

The multiple Hasselmann models—(A), (L), (W), (N),
and (N+)—form a hierarchy of models ordered by the
time scales over which they are expected to provide good
approximations to the climate variables (cf. Table 1). For
a general dynamical system, it is theoretically justified
that the (L) and (N) models are superior to the (A)
model on bounded time scales and that only the (N+)
model is valid on the long, \( O(\epsilon^{-1}) \) time scale. Of course,
for a system not perfectly satisfying the theoretical as-
sumptions, the (A), (L), and (N) approximations may be
accurate over longer time scales. The (L) approximation
in particular is given by the solution of the (A) model
plus a correction given by a Gauss–Markov process with
mean zero [Eq. (7)]. Special cases of the (L) model have
been used to successfully simulate atmospheric LFV in,
for example, Farrell and Ioannou (1995), Penland and
Sardeshmukh (1995), and Whitaker and Sardeshmukh
(1998). However, in application to the KRG05 model,
it can be deduced from the relevant (A) approxima-
tions (below) that the (L) approximations are unable to
capture the (pronounced) non-Gaussian variability of
the jet position. Thus, the (L) approximations are not
considered in this study. The (N+) SDE is also not
solved because of the computational complexity asso-
ciated with derivatives in its forcing. The (A), (W), and
(N) approximations analyzed below are found using as
small time steps as are required to obtain convergence
of the numerical solutions of the reduced models. A
future publication will be concerned with maximizing
efficiency in deriving the Stratonovich-interpreted (N+)
approximation in particular.

a. Averaged approximations (A)

Averaging is considered first since it serves as the basis
for all models in the Hasselmann hierarchy of approxi-
mations. The one-variable (A) ODE in the stationary
mode alone [i.e., with \( a = \) (stationary PC)] has a fixed
point attractor, but the dynamics of the three-variable
(A) approximation [i.e., with \( a = \) (wave-4 PC1, wave-4
PC2, stationary PC)] is nontrivial. Its wave-4 mode is
dominated by a low-frequency oscillation on the order of
200 days, which corresponds to the “fundamental
frequency” of wave 4 of the unreduced model (KRG05);
however, the averaged wave-4 mode does not have a
high-frequency (noisy) component (Fig. 4). The aver-
ged model jet resides at latitudes corresponding to the
high-latitude regime of the full model jet and does not
display transitions to a low-latitude regime, independent
of whether the simulation is initialized in the low- or
high-latitude regime (as reflected in the narrow PDF of

![Fig. 3](image1.png)

![Fig. 4](image2.png)
the stationary mode PC; not shown). In deriving this averaged solution, the macrotime step is \( \Delta t = 3 \text{ h} \), the microtime step is \( \delta t = 10 \text{ min} \), the spinup is \( N_s = 5\delta t \), the number of ensemble members is \( R = 18 \), and the number of averaging iterates per ensemble member is \( N = 1 \) for a gain of a factor of 3 in computation time over the full model using \( R = 18 \) processors in parallel. The number of weather ensemble members over which the average is taken is large but necessary to compute a good approximation of the average. In particular, spurious excursions to the low-latitude regime not characteristic of the true averaged dynamics are altogether eliminated using a sufficiently large \( R \).

As discussed previously, we do not expect that the (A) approximation will generally capture the correct long-term behavior of the climate variables; for this, noise (and noise-induced drift) must be included. Nevertheless, investigation of this forcing can provide dynamical insight. Indeed, the potential function \( F(a) = -\int \bar{F}(a) \, da \) of the one-variable (A) ODE reveals the importance of the averaged function to the non-Gaussianity of the jet dynamics (in the one-variable case). As depicted in Fig. 5, a trough in the potential approximately corresponds to the location of the high-latitude regime of the unreduced model, at which the jet mostly resides, and a plateau in the potential approximately corresponds to the location of the low-latitude regime. The shape of the potential reflects that the meridional shifting of the jet slows down (or reaches a “sticking point”) in a low-latitude band, which is consistent with the skewness of the unreduced model jet. Based on the potential of the one-variable averaged (A) equation, it is expected that this multiple regime behavior can be generated in the one-variable stochastic models with state-independent noise.

**b. Diffusion approximations (W) and (N)**

Indeed, a one-variable (W) SDE, consisting of averaged forcing and additive Gaussian white noise, suffices to generate the full model jet statistics, which are characterized by infrequent and persistent visits of the jet to the low-latitude regime. In particular, it is evident from the pronounced tail of the reduced model PDF and the long lag-correlation time scale of its autocorrelation function (ACF) that the multiple regime behavior of the jet is captured (Fig. 5). The climate mean is also well captured, with a climate drift of 0.17\( \sigma \), where 0.17 is the ratio of the mean of the reduced model stationary mode PDF to the standard deviation \( \sigma \) of the full model stationary mode PDF. The statistics of the one-variable multiplicative noise (N) SDE do not differ significantly from those of the additive noise SDE. The (N) model...
statistics do not change by using a larger upper time lag of 6 h (and correspondingly smaller extrapolation factor) to determine the diffusion coefficient, nor do the statistics change by averaging over more iterations (e.g., with $R = 30$). Note that a one-variable (L) stochastic approximation cannot yield the jet regime behavior because it is determined as the sum of the (fixed point) averaged solution and a Gaussian-distributed correction.

It is suggested by the nature of the potential of the one-variable averaged equation that nonlinearity in the averaged forcing function has a dominant role in driving the regime behavior, which is further supported by the quality of the (W) approximation in simulating jet shifting. That the diffusion matrix is in fact effectively independent of the state variable is illustrated in Fig. 6. Thus, in this model, jet regime behavior is governed by nonlinearity in the deterministic, averaged forcing of the stationary mode. The averaged forcing captures the important climate–weather interactions between the stationary mode and the unresolved/weather modes (i.e., wave-4 and the synoptic-scale eddies) that give rise to the jet shifting. Additive noise is the source of variance or energy exciting the reduced system to sample the full range of these interactions. Note that the success of the one-variable averaged equation does not contradict the results of appendix B, in which it is determined that the stationary mode self-interaction is unimportant to the jet regime behavior in the unreduced model. Rather, averaging induces additional forcing terms in the stationary mode, on top of the stationary mode self-interaction, that account for the important effects of the weather modes in driving this behavior.

With wave-4 defined along with the stationary mode as climate modes [i.e., with $a = (\text{wave-4 PC1, wave-4 PC2, stationary PC})$, there is a clearly defined scale separation between the climate and weather variables. Nevertheless, the untuned three-variable (N) approximation is unstable. It is possible to obtain good three-variable (N) approximations by scaling down the noise strength. In particular, a scaling parameter $\lambda$ will be assigned to the noise forcing, such that the noise scaling in the three-variable stochastic models becomes $\sqrt{\lambda}\|\sigma\|$, and $\lambda$ will be adjusted toward finding the optimal correspondence between the unreduced and reduced model statistics. This procedure is consistent with a tuning of one parameter according to the MTV “minimal regression” tuning procedure (Franzke and Majda 2006; Strounine et al. 2010).

The (tuned) three-variable (N) model with $\lambda = 0.1$ yields a reasonably good approximation of the climate statistics of the unreduced model. Jet regime behavior is generated, but the variance and skewness of the stationary mode statistics that characterize the regime behavior. With $\lambda = 0.1$, both additive noise approximations by scaling down the noise strength. In particular, a scaling parameter $\lambda$ will be assigned to the noise forcing, such that the noise scaling in the three-variable stochastic models becomes $\sqrt{\lambda}\|\sigma\|$, and $\lambda$ will be adjusted toward finding the optimal correspondence between the unreduced and reduced model statistics. This procedure is consistent with a tuning of one parameter according to the MTV “minimal regression” tuning procedure (Franzke and Majda 2006; Strounine et al. 2010).

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noise term is not intrinsic to this behavior. The (W1) model is valid if with respect to the joint distribution of the lagged weather variables the cross correlations of the three climate forcings are zero (simply, the crossed climate forcings are not statistically linked via the weather) and the autocorrelation scales are equal to each other. However, these are very strong conditions; instead, it is more likely that the stochastic reduced models are not (very) sensitive to the structure of the noise term as it is not of first-order importance to the jet regime behavior. Rather, as in the one-variable case, the deterministic averaged function captures the interactions among the planetary- and synoptic-scale modes (cf. appendix B) that generate the multiple jet regimes.

6. Conclusions and discussion

In this study, a parameterization method (Hasselmann’s method) for use in complex climate models was developed from limiting theorems that predict the effective slow dynamics in systems with coupled slow and fast variables. A quasigeostrophic model of midlatitude low-frequency variability (the KRG05 model) was used to derive deterministic and stochastic models in the slowly evolving modes alone. Adaptable, efficient algorithms for integrating these reduced models were developed that require minimal changes to the unreduced KRG05 model algorithm. In particular, the parameterized approximations, which are a function of the fast modes and the tendency forcing of the slow modes of the unreduced model, were computed without the need to explicitly resolve these terms.

The average of the climate forcing with respect to the weather distribution conditioned on the state of the climate captured the important interactions between resolved and unresolved processes. Consistent with the potential of the one-variable averaged forcing, the one-variable additive noise (W) model generated the correct jet regime statistics, despite the absence of scale separation. With an additional resolved mode (wave 4) and with a larger scale separation between resolved and unresolved modes, the additive noise models sufficed to generate the regime behavior. Thus, in both the one-variable and three-variable cases, deterministic nonlinearity (through averaged forcing), and not multiplicative noise, is responsible for the model LFV. Previous model reduction studies using the quasigeostrophic model of Marshall and Molteni (1993) (e.g., D’Andrea and Vautard 2001; Kondrashov et al. 2006; Franzke and Majda 2006) agreed on the point that a state-dependent closure was
needed in the case that the set of resolved modes was of a sufficiently small size, but only the latter study found that a state-dependent noise term was important. However, the analysis of a similarly constructed reduced model (of the same dimension) in Strouline et al. (2010) provided evidence that multiplicative noise was not needed to reproduce the model behavior. Based on analysis of real atmospheric data, Sura et al. (2005) suggested that a multiplicative noise SDE represents a dynamical paradigm of LFV; the results of the present study and those described above suggest that this is not the case for the quasigeostrophic models considered (nor was it the case in a GCM; Berner 2005).

Although the untuned one-variable stochastic models generated excellent approximations without scale separation between the resolved stationary mode and the unresolved wave-4 mode, there is an order of magnitude scale separation between these two planetary-scale modes and the synoptic-scale eddies. This large-scale separation, which may have been an important factor in the success of the one-variable reduced models (and in the success of the tuned three-variable models), is not present in the real atmosphere. However, the MTV models of LFV generated reasonably good approximations of the full model LFV despite the scale separation approaching unity (Franzke et al. 2005; Franzke and Majda 2006). An application of Hasselmann’s method in Monahan and Culina (2010, manuscript submitted to J. Climate) shows for a model of coupled atmosphere–ocean boundary layers that there is good statistical agreement between the reduced and unreduced models despite a scale separation of \( \epsilon \approx 0.7 \). In this boundary layer model, the slowly evolving surface temperatures are brought into excellent agreement with the full model temperatures by (artificially) reducing the scaling of the diffusion term, as was the case in reduction of the KRG05 model. That such a retuning, similar to the “minimal regression” rescaling used in the MTV method to eliminate climate drift, is necessary is a consequence of applying Hasselmann’s method to systems without arbitrarily large scale separation.

A more thorough examination of the reduced dynamics would entail integration of the full suite of reduced models—the (A), (L), (W), (N), and (N+) models. These models form a hierarchy from the least accurate (A) model to the (in principle) most accurate (N+) model, which best accounts for the climate–weather interactions. The one-variable (W) model was sufficient to generate the climate statistics of the unreduced model, but there is room for improvement in the three-variable case; in particular, the untuned three-variable (N) approximation

![Stationary mode statistics of (W1) & (W2) PDF](image1)

![Wave-4 statistics of (W1) & (W2) PDF](image2)

**Fig. 8.** As in Fig. 7, but for the (W1) and (W2) approximations with noise rescaled by \( \sqrt{\lambda} = 0.1 \).
is unstable and the tuned approximations have a climate drift that is strongly dependent on the scaling of the noise term. These results beg the question of whether the \((N+)\) model, with noise-induced drift correction, would better capture the climate statistics. The \((N+)\) SDE can be simulated using numerical methods for Stratonovich SDEs (e.g., Ewald and Penland 2009), but the derivatives associated with these methods may be costly to compute in high-dimensional models (as at each step the diffusion \(\sigma\) would need to be computed not just at the present climate state but at neighboring states). A future study will be concerned with addressing these stability and climate drift problems by analyzing the \((N+)\) approximation of the KRG05 model and comparing the results obtained by fitting the climate data to a Fokker–Planck equation (e.g., Berner 2005).

As developed in this paper, Hasselmann’s method does not have the computational efficiency of a traditional parameterization scheme. It is instead analogous to the superparameterization method developed in Grabowski (2001), the goal of which was to improve the computationally efficient but idealized parameterization of cloud processes through online sampling of a high-resolution cloud-resolving model. Analogously, the Hasselmann models can efficiently generate accurate approximations of the climate through on-the-fly sampling of the fast-evolving weather processes. Computational efficiency is obtained through ensemble averaging of the weather attractor, which can be reached with a small spinup time compared to the climate macrotime step. The focus of this paper was on accuracy of the Hasselmann approximations; a future study will be concerned with maximizing efficiency of these approximations in application to complex climate models (see also Culina 2009).

With additional assumptions on the climate–weather system equations [Eqs. (1) and (2)], significant computational efficiency gains are possible by applying the MTV method following Franzke et al. (2005) and Franzke and Majda (2006). Under the MTV assumption that the weather–weather forcing of the weather variable becomes dominant with increasing scale separation (among other assumptions), the limiting weather measure is independent of the climate, which allows for the explicit determination of the reduced model coefficients prior to integration of the effective equation. However, in applying the MTV method to the KRG05 model in Culina (2009), the multiple regime behavior could not be generated by the untuned (one-variable or three-variable) reduced models. Moreover, only in the one-variable case were multiple jet regimes generated by tuning, and then only with a significant amount of tuning. The additional assumptions of the MTV method appear to be too stringent for its application to the KRG05 model.

In Hasselmann (1976), Klaus Hasselmann showed that the slowly evolving component of the climate system under the influence of fast-evolving turbulent processes can be modeled as a stochastic process in a systematic fashion. Subsequent stochastic climate modeling studies utilized SDEs of a simpler form, not explicitly obtained as the limiting dynamics of a coupled fast–slow (e.g., climate–weather) system. Modern computational resources have allowed us to directly apply rigorously justified limiting theorems to model high-dimensional climate systems—but they require the development of new algorithms for their practical implementation. This paper is one of the first studies to do this, and it is the first using the limiting theorems of Hasselmann’s method. The success in this paper in applying this method and the potential for application to models of greater complexity hold promise for systematic approaches to stochastic parameterization in future studies.

Acknowledgments. SK was supported by the Office of Science (BER), U.S. Department of Energy (DOE), under Grant DE-FG02-07ER64428 and by NSF Grant ATM-0852459 at UWM. AHM was supported by the Natural Sciences and Engineering Research Council of Canada and AHM and JC were supported by the Canadian Foundation for Climate and Atmospheric Sciences.

APPENDIX A

Averaging (A) Algorithm

Here we summarize the seamless algorithm for integrating the averaged (A) equation. The initial values include time \(t_0\), streamfunction \(p_{\delta(t_0)}(0)\), and the climate state \(e_i\cdot[p_{\delta(t_0)}(0)-\bar{p}]=a_i(t_0)\). First, integrate the weather equation for fixed climate variable. This is accomplished by first advancing the streamfunction one microtime step, according to the full model:

\[ p^*(\delta t). \]

Then subtract out the climate state and add in the original climate state:

\[ p_{\delta(t_0)}(\delta t) = p^*(\delta t) - \sum e_i \cdot [p^*(\delta t) - p_{\delta(t_0)}(0)]. \]

Repeat this procedure \(N-1\) times:

\[ p^*(2\delta t) \]
The integration of the effective climate equation gives
\[ p_{a(t_0)}(2\delta t) = p^*(2\delta t) - \sum e_i \cdot \left[ p^*(2\delta t) - p_{a(t_0)}(0) \right] \]
and
\[ p_{a(t_0)}(N\delta t) = p^*(N\delta t) - \sum e_i \cdot \left[ p^*(N\delta t) - p_{a(t_0)}(0) \right]. \]

The integration of the next weather equation is initialized with
\[ p_{a(t_0+\Delta t)}(0) = p_{a(t_0)}(N\delta t) - \sum a_i(t_0)e_i + \sum a_i(t_0 + \Delta t)e_i. \]

**APPENDIX B**

Suppressing Interactions in the Full Model

The algorithm for the (A) approximation can be modified to isolate the effect of particular nonlinear interactions on jet regime behavior in the unreduced KRG05 model. In particular, the unreduced KRG05 equations are obtained with the settings \( R = N = 1, N_1 = 0, \) and \( \Delta t = \delta t \) and interactions of interest are replaced by their climatologies. If the multiple regime behavior persists without a particular set of nonlinear interactions, then these interactions are not necessary to its existence. Only nonlinear interactions are suppressed because the regime behavior is a nonlinear phenomenon (among slowly evolving modes and/or between slowly evolving and fast-evolving modes).

As the one-variable additive noise (W) model in the stationary mode alone generated the full model jet statistics (cf. section 5), it is of interest to determine the importance of the stationary mode self-interaction. In the schematic KRG05 model [Eq. (15)], the stationary mode
self-interaction \(-J(\sum a_{ij}e_{ij}, q(\sum a_{ij}e_{ij}))\) is replaced by its climatology \(-\overline{J(\sum a_{ij}e_{ij}, q(\sum a_{ij}e_{ij}))}\), where \((\cdot)\) is the time average over a long integration of Eq. (15). Based on the similarity of the model statistics with and without the stationary mode self-interaction (Fig. B1, left panels), this interaction does not impact the jet regime behavior. Figure B1 (left panels) demonstrates that the interaction between the stationary and wave-4 modes is also not necessary to the existence of jet regimes, in contrast to the conclusions of KRG05 that the multiple regimes arise from a “weak” nonlinear interaction (i.e., interaction on long time scales) between these modes.

If the jet regimes persist without all nonlinear interactions and self-interactions of the two planetary-scale modes (i.e., without the stationary mode self-interaction, the wave-4 mode self-interaction and the stationary wave-4 interaction), then this would imply that synoptic eddies are of first-order importance to the nonlinear dynamics of the model jet. Although without these interactions the stationary mode distribution is approximately Gaussian (suggesting that regime behavior is eliminated), it also has a strong climate drift (not shown). However, by instead suppressing the projection of these interactions onto the planetary-scale modes, the climate drift is eliminated, but multiple regimes are recovered (Fig. B1, right panels). This set of (projected) interactions is the largest set of nonlinear interactions involving only the planetary-scale modes. Thus, the jet regime behavior of the KRG05 model is determined by interactions among the planetary- and synoptic-scale modes (in contrast to the conclusions of KRG05).

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