Finescale Instabilities of the Double-Diffusive Shear Flow*

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ABSTRACT

This study examines dynamics of finescale instabilities in thermohaline–shear flows. It is shown that the presence of the background diapycnal temperature and salinity fluxes due to double diffusion has a destabilizing effect on the basic current. Using linear stability analysis based on the Floquet theory for the sinusoidal basic velocity profile, the authors demonstrate that the well-known Richardson number criterion (Ri ≤ 1/4) cannot be directly applied to doubly diffusive fluids. Rigorous instabilities are predicted to occur for Richardson numbers as high as—or even exceeding—unity. The inferences from the linear theory are supported by the fully nonlinear numerical simulations. Since the Richardson number in the main thermocline rarely drops below 1/4, whereas the observations of turbulent patches are common, the authors hypothesize that some turbulent mixing events can be attributed to the finescale instabilities associated with double-diffusive processes.

1. Introduction

Within the stratified interior of the ocean, small-scale mixing processes are generally thought to originate through either double-diffusive or shear-generated instabilities (e.g., Gregg 1987; Gargett 1989; Thorpe 2005).

Double diffusion is a mixing process driven by the difference in the molecular diffusivities of heat and salt—the two major components of seawater density. This is a fairly common phenomenon. Conditions favoring double diffusion are met in nearly half of the World Ocean (You 2002), and it is particularly widespread in the main thermocline. For instance, more than 90% of the Atlantic thermocline is double-diffusively unstable (Schmitt 1994). The linear instability theory for double diffusion is well developed (Stern 1960; Schmitt 1979; Baines and Gill 1969). Numerous field observations have also been made, as summarized by Schmitt (2003), particularly for the salt finger regime of double diffusion considered in our study. The intensity of double-diffusive convection depends on the density ratio (e.g., Schmitt 1981) and, to a lesser extent, on the background shear (Linden 1974). Also generally accepted is the link between the primary double-diffusive instabilities acting on the microscale and finescale phenomena, which include collective instability (Stern 1969; Stern et al. 2001), thermohaline intrusions (Stern 1967; Toole and Georgi 1981) and thermohaline staircases (Stern and Turner 1969; Radko 2003, 2005). Several studies attempted to quantify the contribution of double diffusion to the ocean mixing at various locations (St. Laurent and Schmitt 1999; Inoue et al. 2008; Alford and Pinkel 2000a; Mack 1985, 1989; Schmitt et al. 2005) and concluded that double-diffusive diapycnal fluxes of heat and salt may be comparable to or exceed those associated with mechanical shear-driven turbulence. The relative importance of double diffusion and turbulence in the global budgets of heat and salt and its role in maintenance of the thermocline is still uncertain and much debated (e.g., Ruddick and Gargett 2003).

Unlike double diffusion, shear-generated turbulence is highly intermittent in space and time and is generally attributed to the action of internal inertia–gravity waves. The propagation of internal waves through inhomogeneous stratification and the nonlinear interactions between various modes occasionally produce particularly strong local shears susceptible to Kelvin–Helmholtz instability. The
collapse of Kelvin–Helmholtz billows—shear-driven overturns with a characteristic spatial pattern—produces localized sites of intense turbulent mixing. Oceanographic measurements confirmed the connection between turbulent mixing and Kelvin–Helmholtz instabilities of internal waves, as discussed, for example, in a review by Gregg (1987). Particularly convincing are the visualizations of the internal waves by dye injection (Woods 1968), which document their propagation through the seasonal thermocline and the destabilization of some waves, manifested by the billows growing on their crests. Kelvin–Helmholtz billows are routinely observed in the laboratory (Thorpe 1971, 1973; Koop and Browand 1979) and realized in numerical simulations (Smyth and Moum 2000; Smyth et al. 2001; Fringer and Street 2003).

Nevertheless, some aspects remain controversial. The classical theory (Richardson 1920; Miles 1961; Howard 1961) requires the Richardson number to drop below $\frac{1}{4}$ for instability. However, typical values of the Richardson number in the ocean generally exceed this nominal stability threshold (e.g., Munk 1981) and significant turbulent dissipations are found for $\frac{1}{4} < \text{Ri} < 1$ (Polzin 1996). Repeated profiles made from the floating instrument platform (Alford and Pinkel 2000b) in the slope regions, characterized by the abundance of turbulent overturns, indicate that, although overturns are likely to occur at low finescale Ri, this condition may not be essential. These observations seem to imply that turbulent overturns in the ocean are influenced by effects that are beyond the scope of the classical Kelvin–Helmholtz theory.

Several explanations have been put forward to rationalize the appearance of density overturns in low-shear environments. The finescale estimates of the Richardson number reported in most oceanographic studies may not be representative of the local Ri involved in the dynamic instabilities (Kunze et al. 1995). The unstable shear conditions are typically confined to vertical scales less than 1 m in most of the thermocline (Kunze et al. 1990), and therefore the finescale estimates provide only an upper bound for the minimum Ri of a fully resolved flow. The temporal and spatial intermittency of density overturns further complicates the interpretation of field measurements (Mack 1985, 1989) since the observed turbulent microstructure patches may reflect the unstable conditions in the past history of shear at a given location. Another consideration invokes the preexisting nonlinear background noise (Brucker and Sarkar 2007; Thorpe 2005), which is present in nature but not taken into account by the classical linear theory. Laboratory experiments suggest that the latter effect has a potential to increase the critical Richardson number slightly to $\text{Ri} = 0.32$. Finally, it should be also kept in mind that, in addition to the frequently invoked Kelvin–Helmholtz instabilities, overturns can be produced by the convective instabilities of nonlinear internal waves and by several other mechanisms. In practice, establishing the precise processes responsible for the formation of turbulent patches on the basis of field measurements is difficult, if not impossible (Thorpe 2005).

In this paper we examine a mechanism for destabilization of laminar shear flows—and, potentially, for the transition to turbulence—that involves the interaction between shear and double diffusion. While it has never been demonstrated explicitly, it is often assumed that Kelvin–Helmholtz and double-diffusive instabilities operate independently and could be considered in isolation of each other (e.g., Smyth 2008). Our study questions this assumption. We point out that the presence of background double-diffusive transports has a destabilizing effect on stratified shear flows. Linear stability analysis of the sinusoidal basic velocity profile in the doubly stratified ocean reveals an instability that is driven by the interplay between double diffusion and large-scale advection. This new thermohaline-shear mode operates on relatively large scales, greatly exceeding the size of the individual salt fingers. Therefore, its nonlinear development has a potential to produce turbulent overturns. Most notably, the rigid constraint on the Kelvin–Helmholtz instability, requiring the Richardson number to be less than $\frac{1}{4}$, no longer applies directly. The largest growth rates are still found for $\text{Ri} \to 0$. However, in the presence of double diffusion, we find no specific cutoff point for the instability of parallel flows.

The new thermohaline-shear mode is analogous to the well-known oscillatory collective instability (Stern 1969), which is driven by the interaction between salt finger transport and internal waves. The vertical temperature ($T$) and salinity ($S$) fluxes due to salt fingering are controlled by the vertical finescale $T-S$ gradients. Since the gradients in gravity waves are nonuniform, the fluxes also vary in space. The convergences/divergences of fluxes, in turn, modify the initial stratification. The forcing of a gravity wave by the salt finger fluxes results in its amplification provided that the Stern number $A = F_p/\nu|\rho_s|$ exceeds unity, where $F_p$ is the salt-finger density flux, $\rho_s$ is the density gradient, and $\nu$ is the viscosity. This condition is met for the typical conditions in the Atlantic thermocline, where the density ratio $R_s \sim 2$ (e.g., Stern et al. 2001). The two-dimensional simulations by Stern et al. (2001) and Stern and Simeonov (2002) confirmed linear theory of collective instability and revealed that its amplification ultimately results in density overturns. However, these overturns are relatively small scale and have generally limited impact on...
mixing. The thermohaline-shear instability considered in the present study appears to be more effective in this regard owing to its interaction with the background flow. Our nonlinear simulations suggest that the inclusion of the background shear increases the fraction of the volume occupied by density overturns by an order of magnitude relative to pure collective instability and results in larger diapycnal transports.

It should be emphasized that the analysis of the thermohaline-shear system in this paper is focused on the finescale dynamics. The microscale processes are introduced by parameterizing the vertical double-diffusive transport as a function of the background gradients. The specific parameterizations of double diffusion remain a significant source of uncertainty. However, the eddy fluxes of heat and salt by salt fingers typically exceed their molecular counterparts by two and four orders of magnitude, respectively. Therefore, inclusion of the double-diffusive parameterizations in theoretical models seems to be essential in terms of predicting the finescale evolution of our system. In this respect, our study should be clearly distinguished from that in Smyth and Kimura (2007), who performed linear stability analysis of the shear flow using the original Boussinesq equations with molecular dissipation. The latter work brings much insight into the microscale dynamics of salt fingers in shear but excludes from consideration effects of the double-diffusive transport on larger scales of motion.

This paper is organized as follows. The linear stability problem is formulated in section 2 where we consider a horizontally uniform basic state consisting of the sinusoidal vertical shear and uniform temperature/salinity gradients susceptible to double-diffusive (salt finger) convection. Next, we use a numerical technique based on the Floquet theory (reviewed in section 3) to solve the linearized system of governing equations (section 4). The nonlinear dynamics of the thermohaline-shear modes are examined by numerical simulation in section 5. We summarize and discuss the key findings in section 6.

2. Formulation

The temperature and salinity fields \((T_{\text{tot}}, S_{\text{tot}})\) are separated into the uniform vertical background gradients \((T_{\text{bg}}, S_{\text{bg}})\) and a departure \((T, S)\) from it. In the present version of the theory we ignore planetary rotation, the nonlinearity of the equation of state, and the effects of friction—molecular and eddy induced—on the large-scale flow. In this case, the Boussinesq equations of motion reduce to

\[
\begin{align*}
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T + w \frac{\partial T_{\text{bg}}}{\partial z} &= -\frac{\partial F_T}{\partial z} \\
\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S + w \frac{\partial S_{\text{bg}}}{\partial z} &= -\frac{\partial F_S}{\partial z} \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + g(\alpha T - \beta S) \mathbf{k} \\
\nabla \cdot \mathbf{v} &= 0,
\end{align*}
\]

where \(\mathbf{v} = (u, v, w)\) is the velocity; \(p\) is the dynamic pressure, \((\alpha, \beta)\) are the thermal expansion and haline contraction coefficients of the linear equation of state; \(g\) is gravity, \(F_T = -K_T \partial T/\partial z\) and \(F_S = -K_S \partial S/\partial z\) are the vertical temperature and salinity fluxes that we attribute to double diffusion, and \((K_T, K_S)\) are the corresponding eddy diffusivities.

Consider an exact steady solution of system (1) representing unidirectional shear flow with a sinusoidal velocity profile,

\[
\mathbf{u}(z) = U_0 \sin\left(\frac{z}{H}\right), \quad \mathbf{v} = \mathbf{T} = \mathbf{S} = 0. \tag{2}
\]

For convenience, the governing equations are nondimensionalized using \(H\) as the unit of length and \(U_0\) as the unit of velocity. The expansion/contraction coefficients \((\alpha, \beta)\) are incorporated in \((T, S)\). The nondimensionalization is implemented as follows:

\[
\begin{align*}
(u, v, w) &\rightarrow U_0 (u, v, w), \quad p \rightarrow \rho_0 U_0^2 p \\
(x, z) &\rightarrow H(x, z), \quad t \rightarrow \frac{H}{U_0} t \\
\alpha T &\rightarrow (\alpha T_{\text{bg}}) \frac{H}{U_0}, \quad \beta S \rightarrow (\beta S_{\text{bg}}) \frac{H}{U_0} H,
\end{align*}
\]

which transforms the governing equations to

\[
\begin{align*}
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T + w &= \frac{\partial}{\partial z}\left(\frac{K_T}{U_0 H} \frac{\partial T}{\partial z}\right) \\
\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S + w &= \frac{\partial}{\partial z}\left(\frac{K_S}{U_0 H} \frac{\partial S}{\partial z}\right) \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \frac{R_0}{R_0 - 1} \left(T - S\right) \mathbf{k} \\
\nabla \cdot \mathbf{v} &= 0,
\end{align*}
\]

where

\[
R_0 = \frac{(\alpha T_{\text{bg}}) H}{(\beta S_{\text{bg}}) H}
\]

is the background density ratio;

\[
\text{Ri} = \frac{N^2 H^2}{U_0^2} = \frac{g(\alpha T_{\text{bg}}) H^2}{U_0^2} \left(1 - \frac{1}{R_0}\right)
\]
is the Richardson number; and $N_b$ is the buoyancy frequency. The basic state (2) reduces in nondimensional units to

$$\pi(z) = \sin(z), \quad v = w = p = T = S = 0.$$  \hfill (5)

At this point, it becomes necessary to parameterize the diapycnal diffusivities ($K_T, K_S$) in (4) as a function of large-scale fields. The single most significant parameter controlling diapycnal mixing in regions of active double diffusion is the local density ratio

$$R_p = \frac{(\alpha T_{\text{vol}}) z}{(\beta S_{\text{vol}}) z}.$$  

See, for example, the reviews by Turner (1985), Kunze (2003), and Schmitt (2003). Double-diffusive transport is thus commonly parameterized by assuming that the eddy diffusivities of heat and salt are specified functions of $R_p^\gamma$. Another frequently used, although less justified, approximation sets the ratio of thermal and haline density fluxes to a constant value. Particular examples of such parameterizations can be found in Schmitt (1981), Zhang et al. (1998), Merryfield et al. (1999), and Radko (2008). For salt diffusivity, the existing evidence (see the review by Schmitt 2003) indicates that $K_S$ is a decreasing function of $R_p$, which we describe using the following analytical relation:

$$K_S = \frac{K_0}{R_p - 1}.  \hfill (6)$$

The temperature diffusivity is parameterized accordingly:

$$K_T = \frac{\gamma K_S}{R_p} = \frac{\gamma K_0}{R_p(R_p - 1)}.  \hfill (7)$$

Next, parameterizations (6) and (7) are substituted into (4) and the resulting system is linearized with respect to the basic state (5):

$$\begin{align*}
\frac{\partial T^\prime}{\partial t} + \sin(z) \frac{\partial T^\prime}{\partial x} + w^\prime + \gamma \frac{1}{\text{Pe}} \frac{\partial^2}{\partial z^2} [(2R_0 - 1) S^\prime - T^\prime] \\
\frac{\partial S^\prime}{\partial t} + \sin(z) \frac{\partial S^\prime}{\partial x} + \frac{1}{R_0} w^\prime = \frac{1}{\text{Pe}} \frac{1}{(R_0 - 1)^2} \frac{\partial^2}{\partial z^2} [(2R_0 - 1) S^\prime - T^\prime] \\
\frac{\partial u^\prime}{\partial t} + \sin(z) \frac{\partial u^\prime}{\partial x} + w^\prime \cos(z) = -\frac{\partial p^\prime}{\partial x} \\
\frac{\partial v^\prime}{\partial t} + \sin(z) \frac{\partial v^\prime}{\partial x} = -\frac{\partial p^\prime}{\partial y} \\
\frac{\partial w^\prime}{\partial t} + \sin(z) \frac{\partial w^\prime}{\partial x} = -\frac{\partial p^\prime}{\partial z} + \text{Ri} \frac{R_0}{R_0 - 1} (T^\prime - S^\prime) \\
\n\n\n\begin{bmatrix} T^\prime \\ S^\prime \\ u^\prime \\ v^\prime \\ w^\prime \\ p^\prime \end{bmatrix} = \exp(ikx + l y + imz + \lambda t) \sum_{n=-N}^{N} \begin{bmatrix} T_n \\ S_n \\ u_n \\ v_n \\ w_n \\ p_n \end{bmatrix} \exp(inz),  \hfill (9)
\end{align*}$$

where $\lambda$ is the growth rate, $k$ and $l$ are the horizontal wavenumbers, and $m$ is the Floquet coefficient, which controls the fundamental wavelength in $z$. Substituting (9) into the linear system (8) and collecting the individual Fourier components allows us to express the governing equations in the matrix form.
\[ \lambda \xi = A \xi, \quad (10) \]

where \( A \) is the square matrix whose elements are functions of \( k, l, m, Pe, R_0, Ri, N, \) and

\[ \xi = (T_{-N}, S_{-N}, u_{-N}, v_{-N}, w_{-N}, p_{-N}, \ldots, T_{N}, S_{N}, u_{N}, v_{N}, w_{N}, p_{N}). \quad (11) \]

The growth rates of the normal modes of the linear system (8) correspond to the eigenvalues of the matrix \( A \). For each set of governing parameters, we determine the eigenvalue with the maximum real part, which represents the fastest growing mode, and the corresponding eigenvector \( \xi \). The spatial pattern of the perturbation is then reconstructed from \( \xi \),

\[ \tilde{T} = \text{Re}\left[ \exp(ikx + ily + imz) \sum_{n=-N}^{N} T_n \exp(inz) \right], \quad (12) \]

where \( \tilde{T} \) represents the temperature distribution in the fastest growing mode. The patterns of \( S, v, \) and \( p \) in the normal modes are reconstructed in a similar manner.

Extensive experimentation with the Floquet model suggests that the most rapid growth occurs when the Floquet factor is zero \((m = 0)\), which will be used in all subsequent calculations. Physically, \( m = 0 \) implies that the vertical scales of the instability modes identified by the Floquet calculations do not exceed the \( z \) wavelength of the basic shear. The dependencies of the growth rate on the horizontal wavenumbers are conveniently discussed in terms of the orientation of the normal modes \( \theta \) and their two-dimensional wavenumber \( \kappa \) such that

\[ \begin{cases} k = \kappa \cos \theta \\ l = \kappa \sin \theta. \end{cases} \quad (13) \]

In all calculations presented below, we use \( N = 100 \). Comparison of the selected calculations with those where \( N = 200 \) indicate that the resolution is sufficient for an accurate representation of the thermohaline-shear instabilities.

4. Linear results

The essential effect of double diffusion on shear flow instability is revealed by a series of calculations in which the Richardson number is systematically varied within the range \( 0 < Ri < 10 \). The stratification, diffusivity magnitude, vertical scale of shear, background density ratio, and flux ratio are kept constant at

\[ N_b^2 = 10^{-5} \text{ s}^{-2}, \quad K_0 = 2 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}, \quad 2\pi H = 5 \text{ m}, \]

\[ R_0 = 2, \quad \gamma = 0.7. \quad (14) \]

The chosen value of \( K_0 \) and \( \gamma \) yield, for the density ratio of 2, the dimensional diffusivity of heat of \( K_T \sim 0.7 \times 10^{-5} \text{ m}^2 \text{ s}^{-1} \) and diffusivity of salt \( K_S \sim 2 \times 10^{-5} \text{ m}^2 \text{ s}^{-1} \), which is broadly consistent with the microstructure measurements in St. Laurent and Schmitt (1999).

For each \( Ri \) and \( \theta \) we computed the eigenvalues of the linear system (8) using the Floquet method for a wide range of horizontal wavenumbers \((0 < \kappa < 10)\) and identified the most rapidly growing mode. For \( \theta = 0 \)—the normal mode aligned with the \( x \) direction—the maximum growth rate is plotted as a function of the Richardson number in Fig. 1 (blue curve) in logarithmic coordinates. Note that, since the time scale in our non-dimensionalization (3) is \( H/U_0 \), the scale of the growth rate is \( U_0/H \).

For low \( Ri \) the pattern of the growth rate conforms to the \( \lambda(Ri) \) relation expected for the classical (non-diffusive) Kelvin–Helmholtz instability: The most rapid growth occurs for \( Ri = 0 \) and monotonically decreases within the interval \( 0 < Ri < 0.23 \). This pattern changes dramatically for \( Ri > 0.23 \). The growth rate starts to increase with the Richardson number. An inspection of
the imaginary part of the eigenvalue (not shown) reveals another fundamental change in the stability properties of system (8) at low and high R\text{i}. For R\text{i} < 0.23, the instability is direct: the eigenvalue is real. However, for larger R\text{i}, \lambda acquires a finite imaginary component, implying an oscillatory (overstable) instability.

To emphasize the differences brought by the inclusion of double-diffusive effects, we present the \lambda(R\text{i}) relation for the nondiffusive case (indicated by the green curve in Fig. 1), which was obtained by setting \(K_T = K_S = 0\) in (4). This solution reproduces the earlier models of the one-component Kelvin–Helmholtz instability for the sinusoidal shear flow in Hazel (1972) and Balmforth and Young (2002). In addition, a series of calculations were performed with uniform and equal diffusivities of heat and salt (\(K_T = K_S = 10^{-5} \text{ m}^2 \text{s}^{-1}\)); the resulting growth rate pattern is indicated by the red curve in Fig. 1. The latter model is meant to represent the preexisting background turbulence of non-double-diffusive origin.

While all three models agree very closely for R\text{i} < R\text{i}_{cr} = 0.23—their growth rate patterns in Fig. 1 are nearly identical—these solutions rapidly diverge for larger R\text{i}. As expected, the region of instability in the nondiffusive model (the green curve) is limited to R\text{i} < 0.25. The inclusion of finite and equal diffusivities of heat and salt in the turbulence model (the red curve) has a slightly destabilizing effect, pushing the stability boundary to R\text{i} = 0.4. This somewhat counterintuitive tendency has been observed in earlier studies of one-component shear flows (e.g., Balmforth and Young 2002). However, the diffusive instability is rather weak: the growth rates for 0.25 < R\text{i} < 0.4 are typically two orders of magnitude less than for R\text{i} < 0.25. Therefore, it is unlikely that the diffusive destabilization has a substantial impact on vertical mixing in the ocean. On the other hand, double diffusion clearly has the potential to profoundly affect stability of shear flows for R\text{i} > R\text{i}_{cr}.

The spatial temperature pattern for the fastest growing modes in the double-diffusive model is shown in Fig. 2. For a relatively low Richardson number of R\text{i} = 0.15 and \(\theta = 0\) (Fig. 2a), the unstable mode takes a form expected for the traditional Kelvin–Helmholtz instability. In this case, the growth rate has a well defined maximum at \(\kappa_{max} = 0.77\). The perturbation in this case is localized in the region of maximum shear (Fig. 2a), located at \(z = 0, \pi, 2\pi\). For larger Richardson numbers, the perturbation pattern changes considerably and shifts to the region of low shear at \(z \approx 3\pi/2\), as exemplified by the calculation for R\text{i} = 0.5 in Fig. 2b, or to the equally likely location at \(z \approx \pi/2\). The dependence \(\lambda(\kappa)\) also takes a different form, analogous to that of collective instability (Stern et al. 2001). The growth rate increases monotonically with \(\kappa\) for small wavenumbers, approaching a constant value for relatively large \(\kappa > 3\).

To quantify the dynamics of the thermohaline-shear instability, we examine its source of energy. In our nondimensional units, the derivation of the perturbation energy equation involves (i) subtracting the temperature and salinity equations in (8) and multiplying the result by Ri\(\bar{R}_0^2(R_0 - 1)^{-2} (T' - S'*)^*\) to form the density variance equation (asterisks here denote complex conjugate); (ii) multiplying the momentum equation by \((v')^*\) and subtracting the result from the density variance equation; and (iii) integrating the result over one fundamental wavelength in \(x\) and \(z\). The resulting expression is further simplified, yielding

\[
\frac{d}{dt} E = P_{sh} + P_{dd},
\]

where

\[
E = \int_0^{2\pi/k} \int_0^{2\pi/\delta} (Ri\bar{R}_0^2(R_0 - 1)^{-2} (T' - S')^2 + |u'|^2 + |w'|^2) \, dx \, dz,
\]

\[
P_{sh} = -\text{Re} \int \bar{u}w' \, dx \, dz,
\]

\[
P_{dd} = \text{Re} \int \text{Ri}(\gamma - 1)P_e^{-1}\bar{R}_0^2(R_0 - 1)^{-4} [(2R_0 - 1)S_{zz}' - T_{zz}'] (T' - S')^* \, dx \, dz.
\]

Equation (15) is used to examine processes influencing the variation of the perturbation energy \(E\) in time. The first term on the right-hand side \(P_{sh}\) is interpreted as the production of energy by shear, the rate of transfer of the basic kinetic energy to the perturbation. The \(P_{dd}\) term represents the energy production by the diapycnal mixing. Because of the difference in the vertical flux of heat and salt in doubly diffusive fluids, the loss of potential energy stored in the salinity component of density can exceed the gain of energy by the heat component. A substantial part of the difference transforms into perturbation energy, forcing the growth of the unstable modes. Note that the latter mechanism does not require background flow and can operate in an ocean initially at rest. Such dynamics are also at work in the collective instability (Stern 1969; Holyer 1981; Stern et al. 2001).
The energy budget (15) is now quantified for the calculations in Figs. 2a,b. For the low Richardson number case (Fig. 2a), both production terms ($P_{sh}$ and $P_{dd}$) are positive, thereby driving the instability, with $P_{sh}$ exceeding $P_{dd}$ by a factor of 120. For the calculation in Fig. 2b, the double-diffusive and shear terms act in the opposite sense: the $P_{dd}$ term is large and positive, whereas the shear term is less (by a factor of 12.7) and acts against the instability. This suggests that the background current is actually accelerated by its interaction with double-diffusive fluxes. It should be mentioned that the tendency of double diffusion to accelerate horizontal currents has been noted in studies focused specifically on the dynamics of individual salt fingers (Holyer 1984; Stern and Simeonov 2005; Radko 2010). In the present model, the same effect appears as a byproduct of the large-scale instability of the parameterized Eqs. (8). This tendency is relatively weak and therefore, in the oceanic (heat–salt) case, is effectively damped by molecular friction. In other media, such as the astrophysically relevant low Prandtl number fluids,
the ability of double diffusion to accelerate horizontal currents can have a profound effect on the flow evolution (Radko 2010).

Overall, the inspection of instabilities in Fig. 2 suggests the following interpretation: For relatively low Richardson numbers ($\text{Ri} < \text{Ri}_{cr}$), the fastest growing instability of the double-diffusive shear flow is, by and large, the classical Kelvin–Helmholtz instability, which is only slightly modified by double-diffusive transports. However, when the Richardson number is increased, the fastest growing instability shifts to a different oscillatory branch. In this regime, the instability can be thought of as a form of the collective instability, distorted and localized by the background shear. The growth rates of the thermohaline-shear and pure collective instabilities are comparable but different. The spatially periodic wave-like pattern of the collective instability (e.g., Stern et al. 2001; Stern and Simeonov 2002) is in contrast with the characteristically isolated thermohaline-shear structures (Fig. 2b). The physical mechanisms of both instabilities are similar. One of the key properties of salt fingering is the larger (smaller) eddy diffusivity of salt (heat). Thus, the parameterized Eqs (4) are formally analogous to the well-known laminar diffusive systems, representing a stable (unstable) gradient of a slower (faster) diffusing density component (Baines and Gill 1969). Such systems are susceptible to oscillatory instability. We believe that these dynamics are at work in both thermohaline-shear and collective instabilities. However, as will be seen shortly (section 5), the nonlinear evolution of the former is more violent and its large-scale consequences are more profound.

Figure 3 examines the variation in the growth rate with $\theta$. The general pattern of the $\text{A(Ri)}$ relation is qualitatively similar for various orientations of the normal modes. For relatively small values of the Richardson number ($\text{Ri} < \text{Ri}_{cr}$), the fastest growing instability is direct. It closely resembles the traditional Kelvin–Helmholtz modes and the growth rate decreases with increasing $\text{Ri}$. For relatively large $\text{Ri} > \text{Ri}_{cr}$, the instability is dominated by the oscillatory thermohaline modes, and the non-dimensional growth rate increases with the Richardson number. However, the critical Richardson number $\text{Ri}_{cr}$ representing the point of transition between these regimes depends strongly on the spatial orientation of the normal mode. As $\theta$ increases from $0$ to $\pi/2$, $\text{Ri}_{cr}$ monotonically decreases from $0.23$ to $0$. In the dynamic instability regime ($\text{Ri} < \text{Ri}_{cr}$), the growth rates decrease with increasing $\theta$. This is a generic property of Kelvin–Helmholtz instabilities, elegantly expressed by the Squire theorem (Squire 1933): the rapidly growing perturbations of parallel flows are aligned in the direction of the basic current. In this respect, thermohaline instabilities ($\text{Ri} > \text{Ri}_{cr}$) behave very differently. Their growth rates monotonically increase with $\theta$. For low $\theta$ thermohaline instabilities interact fully with the background shear flow, which tends to suppress them. This interaction gradually weakens for larger $\theta$ and finally in the limit $\theta = \pi/2$ (Fig. 3d) the perturbation becomes completely decoupled from the background flow. In the cross-current direction, dynamic instabilities no longer operate and thermohaline modes reduce to the traditional (nonsheared) collective instabilities (Stern 1969; Stern et al. 2001).

5. Nonlinear simulations

Having established the persistence of linear thermohaline instabilities for moderately strong but dynamically stable shear flows ($\text{Ri} \sim 1$), it is natural to inquire into their nonlinear consequences. This is accomplished by solving the governing equations (4) numerically.
Following Radko and Stern (1999, 2000), we assume triply periodic boundary conditions for \((T, S, v)\) and integrate the governing equations in time using a pseudospectral method. An important difference between this and earlier double-diffusive simulations is related to our current use of the parameterized flux-gradient model. In contrast with the direct numerical simulations (DNS), we make no attempt to explicitly resolve the dissipation scales. Therefore, in anticipation of the appearance of convective overturns and associated turbulent cascade of energy toward small scales, it becomes necessary to introduce a subgrid model that selectively damps energy at smallest resolved scales. For that, we add the biharmonic damping operator \(\mu \nabla^4(\ldots)\) acting on \((T, S, v)\) to the temperature, salinity, and momentum equations, respectively. The value of the damping coefficient \(\mu = 3 \times 10^{-5}\) in all following simulations is chosen to limit dissipation to the marginally resolved scales, leaving longer wavelengths essentially unaffected. Thus, the proposed numerical method contains elements of both flux-gradient modeling (e.g., Stern and Simeonov 2002; Radko 2005) used for the salt finger regions and large-eddy simulations (e.g., Winters and D’Assaro 1994; Lelong and Dunkerton 1998) for density overturns. The critical advantage of these simulations relative to their DNS counterparts is related to their efficiency, making it possible to easily integrate the governing equations in a dimensional computational volume of \(\sim 1000\ m^3\) over periods of several days. Simulations of this scale are currently beyond the capabilities of the DNS models. Also important is the relative transparency of the parameterized models, allowing the direct and unambiguous interpretation of the processes at play.

The discussion in this section is based on the comparison of four representative experiments:

- Experiment 1: Dynamically stable flow \((\text{Ri} = 0.5)\) with double diffusion;
- Experiment 2: Dynamically unstable flow \((\text{Ri} = 0.15)\) with double diffusion;
- Experiment 3: Dynamically unstable flow \((\text{Ri} = 0.15)\) without double diffusion;
- Experiment 4: Double diffusion in the absence of the background flow \((\text{Ri} = \infty)\).

In all experiments, we use the identical mesh of \((N_x, N_y, N_z) = (256, 128, 64)\) grid points with \(\Delta x = \Delta y = \Delta z = 2\pi/N_z\), identical representation of double diffusion in (6) and (7), and the background parameters in (14). In dimensional units, the size of the computational domain is equal to \(20\ m \times 10\ m \times 5\ m\). The experiments were initialized by the sinusoidal flow (5) slightly perturbed by small amplitude random noise, except for experiment 4 in which the background flow was set to zero.

The temporal evolution of the flow field for experiment 1 is illustrated in Fig. 4, where we plot the cross-current (left panels) and along-current (right panels) sections of the perturbation temperature. The first stage involves development of thermohaline instability. Since the largest growth rates at \(\text{Ri} = 0.5\) are realized for the cross-current modes (see Fig. 3), the temperature pattern at \(t = 2000\) is characterized by strong variation in \(y\) and \(z\) but is almost uniform in \(x\) (Fig. 4a). An analogy could be made at this point between the development of the secondary finescale double-diffusive instabilities in Fig. 4a and the evolution of primary salt fingers in shear (Linden 1974; Kimura and Smyth 2007). In the latter case, the fluid also tends to homogenize in the direction of the background flow, forming so-called salt sheets. This pattern changes dramatically by \(t = 2700\) when the significant along-current variation develops (Fig. 4b). It is interesting to note that the along-current variability is particularly strong at \(z = 0, z = \pi,\) and \(z = 2\pi\) (the bottom, center, and top of each plot in Fig. 4)—locations of the largest background shear. At first, this pattern seems at odds with the linear stability analysis of thermohaline-shear instabilities (Fig. 2b), which suggests that the most rapidly growing perturbation is localized to the regions of relatively low shear. The along-current perturbations seen in Fig. 2b are therefore interpreted as secondary Kelvin–Helmholtz instabilities induced by the thermohaline modes. The growing primary modes introduce, upon reaching finite amplitude, significant spatial variation in the vertical density gradient. The reduction of density gradient in some regions, in turn, makes the flow susceptible to local Kelvin–Helmholtz instabilities, particularly in the zones of high background shear. This effect, however, appears to be transient. Because the ensuing vertical mixing stabilizes the high-shear zones, the along-current variability shifts (by \(t = 3300\)) to the low-shear zones (Fig. 4c) located near \(z \approx \pi/2\) and \(z \approx 3\pi/2\). In the final stage of the experiment \((t = 4200\) in Fig. 4d), the temperature variance is distributed rather uniformly throughout the computational domain. Also apparent is the tendency for a systematic increase of typical spatial scales of the perturbation in time, associated with the sequential coalescence of relatively small eddies into larger and larger structures. Such mergers are common in turbulent stratified fluids (Balmforth et al. 1998; Radko 2007), particularly in the double-diffusive environment (Huppert 1971; Radko 2005; Simeonov and Stern 2008).

A convenient quantitative measure for the intensity of mixing is given by the vertical eddy fluxes of temperature and salinity, which are denoted as
It should be emphasized that (17) represents the transport by the resolved scales of motion, in contrast to \((F_T, F_S)\) that measure the parameterized transport by the double diffusion on the microscale. Analysis of the eddy flux (17) therefore illustrates the influence of double diffusion on finescale dynamics. In Fig. 5a, we present the evolution of the vertical flux in the initial \((t < 1100)\) stage of experiment 1, characterized by the linear growth of the unstable thermohaline-shear perturbations. As expected from linear stability analysis (section 4), the evolutionary pattern is oscillatory. To examine the eddy flux on time scales exceeding the characteristic period of oscillations, in Fig. 5b we smooth the record of heat flux by applying the corresponding running average. The smoothed record reveals the slow evolution of mixing intensity, which first monotonically increases in magnitude, reaching the maximum of

\[
|F_{T,\text{eddy}}| = 4 \times 10^{-4} \quad (18)
\]

at \(t = 2500\). To put these values into an oceanographic perspective, the reader is reminded that throughout this study we use the nondimensionalization system (3). For the background parameters (14) the period of 1000 nondimensional time units is equivalent to 2.5 days and the nondimensional heat flux (18) is equivalent to the eddy diffusivity of \(K_{T,\text{eddy}} \sim 1.2 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}\), substantially less than the parameterized diffusivity \(K_T \sim 7 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}\).

Figure 6 illustrates the qualitative differences in the (smoothed) heat flux record for all experiments discussed in this section. The evolution of the dynamically stable experiment 1 (\(R_i = 0.5\)) is much slower than that of the unstable experiment 2 (\(R_i = 0.15\)). The violent
burst of mixing, induced by the Kelvin–Helmholtz instability in experiment 2, reaches maximum at $t = 200$ and subsides by $t = 700$. In contrast, the dynamically stable case (experiment 1) develops significant levels of eddy activity only after $t = 700$. However, the intensity of mixing for $t > 700$ is consistently higher in experiment 1 than it is in experiment 2. This is attributed to the tendency of energetic mixing driven by the Kelvin–Helmholtz overturn to suppress and delay the development of thermohaline instabilities. Although the maximum eddy flux in experiment 2 exceeds that in experiment 1 by two orders of magnitude, the average fluxes differ only by a factor of 3 (see Table 1).

The comparison of experiments 2 and 3—which differ by the presence (absence) of double diffusion in experiment 2 (experiment 3)—reveals that their first stages are very similar. This similarity is consistent with the linear theory (Fig. 1), which predicts a very limited impact of double diffusion on primary Kelvin–Helmholtz instabilities for low Ri. However, the eddy activity in the non-double-diffusive experiment 3 persists only in the initial phase ($t < 700$) and rapidly decays afterward. Ignoring double diffusion in experiment 3 reduces the average eddy flux by a factor of 2 relative to that in experiment 2.

It is also of interest to compare the evolution of systems with and without the background flow (experiments 1 and 4). Instead of suppressing thermohaline instabilities, the background flow acts as a catalyst, expediting the transition to the irregular turbulent state. The average eddy flux in experiment 1 exceeds that in 4 by 65%. This somewhat counterintuitive tendency is attributed to the interplay between the relatively large-scale basic shear and thermohaline instabilities, operating
on smaller spatial scales. The latter produce isolated regions of reduced vertical density gradient—the traumata effect discussed in Bouret-Aubertot et al. (1995) and Stern and Simeonov (2002)—which, in the presence of background shear, are susceptible to local dynamical instabilities.

The profound role of the stable background flow in mixing is revealed even more clearly in Fig. 7, which presents the time record of the filling factor $\delta$, the fraction of the volume that is convectively unstable ($\partial \rho / \partial z > 0$). In the no-shear calculation (experiment 4) the average and maximum values of $\delta$ are both less, by an order of magnitude, than the corresponding values in the stable shear case (experiment 1). This implies that the presence of the background shear is essential in the final stages of the evolution of thermohaline instabilities—formation of density overturns and consequent transition to turbulence. As indicated in Table 1, the time-averaged filling factor in all experiments with the background shear (experiments 1–3), stable or unstable, are comparable. Their typical values are on the order of 1%. The maximal values of $\delta$ are of course much higher for the dynamically unstable runs (experiments 2 and 3); the violent but short-lived Kelvin–Helmholtz instability at its peak is capable of overturning up to a third of the net volume.

### Table 1. Time mean and maximal values (in parentheses) of the heat flux and filling factor for expts 1–4.

| Expt | Eddy heat flux $|F_{\text{eddy}}|$ average (max) | Filling factor $\delta$ average (max) |
|------|-----------------------------------------------|---------------------------------------|
| 1    | $1.3 \times 10^{-4}$ ($4.1 \times 10^{-4}$)   | 0.012 (0.074)                          |
| 2    | $4.0 \times 10^{-4}$ ($1.1 \times 10^{-2}$)  | 0.014 (0.32)                           |
| 3    | $2.2 \times 10^{-4}$ ($1.1 \times 10^{-2}$)  | 0.009 (0.34)                           |
| 4    | $0.8 \times 10^{-4}$ ($2.5 \times 10^{-4}$)  | 0.001 (0.008)                          |

6. Discussion

This study examines the instability of sinusoidal shear flow in the presence of background double-diffusive transport, which is parameterized as a function of the large-scale vertical temperature/salinity gradients. Stability analysis based on the Floquet method reveals the existence of mixed thermohaline-shear instabilities. Unlike Kelvin–Helmholtz waves, these instabilities are not constrained by the requirement that the Richardson number (Ri) is less than a quarter. Inspection of the fastest growing normal modes reveals that the spatial and temporal patterns of instabilities in a double-diffusive shear flow vary substantially with Ri. For low Ri the unstable modes retain the major characteristics of the nondiffusive Kelvin–Helmholtz instability. However, when the Richardson number exceeds a critical value (Ri = 0.23 for the parameters used in this study), the structure of the fastest growing mode becomes very different. The growth rate acquires a finite imaginary component, which is associated with temporal oscillations, and the corresponding normal mode becomes spatially localized in the regions of low shear. Comparison with the corresponding turbulence model, characterized by equal diffusivities of heat and salt, indicates that double-diffusive mixing is particularly effective in destabilizing the background current.

Our preliminary nonlinear simulations of the parameterized equations (section 6) are encouraging. In the presence of double diffusion, the dynamically stable shear undergoes thermohaline destabilization, which ultimately leads to persistent turbulent mixing. The average heat flux by the resolved scales of motion (thus excluding the direct double-diffusive transport) in the experiment with
While these effects are beyond the scope of current studies, questions with regard to rotational effects. Of particular concern is the time-dependent nature of oceanic shears, which contain a substantial oscillatory component at near-inertial frequencies and thus operate on time scales comparable to that of thermohaline–shear instabilities. While these effects are beyond the scope of current investigation, subsequent generalizations of the model should include both time-dependent background flow and planetary rotation. The questions raised by this study motivate a more detailed inquiry into the implications of the thermohaline–shear instability for diapycnal mixing. An obvious path for further investigations involves using direct numerical simulations to resolve salt fingers and the turbulence dissipation scale. An adequate representation of thermohaline–shear instabilities by DNS would require simultaneous resolution of a wide range of spatial scales from millimeters to several meters and time scales of several days, which, unfortunately, is beyond current computational capabilities. However, the tendency for double-diffusive destabilization of shear flows, illustrated here in terms of idealized conceptual models, is suggestive and will likely persist in more realistic models and perhaps in nature as well. It is our belief—and the key thesis of this paper—that the analysis of double-diffusive destabilization will ultimately help to rationalize some observations of turbulent mixing in the main thermocline, an environment characterized by the relatively weak (\(Ri > \frac{1}{2}\)) shears.

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