True Random Source from Integratable Chaotic Circuits

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Physical Random Number Generation

- **Large Quantities of Random Numbers**
  - Monte Carlo simulation
  - encryption
- **Pseudo-Random Is Not Good Enough**
  - potential weakness
  - obscurity relies on computational hardness
- **Physical Random Sources**
  - human interaction (stochastic seed)
  - quantum effects (stochastic process)
  - classical mechanics (chaotic process)

Pseudo-Random Numbers:

$$\mathbf{x}_n = F\left(\mathbf{x}_n \mid \lambda\right)$$

$$s_n = P\left(\mathbf{x}_n\right)$$

where

s_n – pseudo-random sample

$\mathbf{x}_n$ – internal state

λ – seed

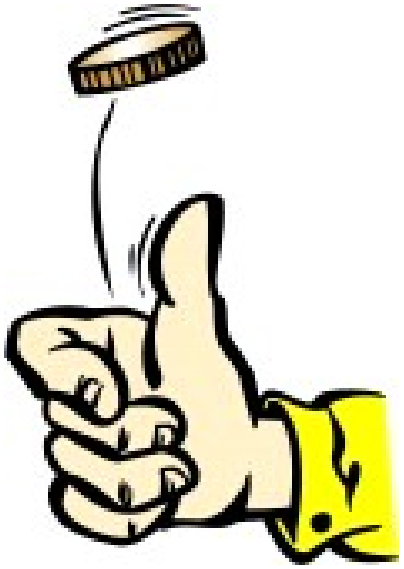
Successive samples are necessarily related

Random Bits for Encryption

- **Ideal Random Process**

$$P(\mathrm{H}) = P(\mathrm{T}) = 1/2$$

H T T H T H H T T T T H T T…



- **Bits**

$$\mathrm{H} \rightarrow 0 \quad \mathrm{T} \rightarrow 1$$

0 1 1 0 1 0 0 1 1 1 1 0 1 1…

Process to generate random bits must be independent and fair.
Otherwise, tendencies can be exploited.

Bias

- **Unfair Coin** $p \neq 1/2$

$$P(\mathrm{H}) = p$$
$$P(\mathrm{T}) = 1-p$$

$$P(\mathrm{HH}) = p^2$$
$$P(\mathrm{HT}) = p(1-p)$$
$$P(\mathrm{TH}) = p(1-p)$$
$$P(\mathrm{TT}) = (1-p)^2$$

von Neumann Bias Corrector:

$$HT \to 0$$
$$TH \to 1$$
$$(HH, TT \to \text{n.s.})$$

… HT TT TT HT TT TH HT TT HH TT TH TT …

… 0 0 1 0 1 …

Correlation

- **Conditional Probability**

$$P\left(\text{next toss}\middle|\text{prior tosses}\right)$$

- **Independent (uncorrelated)**

$$P\left(\text{H}\middle|\text{any sequence}\right)=P\left(\text{H}\right)$$

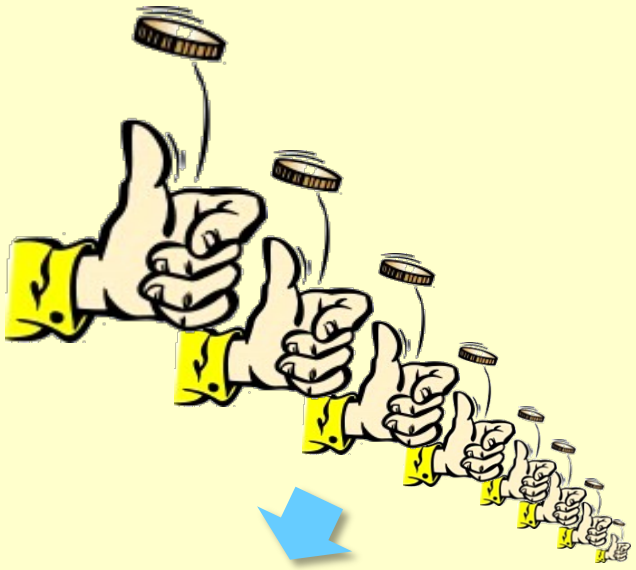
$$P\left(\text{T}\middle|\text{any sequence}\right)=P\left(\text{T}\right)$$

- **Dependent (correlated)**

$$P\left(\text{H}\middle|\ldots\text{H}\ldots\right)\neq P\left(\text{H}\middle|\ldots\text{T}\ldots\right)$$

$$P\left(\text{T}\middle|\ldots\text{H}\ldots\right)\neq P\left(\text{T}\middle|\ldots\text{T}\ldots\right)$$

Random Bit Stream:



H T T H T H H T T …

Lazy sequential tosses could be correlated…

Classical Sources of Randomness

- **Classical Mechanics**
  - motion determined by physical laws and initial conditions

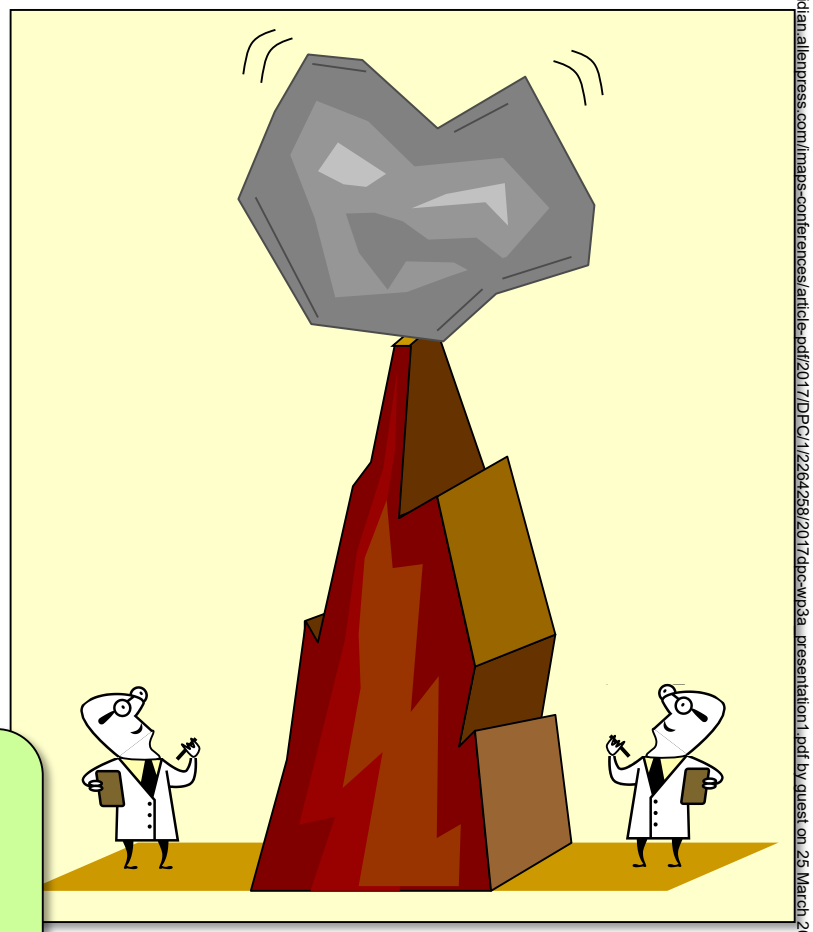
$$F = ma$$

- **Laplace's Demon**
  - if position and velocity of every particle in the universe is precisely known…

Can classical mechanics generate truly random bits?

Determinism and Information

- **No new information comes from deterministic dynamics…**
  - initial conditions
  - system equations
- **… unless your knowledge is incomplete**
  - finite precision
  - model errors
  - unmodeled effects (noise)

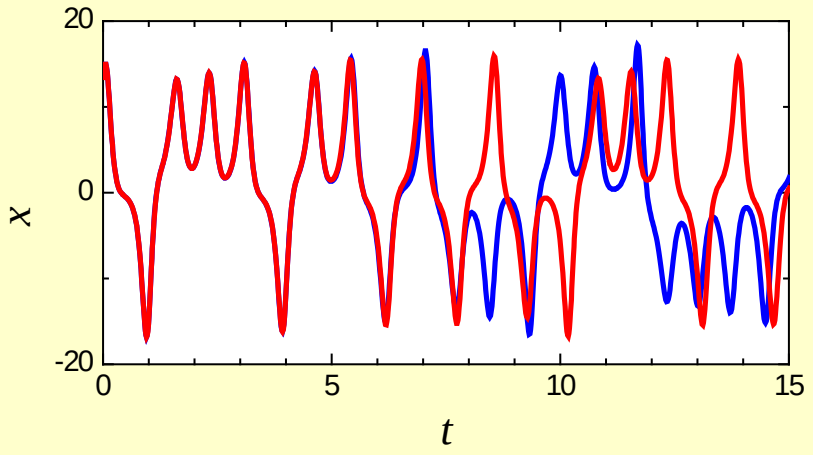


Instability can be an information source when observed with limited precision

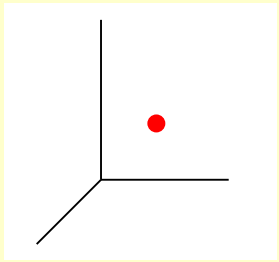
Chaos

- **Dynamic State**
  - aperiodic, bounded oscillations
  - extreme sensitivity to initial conditions
  - deterministic, yet unpredictable
  - continuously unstable

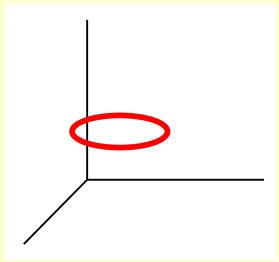
Sensitivity to Initial Conditions



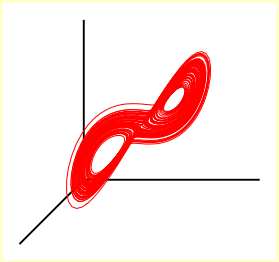
Attractor Hierarchy



equilibrium
(steady state)



limit cycle
(periodic)



strange attractor
(chaos)

Chaos as Information Source

- **Given a chaotic flow**
  - deterministic dynamics

$$\frac{dx}{dt} = f(x)$$

- **Measure state with finite precision**
  - information beyond measurement precision is unknown

$$x(t) \doteq 3.197\_\_\_\_\cdots$$

- **Future state can be predicted**
  - precision lost due to sensitivity to initial conditions

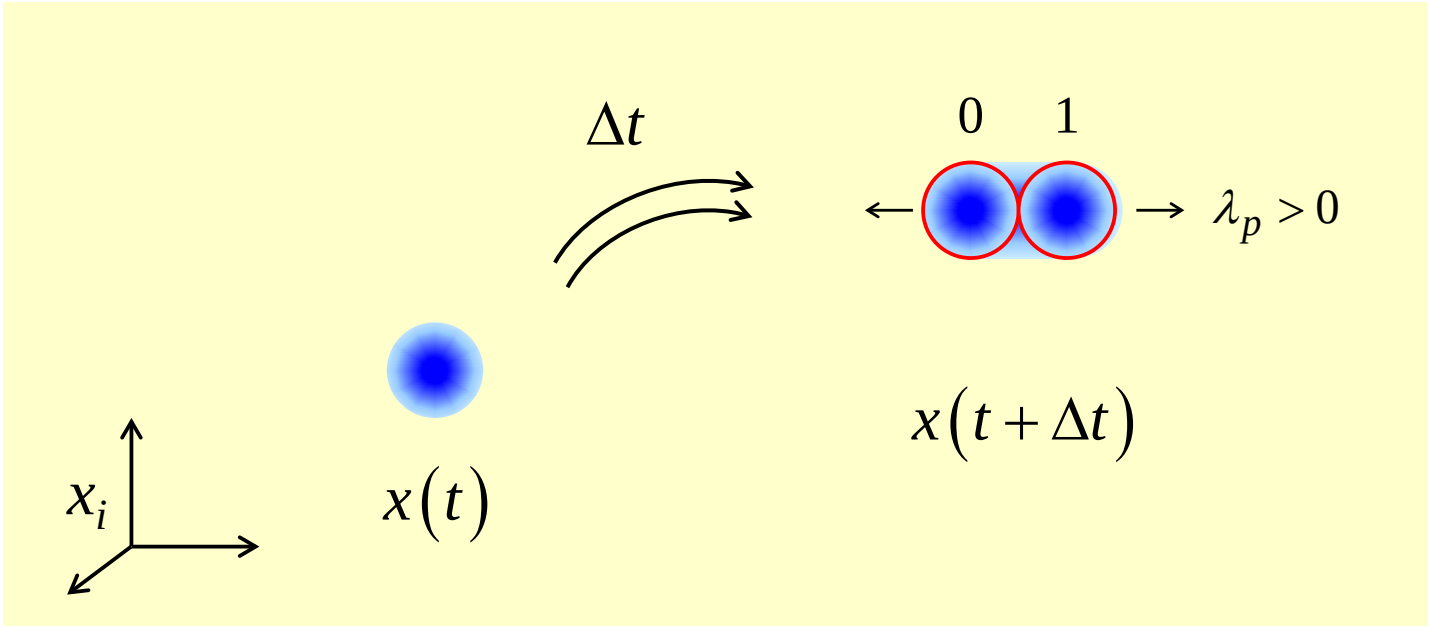
$$x(t+\Delta t) \doteq 1.78?$$

- **Re-measuring state reveals new information**
  - new information derived from previously unresolved digits

$$x(t+\Delta t) \doteq 1.784$$

new information

Entropy Rate in Chaos

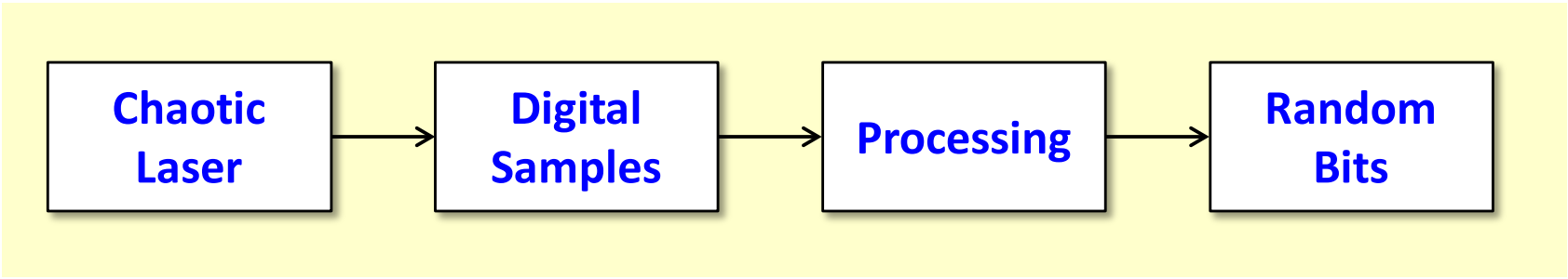


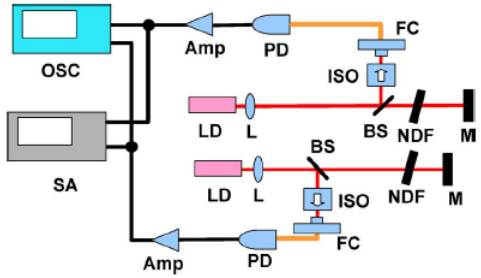
$$\Delta t = \frac{\ln 2}{\lambda_p} \quad \Rightarrow \quad H = \frac{1 \text{ bit}}{\Delta t} = \frac{\lambda_p}{\ln 2}$$

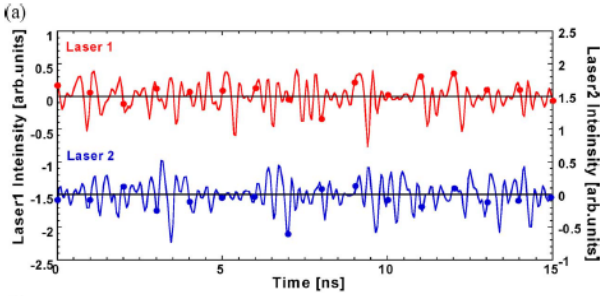
entropy rate

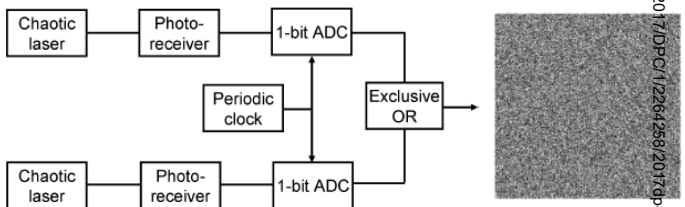
Theoretical Speed Limit for Physical Random Bits

Fast Physical Random Number Generation Using Laser Chaos





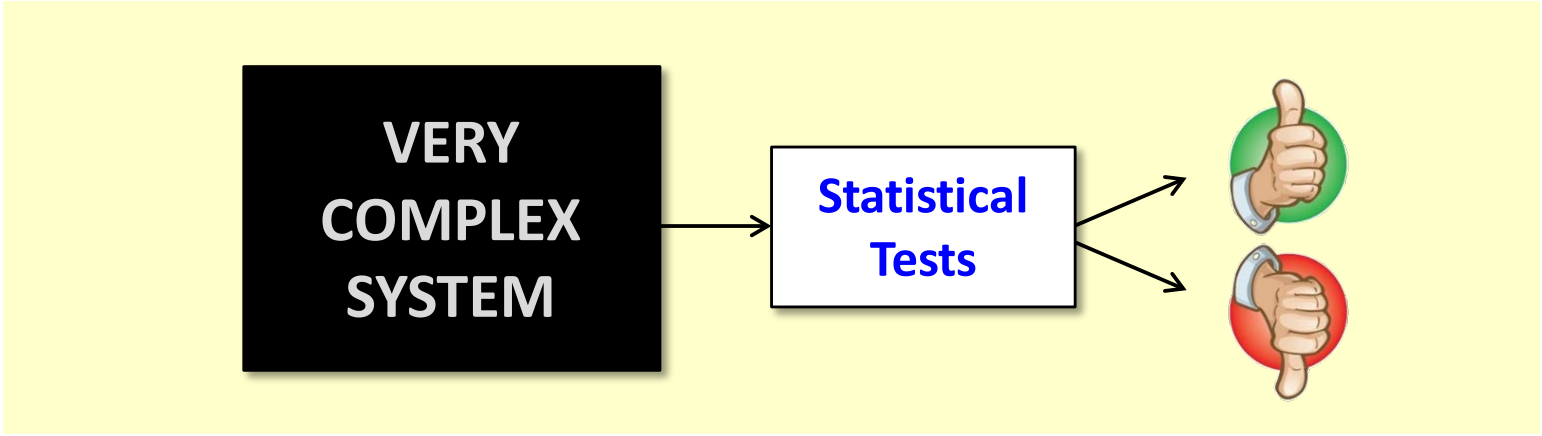




- Uchida et al., *Nature Photonics* (2008), 1.7 Gb/s
- Reidler et al., *Phys. Rev. Lett.* (2009), 12.5 Gb/s
- Argyris et al., *Optics Express* (2010), 140 Gb/s
- Uchida et al., *IEEE Photonics Tech. Lett.* (2012), 400 Gb/s
- Oliver et al., *Optics Lett.* (2013), 480 Gb/s
- Li et al., *Dynamics Days US*, *SPIE* (2014), >1 Tb/s

All passing NIST statistical tests

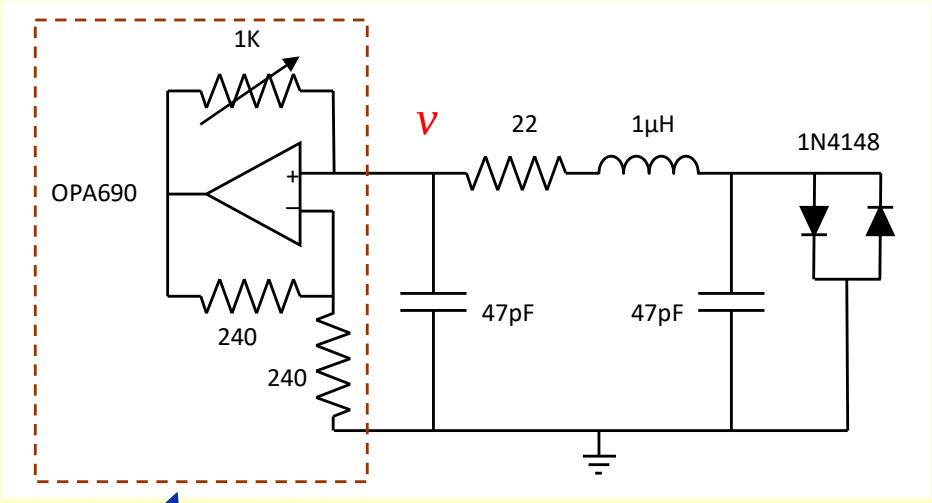
Statistical Testing

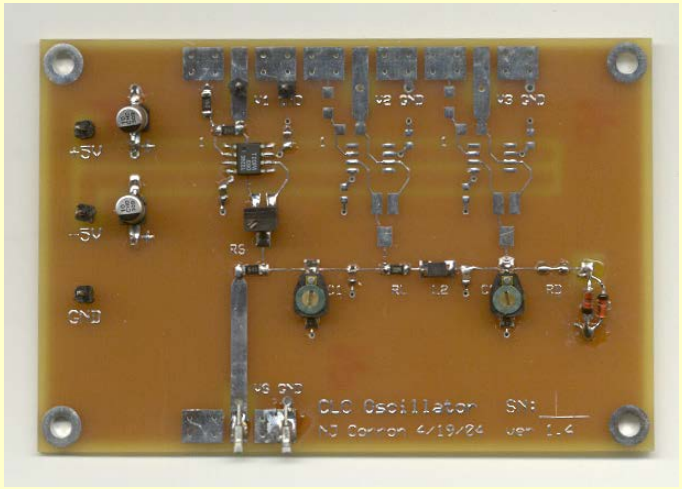


- **Pseudorandom Number Generators**
  - have correlations, yet pass statistical tests
- **Complex Physical System**
  - may also have correlations, yet pass statistical tests

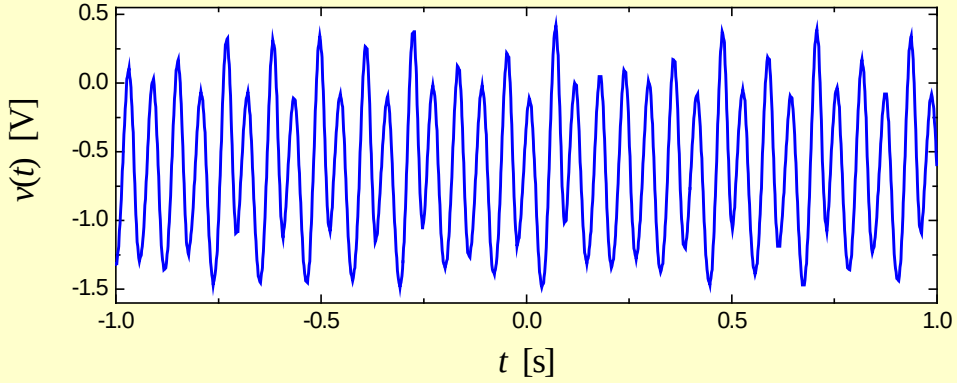
True Random Generators Must be Based on First-Principle Physics

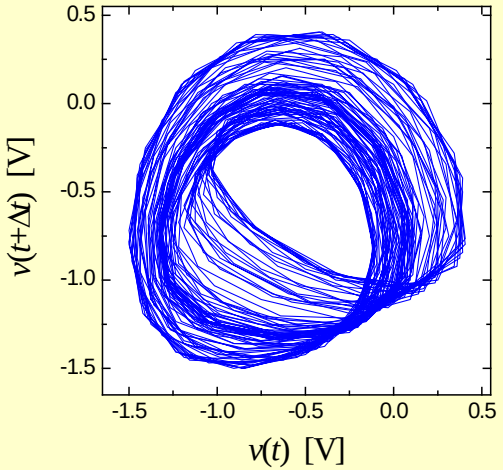
20-MHz Chaotic Oscillator





~20 MHz chaos

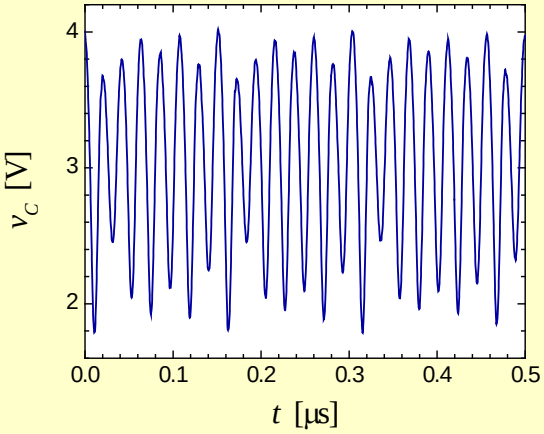


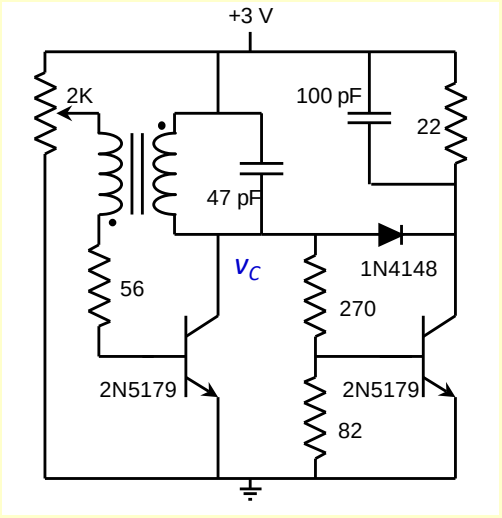


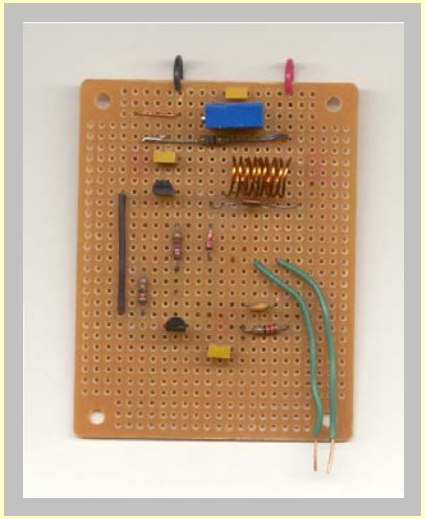
- N. J. Corron, J. N. Blakely, and S. D. Pethel, "Lag and Anticipating Synchronization without Time-Delay Coupling," *Chaos* 15, 023110 (2005)

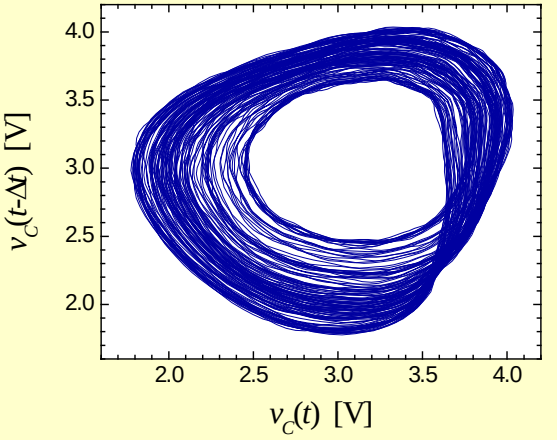
50-MHz Chaotic Transistor Oscillator

- **Two-Transistor RF Oscillator**
  - Modified tuned-collector LC oscillator
  - Exhibits chaotic dynamics
    - nearly sinusoidal waveform
    - peak-to-peak amplitude variations



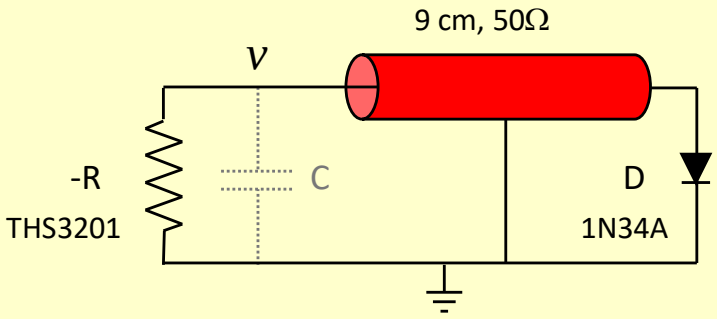


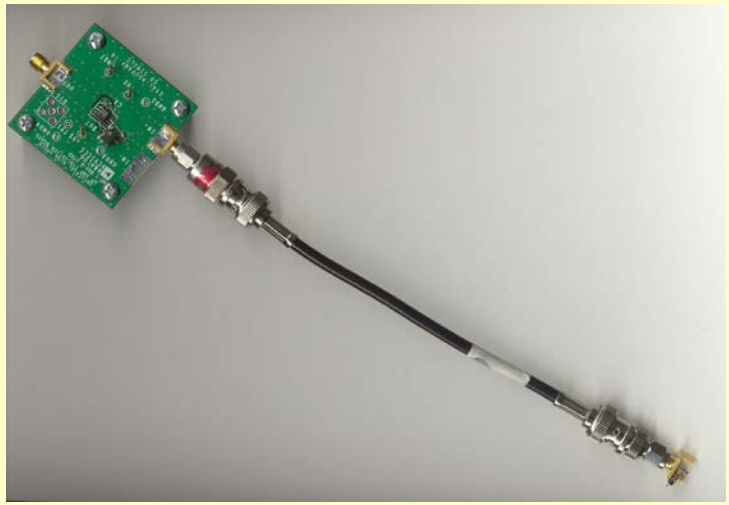


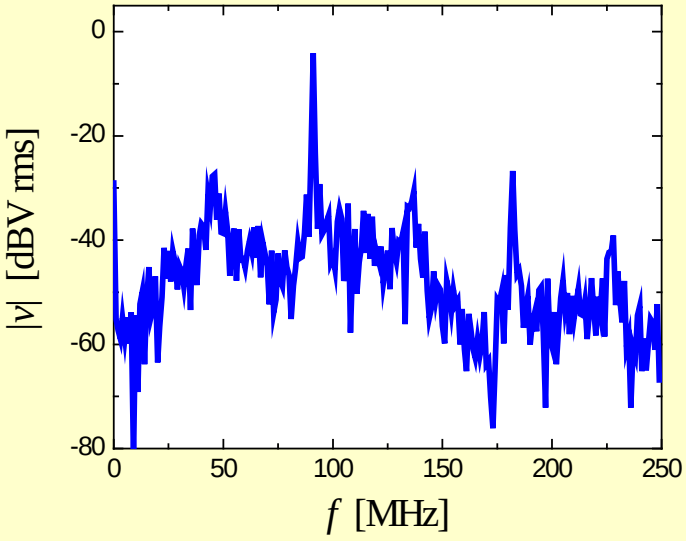


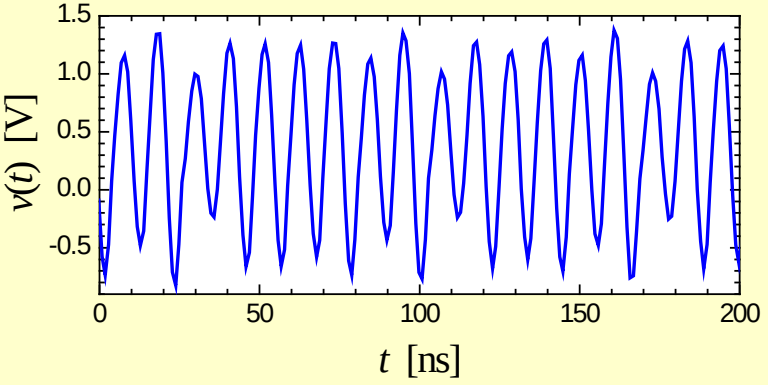
– N. J. Corron, B. A. Hopper, and S. D. Pethel, "Limiter Control of a Chaotic RF Transistor Oscillator," *International Journal of Bifurcation and Chaos* 13, 957 (2003)

90-MHz Chaotic TX-Line Oscillator





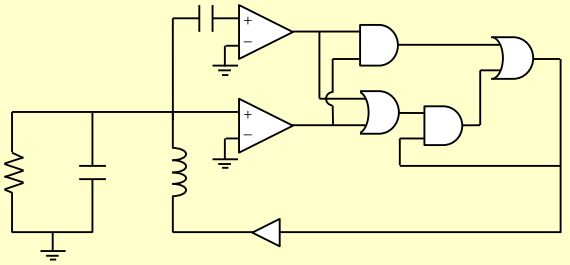




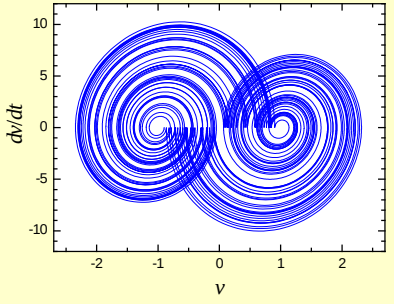
– J. N. Blakely and N. J. Corron, "Experimental Observation of Delay-Induced RF Chaos in a Transmission Line Oscillator," *Chaos* 14, 1035 (2004)

Exactly Solvable Chaos

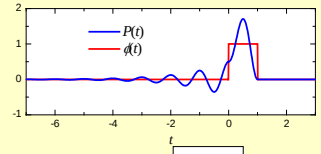
- **Analytically Solvable Nonlinear Systems**
  - piecewise linear dynamics
  - discrete switching events
  - straightforward hybrid circuit realization (analog and digital components)
- **First-Principle Understanding**
  - provably chaotic
  - exact return map
  - generating partition
  - complete symbolic representation
- **Tunable Parameters**
  - Markov partition
  - IID events (possibly biased)

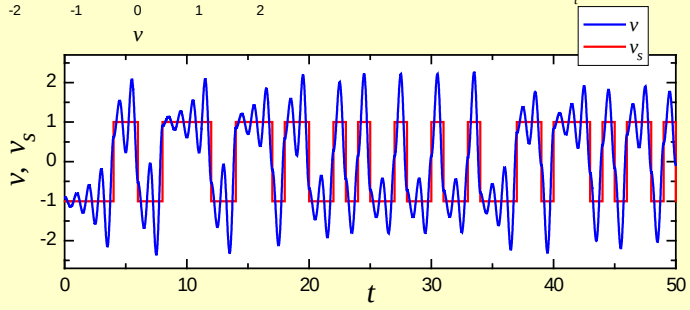


$$\frac{d^2v}{dt^2} - 2\beta\frac{dv}{dt} + \left(\omega^2 + \beta^2\right)\cdot\left(v - v_s\right) = 0$$



$$\frac{dv}{dt} = 0 \quad \Rightarrow \quad v_s = \mathrm{sgn}\left(v\right)$$



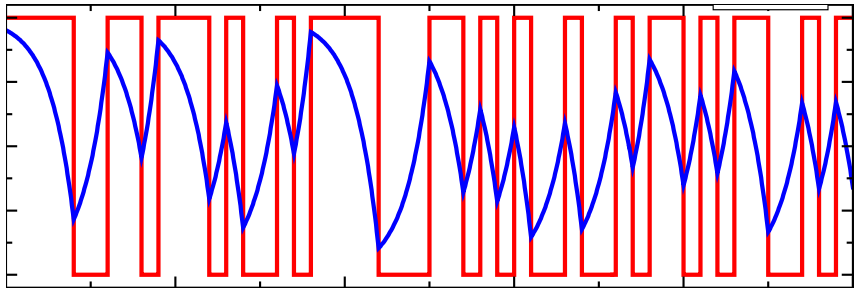


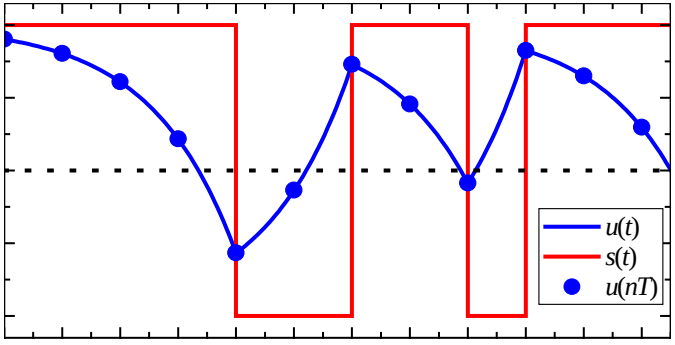
Exactly Solvable First-Order Chaos

- **Dynamical System**

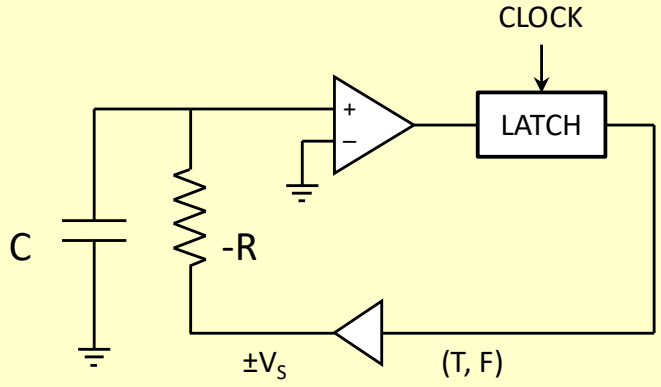
$$\frac{du}{dt} = u - s$$

$$s(nT) = \begin{cases} +1, & u(nT) > 0 \\ -1, & u(nT) \le 0 \end{cases}$$

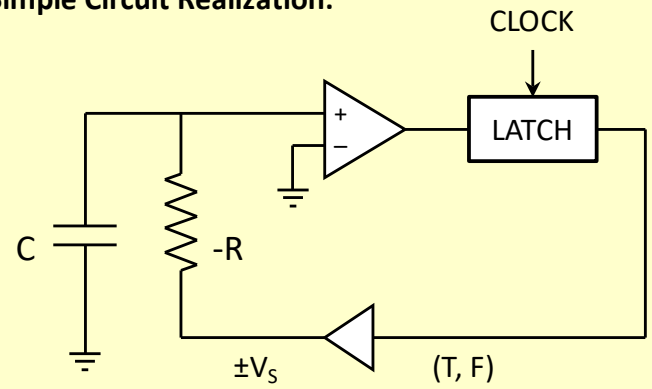


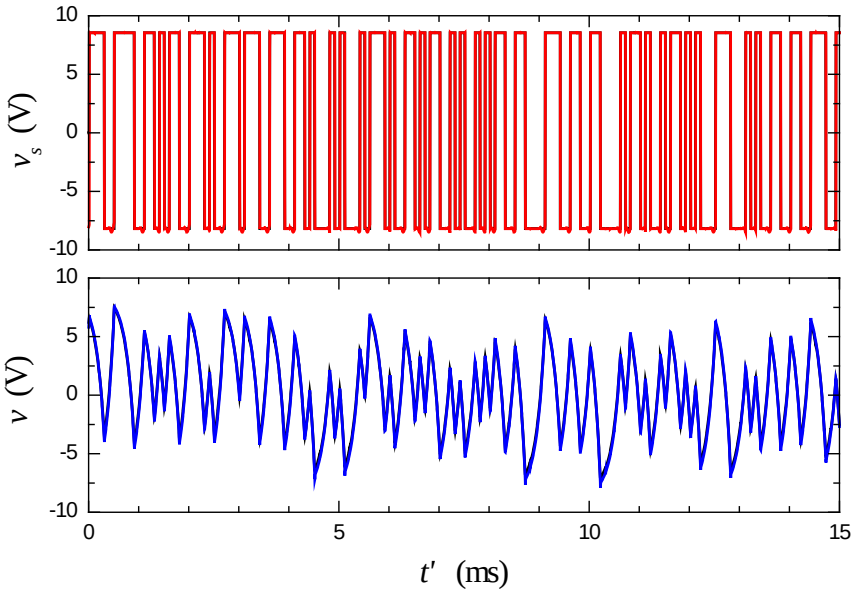


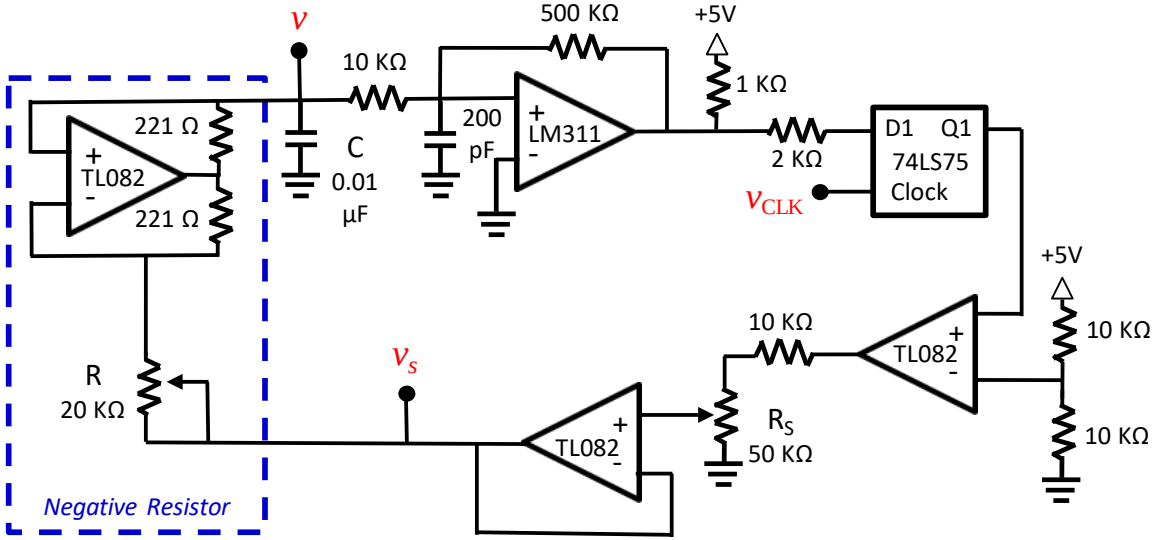
Simple Circuit Realization:

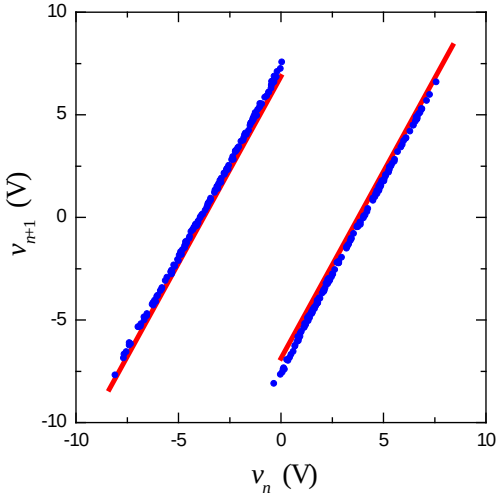


Simple Circuit Realization:









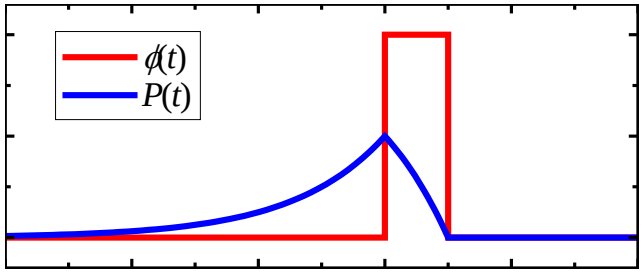
- **Analytic Solution**

$$s(t) = \sum_{n=-\infty}^{\infty} s_n \cdot \phi(t - nT)$$

$$u(t) = \sum_{n=-\infty}^{\infty} s_n \cdot P(t - nT)$$

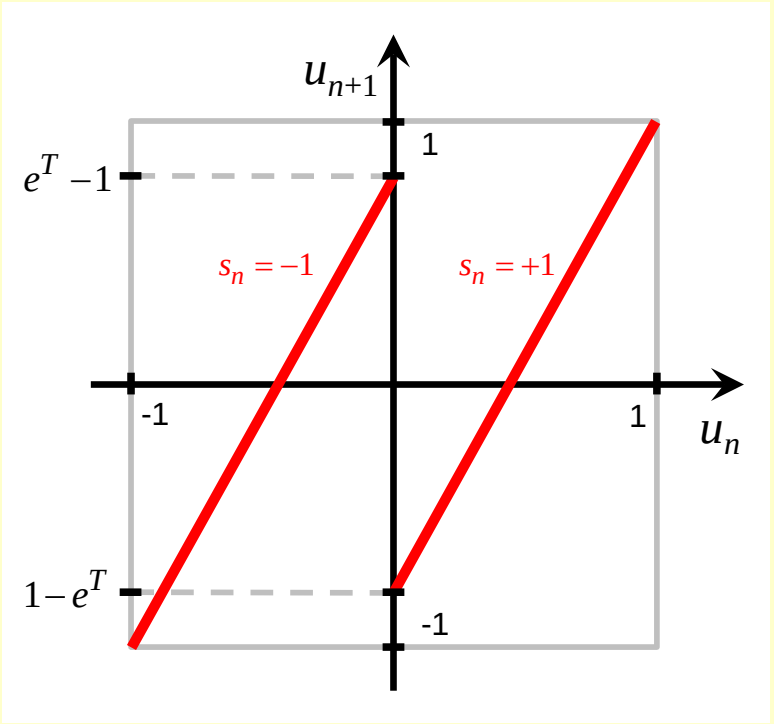
$$\phi(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$$

$$P(t) = \begin{cases} \left(1 - e^{-T}\right) e^{t}, & t < 0 \\ 1 - e^{(t-T)}, & 0 \le t < T \\ 0, & T \le t \end{cases}$$



Return Map:

$$u_{n+1} = s_n + e^{T}\left(u_n - s_n\right)$$



$$t_n = nT$$

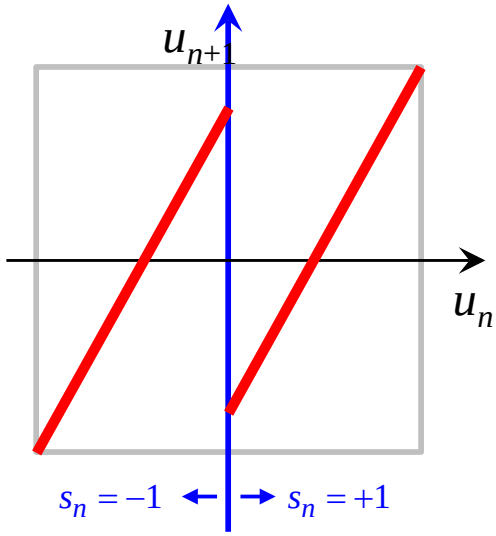
$$u_n = u(t_n) \qquad s_n = s(t_n)$$

Partitioning Random Bits

- **Dynamical System**

$$\frac{du}{dt} = u - s$$

$$s(nT) = \begin{cases} +1, & u(nT) > 0 \\ -1, & u(nT) \le 0 \end{cases}$$



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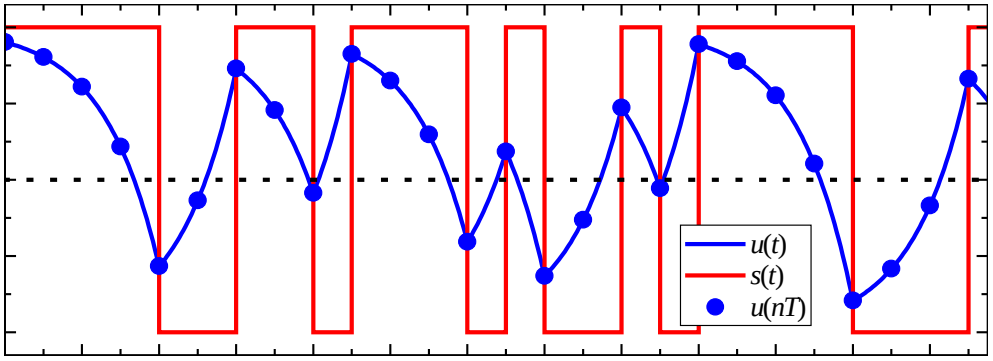
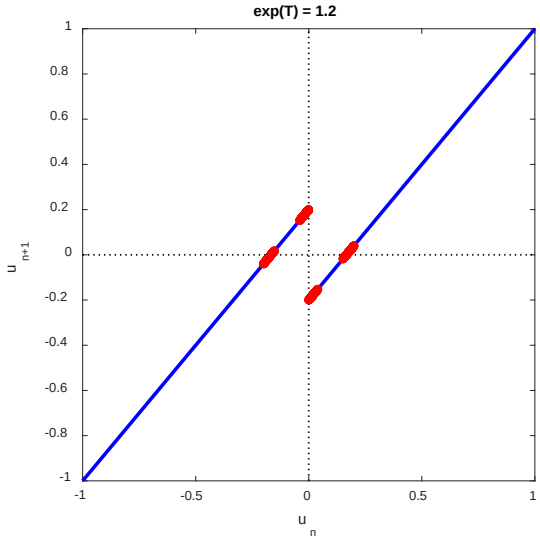
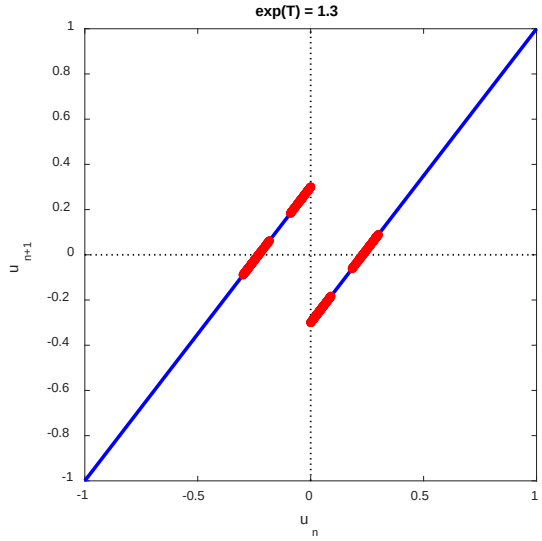
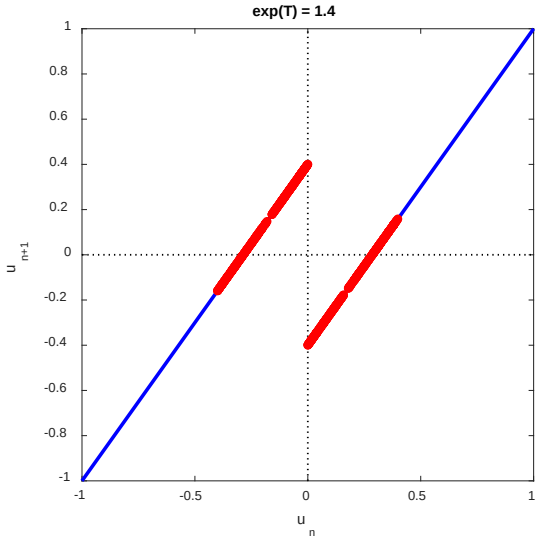
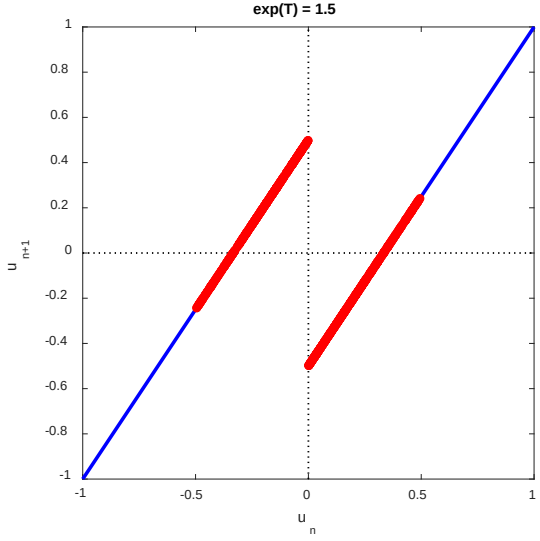


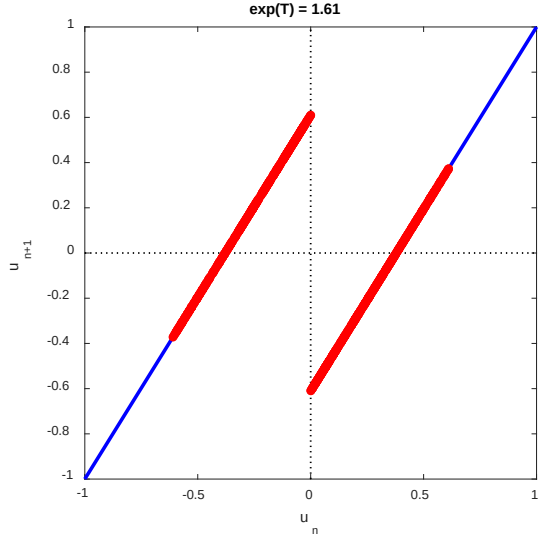
Tunable Return Map

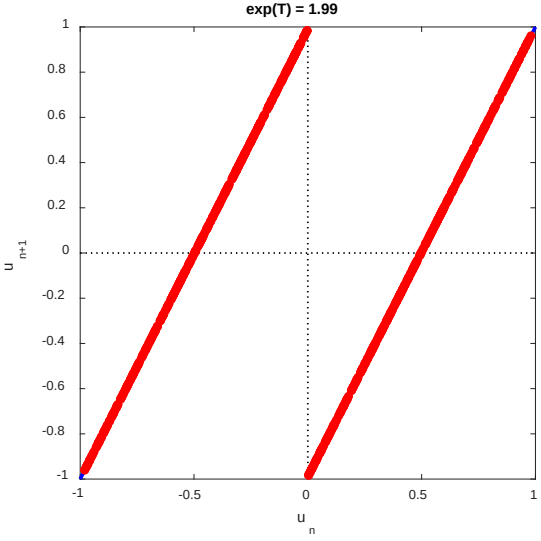






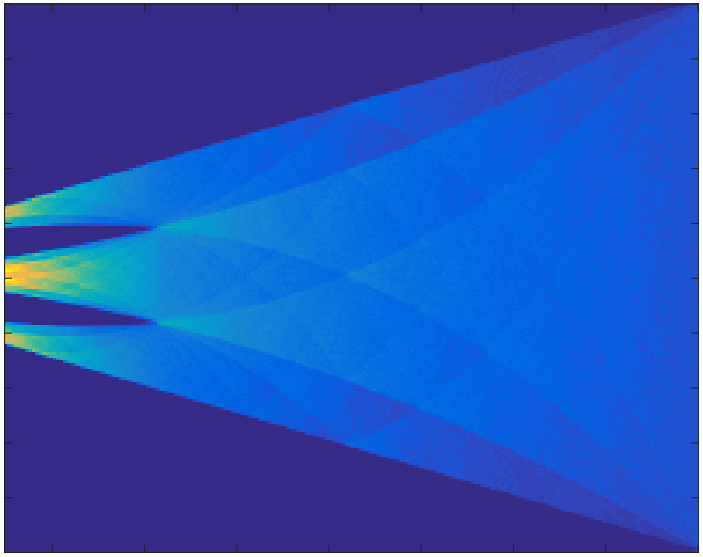


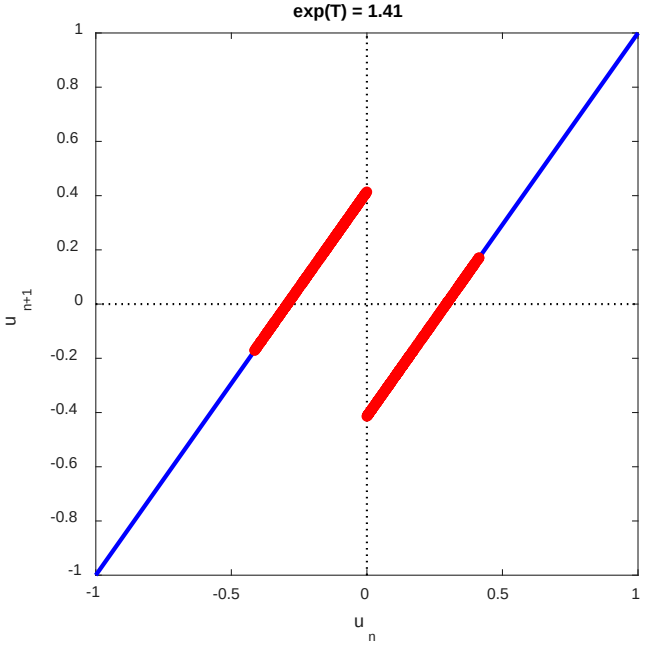




Special Tuning

- **Emergence of Period-1 Orbit**





$$e^T = \sqrt{2}$$

$$u_{n+1} = s_n + \sqrt{2}\left(u_n - s_n\right)$$

$$s_n = \mathrm{sgn}\left(u_n\right)$$

Maximal Sequences

- **Map Function**

$$f(u) = s + \sqrt{2}(u - s)$$

$$s = \operatorname{sgn}(u)$$

- **Closed Set**

$$f\left(1-\sqrt{2}\right) = 2\sqrt{2-3}$$

$$f\left(2\sqrt{2}-3\right) = 3-2\sqrt{2}$$

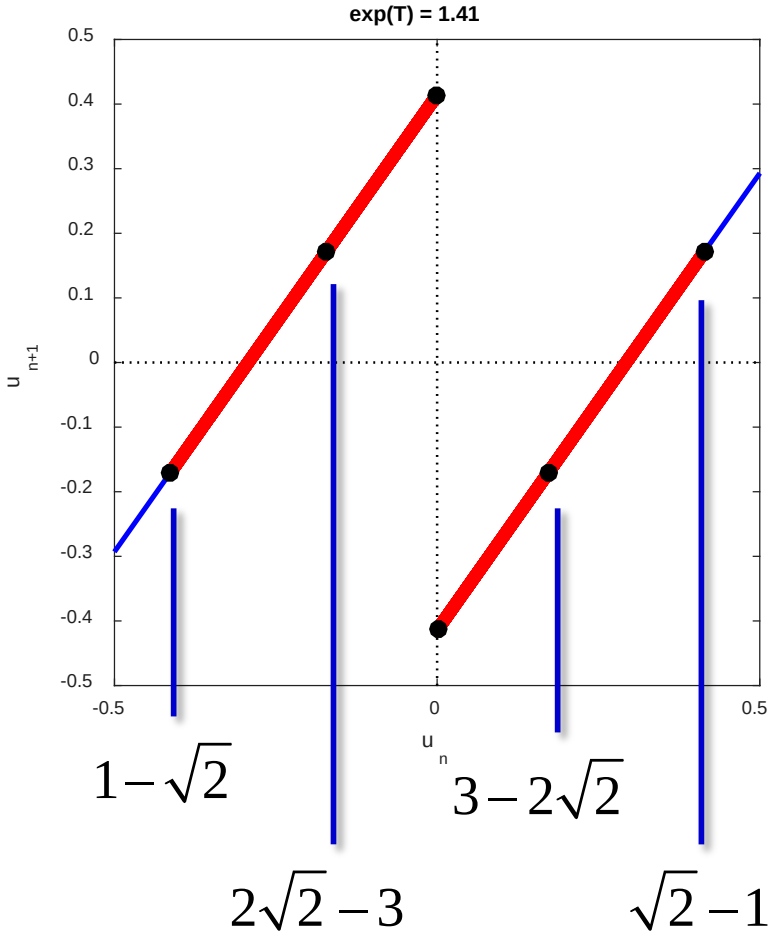
$$f\left(0^{-}\right) = \sqrt{2}-1$$

$$f\left(0^{+}\right) = 1-\sqrt{2}$$

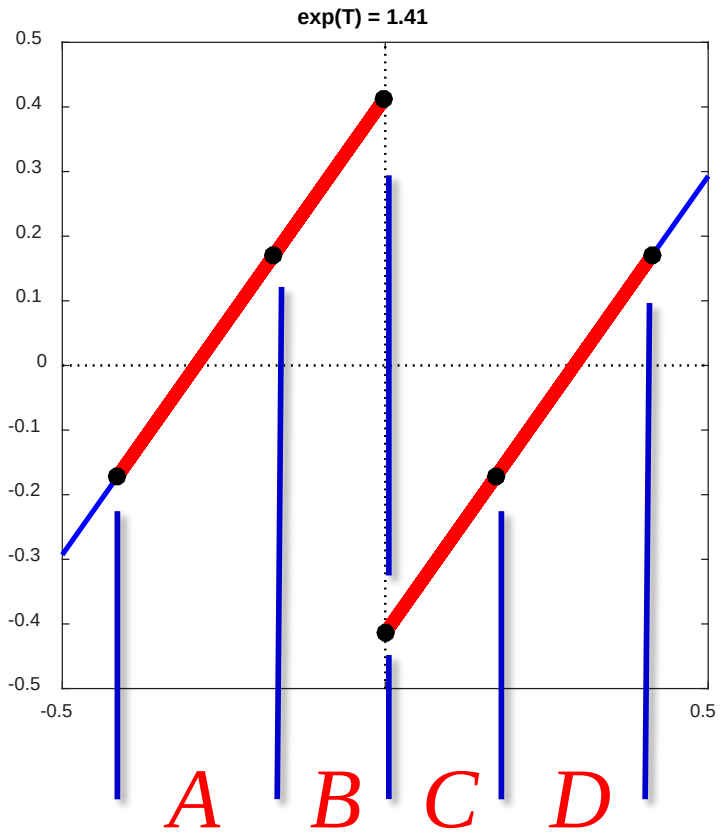
$$f\left(3-2\sqrt{2}\right) = 2\sqrt{2}-3$$

$$f\left(\sqrt{2}-1\right) = 3-2\sqrt{2}$$

$$u_{n+1} = f(u_n)$$

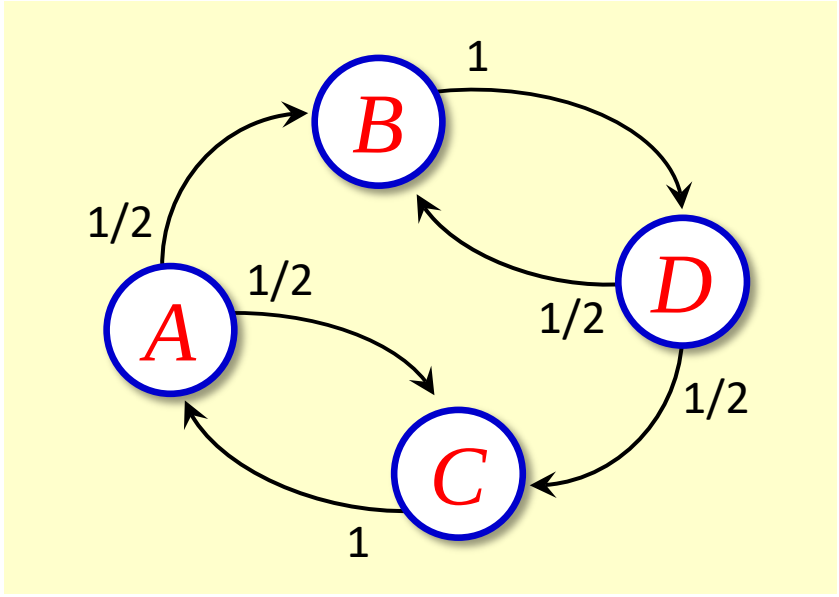


Markov Partition



$$\Phi = \{A, B, C, D\}$$

$$f\left(d\Phi\right) \in d\Phi$$



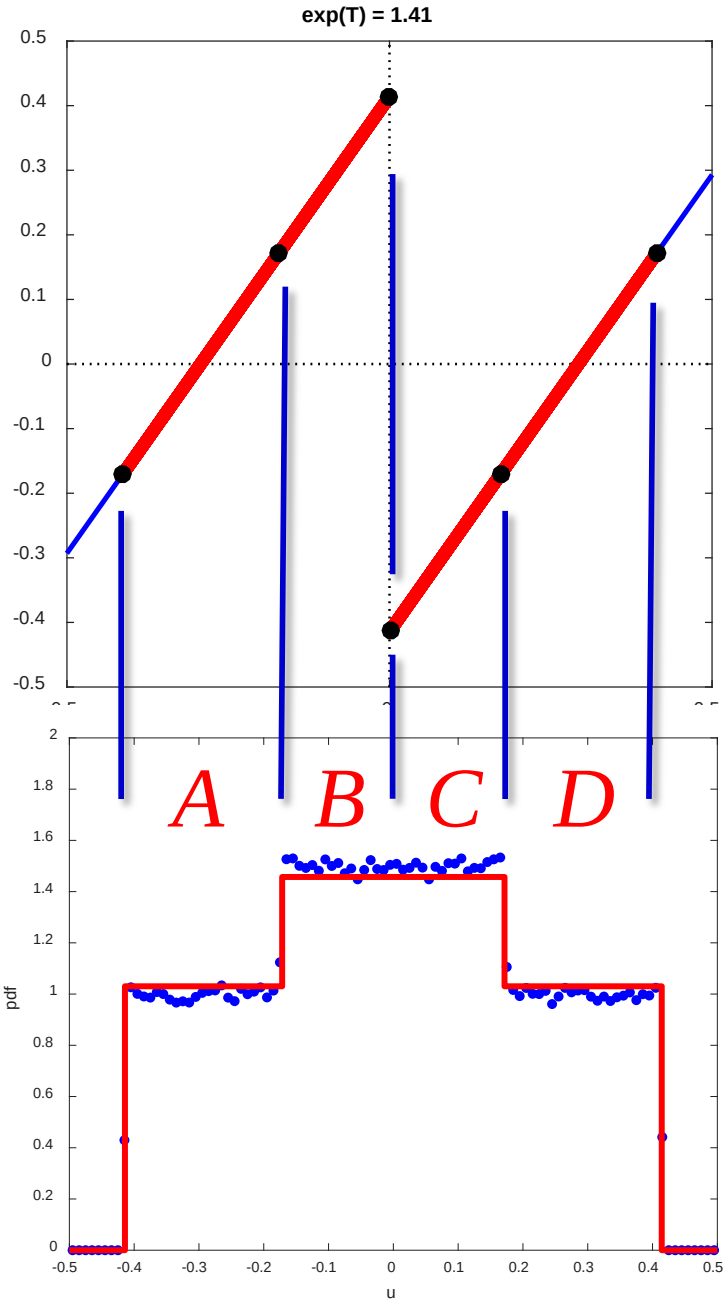
- **Natural Invariant Measure**

$$\mu_I = \sum_{f(J)\subset I} \frac{\mu_J}{\left|f'(J)\right|}$$

$$\begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \\ \mu_D \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \\ \mu_D \end{pmatrix}$$

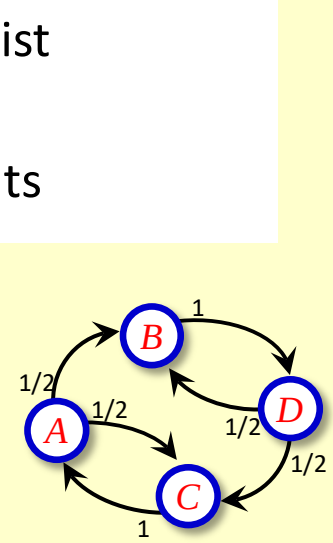
$$\mu_A = \mu_D = \frac{1}{4\left(3\sqrt{2}-4\right)}$$

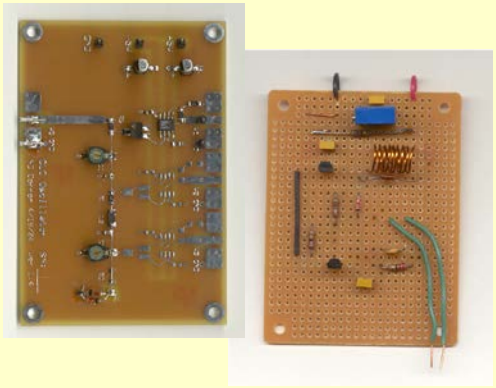
$$\mu_B = \mu_C = \frac{1}{4\left(3-2\sqrt{2}\right)}$$

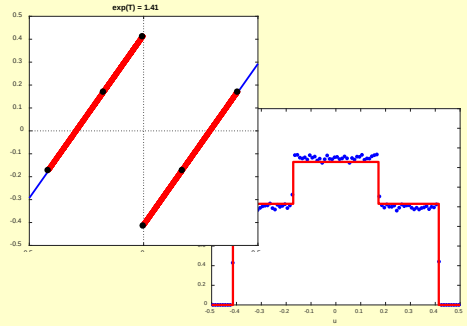


Conclusions

- **Motivation for Physical or True Random Number Generators**
  - it is not enough to be “good enough”
  - 100% assurance no correlations exist
  - transparent path from first-principle physics to random bits
- **Chaotic Electronic Circuits**
  - simple, low-cost devices
  - does not require exotic
- **Complete Analytic Description**
  - natural invariant density
  - independent, identically distributed events
- **Implementation**
  - easily integratable circuits
  - parameter feedback control required







Chaotic Circuit → Random Bits