An asymptotic observer-based monitoring scheme for a class of plug flow reactors
Efrén Aguilar-Garnica, Juan Paulo García-Sandoval, César Arturo Aceves-Lara and Froylán Mario E. Escalante

ABSTRACT
In this paper a monitoring tool is designed for a class of plug flow reactors whose mathematical model is described by a set of first-order partial differential equations with different coefficients in the convective terms. The infinite dimensional structure of such a tool is derived according to the methodology established in the design of the well-known asymptotic observer. As a consequence, it preserves the robustness of the aforementioned observer against the lack of information of the nonlinear terms involved in the model. The original structure of the estimator is then represented as a couple of integral equations by means of the method of characteristics and its behaviour is analyzed through simulation experiments. These simulations show that the mean square observation error is 0.58 when the proposed observer is implemented in a solid-waste anaerobic digestion process to estimate the evolution of biomass concentration.

Key words | method of characteristics, monitoring, partial differential equations, solid-waste anaerobic digestion process

NOMENCLATURE
\( q, \gamma, k_D, k_1, \chi, Y, \rho_m, K_m, m_f, m_g, K_f, K_g, a, L \)
\( z, t \)
\( z_0, t_0 \)
\( A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2, D_3, E_1, E_2, F_1, F_2, F_3, G_1, G_2, G_3, R_1, R_2, R_3, f(\xi), r(S(\xi, t)), g(S(\xi, t)) \)
\( W(\xi, t), S(\xi, t), B(\xi, t) \)

Parameters of the solid-waste anaerobic digestion model
Axial position and time variables, respectively
Points within \( z-t \) plane
Nonlinear functions of the state variables
Waste concentration, Volatile Fatty Acids concentration and methanogenic biomass concentration

\( v, v_1, v_2, K, K_1, K_2, C, C_{11}, C_{12}, C_{21}, C_{22}, \xi(z, t), \xi_1(z, t), \xi_2(z, t) \)
\( \alpha(z), \alpha_1(z), \alpha_2(z), \beta(t), \beta_1(t), \beta_2(t) \)
\( I, H, A, D, E, K_2^1, N, M, N_1, N_2, k \)

Matrices of the general dynamical model
State vector and partitions of the state vector
Initial conditions vector and partitions of the initial conditions vector
Boundary conditions vector and partitions of the boundary conditions vector
Matrices of the asymptotic observer-based scheme
Number of state variables, dimension of the nonlinearities vector, number of estimated variables, number of measured variables and rank of matrix \( K \), respectively

Efrén Aguilar-Garnica (corresponding author)
Froylán Mario E. Escalante
Department of Chemistry, Autonomous University of Guadalajara, 1201 Av. Patria, 44100 Guadalajara, México
E-mail: efren.aguilar@edu.uag.mx
Juan Paulo García-Sandoval
Department of Chemical Engineering, University of Guadalajara, 1451 Blvd. M.G. Barragán, 44430 Guadalajara, México
César Arturo Aceves-Lara
UPS, INSA, INP, LISBP, Université de Toulouse, F-31077 Toulouse, France
and
INRA, UMR792, Ingénierie des Systèmes Biologiques et des Procédés, Toulouse, France
and
CNRS, UMR5504, Toulouse, France 135 Avenue de Rangueil, Toulouse Cedex, F-31077 France

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Abbreviations

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<th>Abbr.</th>
<th>Description</th>
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<tr>
<td>MC</td>
<td>Method of Characteristics</td>
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<tr>
<td>FDM</td>
<td>Finite Difference Method</td>
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<tr>
<td>OCM</td>
<td>Orthogonal Collocation Method</td>
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<td>VFAs</td>
<td>Volatile Fatty Acids</td>
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<td>MSOE</td>
<td>Mean Square Observation Error</td>
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**INTRODUCTION**

In the last 20 years, process monitoring theory has shown a noteworthy evolution. Special attention has been paid to the design of robust estimation algorithms that are able to provide estimates of certain key process variables that are further required to implement efficient control schemes. One of these algorithms is known as the asymptotic observer (Bastin & Dochain 1990). This observer has been successfully implemented in stirred tank reactors (Dochain et al. 1992) and in biotechnological processes such as anaerobic digestion (Alcaraz-González 2001) to mention a few examples. Another robust algorithm is the so-called interval observer (Gouzé et al. 2000). This estimator preserves the robustness of the asymptotic observer against the nonlinearities of the process model and additionally, it is able to handle uncertainties in the process inputs. Satisfactory results have also been obtained with this observer when it is applied in chemical and biological processes (Alcaraz-González et al. 2002; Dochain 2003). In all these contributions, the observers were designed by considering that the process can be described by a mathematical model in ordinary differential equations. In the case of processes whose mathematical models are given by a set of partial differential equations, the asymptotic observer has also provided satisfactory results, as demonstrated by Aguilar-Garnica et al. (2005) and Dochain et al. (1997). More recently, Aguilar-Garnica et al. (2011) have proposed a design procedure for an interval observer for a class of distributed parameter processes which is based on the application of the Method of Characteristics (MC). The MC has found many applications, not only to design estimation algorithms but also to build robust control laws (García-Sandoval et al. 2008; Shang et al. 2005), probably because it has a remarkable property: it can provide an exact representation of the infinite dimensional model at certain specific axial points.

At the same time, a number of efforts have been made to obtain a mathematical model describing anaerobic digestion processes. One of these models is that proposed by Bernard et al. (2001) which was obtained for a continuous fixed bed reactor considering two microbial populations and two substrates uniform spatially distributed along the reactor. Even when this model is reliable and robust (in particular during abnormal operating conditions) it has a potential drawback: it assumes the process to behave like a stirred tank reactor (Fibrianto et al. 2000). In practice, the concentrations of the process components in liquid phase within a fixed bed reactor can be assumed to have uniform spatial distribution but in contrast, the concentration of biomass may exhibit a gradient. Regarding this fact, Schoefs et al. (2004) have proposed and validated a distributed parameter model for the anaerobic digestion process which is today considered as a diagnostic tool to predict process faults such as clogging (Harmand et al. 2002). Another model that takes into account biomass gradients in anaerobic digestion is the one obtained by Vavilin et al. (2003a). Specifically, this model was developed for anaerobic digestion of the organic fraction of municipal solid-waste and it includes a set of partial differential equations with two possible rate-limiting steps: polymer hydrolysis/acidogenesis and acetogenesis/methanogenesis. Even if these infinite dimensional models adequately describe the behaviour of the anaerobic digestion process, nothing has been done to take them as the basis for the design of a robust monitoring scheme.

The main contribution of this work is to propose a design procedure for a monitoring algorithm for a class of distributed parameter processes involving state variables travelling along different characteristic lines. Specifically, we applied the proposed estimation scheme to solid-waste anaerobic digestion whose mathematical model was...
obtained by Vavlin et al. (2003a). This scheme is deduced from the methodology to design an asymptotic observer and it is successfully implemented via simulation runs once the MC is used to derive an integral equation version of the scheme. The paper is organized as follows: the second section describes the general dynamical model for a class of plug-flow (bio) reactors whilst the third section shows the design of the asymptotic observer-based estimation scheme. Finally, the fourth section depicts the mathematical model of the solid-waste, the application of the estimation algorithm for such a process and some guidelines for simulating both the mathematical model and the estimator.

GENERAL DYNAMICAL MODEL

Let us consider a chemical or a biological process carried out in a plug flow reactor whose mathematical model is given by:

\[
\frac{\partial \xi(z, t)}{\partial t} = -v \frac{\partial \xi(z, t)}{\partial z} + Kf(\xi) + C\xi(z, t)
\]

with the following initial and boundary conditions:

\[
\xi(z, 0) = \alpha(z)
\]

\[
\xi(0, t) = \beta(t)
\]

In the above equations, \(\xi(z, t) \in \mathbb{R}^N\) is the state vector, \(\xi(\xi) \in \mathbb{R}^N\) is the nonlinearities vector, \(K \in \mathbb{R}^{N \times M}\) denotes a yield coefficients matrix and \(C \in \mathbb{R}^{N \times N}\) is the state matrix with known elements. Additionally, \(t\) represents the time variable whereas \(z (z \in [0, L])\) is the axial position, \(L\) is the reactor length and \(v \in \mathbb{R}^{N \times N}\) is considered as a positive and known matrix related to the superficial velocity of the inlet stream and suspended solids. In addition, \(\beta(t)\) is a column vector which is a sufficiently smooth function of time and \(\alpha(z) \in \mathcal{H}^N(0, L), \mathbb{R}^N\) where \(\mathcal{H}^N(0, L), \mathbb{R}^N\) is the infinite dimensional Hilbert Space of \(N\)-dimensional-like vector functions defined on the interval \([0, L]\). Once the general dynamical model is described one needs to introduce the following assumptions:

**Assumption 1** The elements of matrix \(K\) are constants and known. Furthermore, it is possible to compute the rank of matrix \(K\) such as rank \(K = k\).

**Assumption 2** There are \(N_1\) state variables that are gathered in \(\xi_1(z, t) (\xi_1(z, t) \in \mathbb{R}^{N_1})\) which are difficult to measure and \(N_2(N_2 = N - N_1)\) state variables that are grouped in \(\xi_2(z, t) (\xi_2(z, t) \in \mathbb{R}^{N_2})\) that can be measured. Besides \(N_2 \geq k\).

Considering Assumption 2, the general dynamical model described in Equation (1) can be split as:

\[
\frac{\partial \xi_1(z, t)}{\partial t} = -v_1 I_{N_1} \frac{\partial \xi_1(z, t)}{\partial z} + K_1 f(\xi) + C_{11} \xi_1(z, t) + C_{12} \xi_2(z, t)
\]

\[
\frac{\partial \xi_2(z, t)}{\partial t} = -v_2 \frac{\partial \xi_2(z, t)}{\partial z} + K_2 f(\xi) + C_{21} \xi_1(z, t) + C_{22} \xi_2(z, t)
\]

with the following initial and boundary conditions:

\[
\xi_1(z, 0) = \alpha_1(z)
\]

\[
\xi_2(z, 0) = \alpha_2(z)
\]

\[
\xi_1(0, t) = \beta_1(t)
\]

\[
\xi_2(0, t) = \beta_2(t)
\]

where \(\alpha_1(z) \in \mathbb{R}^{N_1}, \alpha_2(z) \in \mathbb{R}^{N_2}, \beta_1(t) \in \mathbb{R}^{N_1}, \beta_2(t) \in \mathbb{R}^{N_2}, K_1 \in \mathbb{R}^{N_1 \times M}, K_2 \in \mathbb{R}^{N_2 \times M}, C_{11} \in \mathbb{R}^{N_1 \times N_1}, C_{12} \in \mathbb{R}^{N_1 \times N_2}, C_{21} \in \mathbb{R}^{N_2 \times N_1}, C_{22} \in \mathbb{R}^{N_2 \times N_2}\) are the corresponding partitions of \(\alpha(z), \beta(t), K\) and \(C\). Furthermore \(v_1\) is a positive constant, \(I_{N_1}\) is the identity matrix of dimension \(N_1\) and \(v_2 \in \mathbb{R}^{N_2 \times N_2}\) is a matrix of constant elements. Thus, \(v\) can be written as \(v = \begin{pmatrix} v_1 I_{N_1} & 0 \\ 0 & v_2 \end{pmatrix}\).

ASYMPTOTIC OBSERVER-BASED ESTIMATION SCHEME DESIGN

The design of the proposed estimation scheme is partially based on the methodology to design an asymptotic observer for distributed parameter systems. This methodology states that if Assumptions 1 and 2 hold then it is possible to find a matrix \(H\) such that (Dochain & Vanrolleghem 2001):

\[
K_1 + HK_2 = 0
\]

The previous equation is fulfilled if \(H = -K_1 K_1^I\) where \(K_2^I\) is the generalized pseudo-inverse of \(K_2\) with the following property: \(K_2 K_2^I K_2 = K_2\). Note that in the case when \(K_2\) is square (i.e. \(N_2 = k\)) then \(K_2^I = K_2^{-1}\). Once \(H\) is computed then it is possible to define the following linear combination of the process state variables (Dochain 2000):

\[
\omega(z, t) = \xi_1(z, t) + H \xi_2(z, t)
\]

where \(\omega(z, t) \in \mathbb{R}^{N_1}\) is an
auxiliary variable. The dynamics of this variable is fully independent of the nonlinearities vector \( f(\xi) \) because Equation (8) holds and it is given by:

\[
\frac{\partial w(z, t)}{\partial t} = -v_1 I_N - \frac{\partial w(z, t)}{\partial z} + Aw(z, t) + D \xi_2(z, t) + E \frac{\partial \xi_2(z, t)}{\partial z}
\]

where \( A = C_{11} + HC_{21}, \ D = C_{12} + HC_{22} - AH \), and \( E = v_1 I_N H - H \nu_2 \). In addition, the initial and boundary conditions for the previous equation remain as:

\[
w(z, 0) = \xi_1(z, 0) + H \xi_2(z, 0) = \alpha_1(z) + H \alpha_2(z) \quad (10)
\]

\[
w(0, t) = \xi_1(0, t) + H \xi_2(0, t) = \beta_1(t) + H \beta_2(t) \quad (11)
\]

Usually, the next step of the methodology would be the transformation of Equation (9) into a set of ordinary differential equations by means of diverse methods such as the Finite Differences Method (FDM) or the Orthogonal Collocation Method (OCM). However, there are certain well-known drawbacks associated with the use of these methods. For instance if the FDM is considered to obtain the lumped model, then a large number of discretization points may be required in order to try to capture the dynamic behaviour of the distributed parameter model (Christofides 2001). Even when the OCM is able to substantially reduce the number of discretization points and some efforts have been made to compute such points in an optimal form (Lefèvre et al. 2000), the OCM is only considered an approximation method because the convenient selection of a weight on the error function is still an open problem. In this work we consider an alternative to this step of the methodology. Specifically we propose to represent Equation (9) with the MC (Rhee et al. 2000) as it is able to generate an exact solution for \( w(z, t) \) at certain specific axial points. Thus, if the MC is applied in Equation (9) at \( z = L \) defining \( \tau = L/v_1 \) as the residence time for the unmeasured variables, one obtains the following set of integral equations (see Appendix A, available online at http://www.iwaponline.com/wst/065/572.pdf):

If \( t \leq \tau \)

\[
w(L, t) = e^{At}w(0, t - \tau) + \int_0^t e^{A(t-\lambda)} \left[ D \xi_2(v_1(\tau + \lambda - t), \lambda) + E \frac{\partial \xi_2(v_1(\tau + \lambda - t), \lambda)}{\partial \lambda} \right] d\lambda.
\]

If \( t > \tau \)

\[
w(L, t) = e^{At}w(0, t - \tau) + \int_{t-\tau}^t e^{A(t-\lambda)} \left[ D \xi_2(v_1(\tau + \lambda - t), \lambda) + E \frac{\partial \xi_2(v_1(\tau + \lambda - t), \lambda)}{\partial \lambda} \right] d\lambda.
\]

In these equations, \( w(v_1(\tau - t), 0) \) represents the initial condition along the characteristic lines of the unmeasured variables and \( w(v_1(\tau - t), 0) \) denotes the boundary condition along the same lines. In both cases, they are unknown as they depend on the unmeasured variables (see Equations (10) and (11)). Then, it is necessary to set the following estimates:

\[
\hat{w}(z, 0) = \hat{\xi}_1(z, 0) + H \hat{\xi}_2(z, 0) = \hat{\alpha}_1(z) + H \hat{\alpha}_2(z)
\]

\[
\hat{w}(0, t) = \hat{\xi}_1(0, t) + H \hat{\xi}_2(0, t) = \hat{\beta}_1(t) + H \hat{\beta}_2(t)
\]

where the circumflex accent is considered to label the estimates. An asymptotic observer-based estimation scheme is readily derived by considering the previous estimates and the structure of Equations (12) and (13):

If \( t \leq \tau \)

\[
\hat{w}(L, t) = e^{At}\hat{w}(v_1(t - \tau), 0) + \int_0^t e^{A(t-\lambda)} \left[ D \xi_2(v_1(\tau + \lambda - t), \lambda) + E \frac{\partial \xi_2(v_1(\tau + \lambda - t), \lambda)}{\partial \lambda} \right] d\lambda.
\]

If \( t > \tau \)

\[
\hat{w}(L, t) = e^{At}\hat{w}(0, t - \tau) + \int_{t-\tau}^t e^{A(t-\lambda)} \left[ D \xi_2(v_1(\tau + \lambda - t), \lambda) + E \frac{\partial \xi_2(v_1(\tau + \lambda - t), \lambda)}{\partial \lambda} \right] d\lambda.
\]

In these equations the terms \( \xi_2(v_1(\tau + \lambda - t), \lambda) \) and \( \frac{\partial \xi_2(v_1(\tau + \lambda - t), \lambda)}{\partial \lambda} \) represent the vector of measured variables and its derivative along the characteristic line of the unmeasured variables. Note that the proposed estimation scheme is given by integral equations. This structure is different from those of the monitoring tool reported in literature which is given by a set of differential equations.
Finally, it is important to comment that it has been decided to present the structure of the proposed monitoring tool at the output of the reactor (i.e. at $z = L$) only to consider a specific axial point. As this tool was designed for any axial point $z = a$ (see Appendix A, available online at http://www.iwaponline.com/wst/065/572.pdf) it could be implemented at a different axial position or even at different regions at once. In the following section the asymptotic observer-based estimation scheme will be applied to a particular process: solid-waste anaerobic digestion.

**SIMULATION RESULTS AND DISCUSSION**

**Study case: solid-waste anaerobic digestion process**

**Mathematical model of the solid-waste anaerobic digestion process**

Vavilin et al. (2003a) studied the solid-waste anaerobic digestion process considering that polymer hydrolysis/acidogenesis and acetogenesis/methanogenesis are two possible rate-limiting steps of the overall process, that these steps are being inhibited by an intermediate product and that all transformation reactions involved in the conversion of Volatile Fatty Acids (VFAs) to methane can be gathered together as a single step. Under these assumptions they have proposed and validated a mathematical model for such a process which was carried out in a 1-D bioreactor with waste initially loaded. This model involves five state variables: the solid-waste concentration, the VFAs concentration, the methanogenic biomass concentration, the methane concentration and the sodium concentration. It is straightforward to verify that the equations related to methane and sodium concentrations are decoupled from the rest. Further, one can compute the Peclet number with the reported parameters obtaining relatively high values. As a consequence, the mathematical model proposed by Vavilin et al. (2003a) can be studied as a set of equations describing the behaviour of three state variables neglecting diffusive phenomena. This version of the original model proposed for the solid-waste digestion exactly matches with Equation (1) if

$$
K = \begin{bmatrix}
-k_1 & 0 \\
-\lambda k_1 & -1 \\
0 & Y
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -k_d
\end{bmatrix}
$$

where $W(z,t)$, $S(z,t)$, $B(z,t)$ denote the solid-waste concentration, the VFAs concentration and the methanogenic biomass concentration, respectively. In addition, $q$ is the volumetric liquid flow rate per unit surface area, $\gamma$ is the fraction of biomass transferred by liquid flow, $k_d$ is the specific biomass decay coefficient, $K_1$ is the first-order hydrolysis rate constant, $\lambda$ is a stoichiometric coefficient whereas $Y$ is the biomass yield coefficient. Furthermore, $\rho_m$ is the maximum specific rate of VFAs utilization, $K_s$ is the half-saturation constant for VFAs utilization while $z$ and $t$ denote the axial position and time variables, respectively. Further, $r(S(z,t))$ and $g(S(z,t))$ are dimensionless nonlinear functions describing nonionized VFAs inhibition of polymer hydrolysis/acidogenesis and acetogenesis/methanogenesis, respectively. These functions have the following structure

$$
r(S(z,t)) = \left(1 + \left(\frac{S(z,t)}{K_f}\right)^{m_f}\right)^{-1}
$$

and

$$
g(S(z,t)) = \left(1 + \left(\frac{S(z,t)}{K_g}\right)^{m_g}\right)^{-1}
$$

where $m_f$, $m_g$, $K_f$ and $K_g$ are inhibition degree indexes and inhibition constants, respectively. At this point it is important to remark that in some cases, sodium can be considered as an inhibitor. However, in this work we assume that the alkalinity and VFAs production are in balance (i.e. the process is stable) so that sodium as an inhibitor can be excluded from the model Equation (16) (Björnsson et al. 2001).

**Simulation of the solid-waste anaerobic digestion process**

According to the MC, the state variables involved in Equation (16), travel along three characteristic lines. These lines can be plotted in an $x$-$y$ diagram where the axial position varies along the $x$-axis and the $y$-axis represents time variations (see Figure 1). In this diagram, the characteristic lines depicting the solid-waste concentration are the vertical (i.e. with an infinite slope) dashed-dot lines, the characteristic lines of the VFAs concentration have a slope equal to $1/q$ (dotted lines) and the biomass concentration can be
represented by the characteristic lines with a slope equal to $1/\gamma q$ (continuous lines). The grid generated in Figure 1 is a combination of triangles with the structure depicted in Figure 2.

In this figure: $p_i$ ($i = 1 \ldots 7$) represent certain $z-t$ positions. It can be demonstrated (see Appendix B, available online at http://www.iwaponline.com/wst/065/572.pdf) that the state variables at $p_4$ are obtained with the set of equations:

$$W_{p_4} = W_{p_1} - k_1 W_{p_1} r(S_{p_1})(t_{p_4} - t_{p_1})$$

(17)

$$S_{p_4} = S_{p_1} + \left[ \chi k_1 W_{p_1} r(S_{p_1}) - \rho_m g(S_{p_1}) \frac{S_{p_1} B_{p_3}}{K_S + S_{p_1}} \right] (t_{p_4} - t_{p_1})$$

(18)

$$B_{p_4} = B_{p_2} + \left[ Y \phi_m g(S_{p_2}) \frac{S_{p_1} B_{p_2}}{K_S + S_{p_1}} - k_d B_{p_2} \right] (t_{p_4} - t_{p_2})$$

(19)

where the subscripts $p_1, p_2, p_3$ and $p_4$ in both the state variables and time, are considered to represent their values at such $z-t$ positions. Note that $W_{p_4}$ is computed with Equation (17) from values at $p_1$ (i.e. below $p_4$ along the characteristic line for the solid-waste), $S_{p_4}$ is obtained with Equation (18) from information at $p_3$ (i.e. under $p_4$ along the characteristic line for the substrate concentration) whereas $B_{p_4}$ is generated with Equation (19) from values along the characteristic line for the biomass concentration that are located before $p_4$. Once these variables are obtained, they can be considered as ‘starting points’ to compute their numerical values at $p_5, p_6$ and $p_7$.

The results of this procedure are shown in Figures 3–5 for a period of 160 d using 46 positions along the $z$-axis and were obtained with the parameters depicted in Vavilin et al. (2005a, b): $q = 1 \text{ m d}^{-1}$, $\gamma = 0.1$, $k_D = 0.001 \text{ d}^{-1}$, $k_1 = 0.011 \text{ d}^{-1}$, $\chi = 0.48$, $Y = 0.12$, $\rho_m = 0.31 \text{ d}^{-1}$, $K_S = 1.2 \text{ g L}^{-1}$, $m_f = m_g = 3$, $K_f = 16 \text{ g L}^{-1}$ and $K_g = 10 \text{ g L}^{-1}$. Further, for the sake of simulations we have decided to set $L = 10$ m and to consider the following initial and boundary conditions:

$W(z, 0) = 250 \text{ g L}^{-1}$, $S(z, 0) = 15(1 - 0.08z)$, $B(z, 0) = 4(1 + 0.05z)\text{(gL}^{-1})$, $W(0, t) = 250 \text{ g L}^{-1}$, $S(0, z) = 14 \text{ g L}^{-1}$ and $B(0, t) = 3 \text{ g L}^{-1}$. At this point it is important to say that these conditions were arbitrarily selected. Nevertheless, they can be changed without modifying either the simulation scheme or the estimation algorithm presented in this paper.
In order to understand the behaviour of biomass and substrate concentrations depicted in Figures 4 and 5, respectively, it is necessary to look at Figure 1 again. In this Figure, the continuous line that departs from \( t = 0 \) and \( z = 0 \) represents the very first portion of biomass governed by the boundary condition (i.e. ‘fresh fluid’ of biomass) leaving the reactor at \( t = \tau = 100 \, \text{d} \). The continuous lines below this one are influenced by the initial condition whereas the continuous lines departing from \( z = 0 \) and \( t > 0 \) are governed by the boundary condition. As a consequence, one expects a change in the dynamical behaviour of the biomass concentration just at \( t = \tau \). This change is clearly observed in Figure 5. With regard to the substrate concentration, one also expects a variation not at \( \tau \) but at a different time-point \( \tau^* \). This is because the substrate concentration travels in a different characteristic line from that of the biomass concentration. If we consider the set of parameters already reported, then \( t = \tau^* = 10d \) (as can be seen in Figure 4). Finally note that these variations are not present in the behaviour of the solid-waste (see Figure 3) because it is assumed that it remains within the bioreactor along the digestion process.

**Application of the proposed estimation algorithm to the solid-waste anaerobic digestion**

Let us assume that it is possible to measure the solid-waste concentration and the VFAs concentration (i.e. \( \xi_2(z, t) = [W(z, t) \ S(z, t)]^T \)) in order to estimate the methanogenic biomass concentration (i.e. \( \xi_1(z, t) = B(z, t) \)). In this case, the mathematical model Equation (1) can be split as it is described in Equations (2)–(7) with

\[
\begin{align*}
\nu_1 &= q \gamma, & I_{N_1} &= 1, & K_1 &= [0 \ Y], & C_{11} &= -k_d, & C_{12} &= [0 \ 0],
\end{align*}
\]
\[ v_2 = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}, \quad K_2 = \begin{bmatrix} -k & 0 \\ \lambda k & -1 \end{bmatrix}, \]
\[ C_{21} = 0 \quad \text{and} \quad C_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

Considering these elements, one can compute the following terms: \( H = [x \ y] \), \( A = -k_i \), \( D = [k_2 X \ k_2 Y] \) and \( E = [q q Y q Y - q Y] \). Then, one gets the structure of an asymptotic observer-based estimation scheme for the solid-waste anaerobic digestion process which provides estimates for the methanogenic biomass concentration at \( z = L \) when these terms are included in Equations (14)–(15).

**Simulation of the asymptotic observer-based estimation scheme**

The performance of the proposed estimation scheme is depicted in Figure 6 by means of small circles \((\tilde{B}(L, t))\) whereas the continuous line denotes the ‘real’ evolution of the biomass concentration at \( z = L \) \((B(L, t))\) which is provided by the simulations of the solid-waste anaerobic digestion.

The circles in Figure 6 were obtained with Equation (14) and \( \tilde{B}(z, 0) = B(z, 0) - 0.1 \) if \( t \leq \tau \) and with Equation (15) and \( \tilde{B}(0, t) = B(0, t) - 0.1 \) if \( t > \tau \). Note that both Equations (14) and (15) require the knowledge of integral terms whose arguments involve \( \tilde{z}_2[\tau_1(t + \lambda - \tau), \lambda] \) and \( (\partial\tilde{z}_2[\tau_1(t + \lambda - \tau), \lambda])_{\partial z} \). The first term was obtained directly from the model whereas the second one was computed with finite differences. Then, the areas enclosed by the function \( e^{A(t - \lambda)}[D_{22}\tau_1(t + \lambda - t) + E(\partial\tilde{z}_2[\tau_1(t + \lambda - t), \lambda])_{\partial z}] \) were computed. As these computations are approximations, they cause the zig-zag behaviour of the observer. From the results presented in Figure 6, it is possible to obtain the observation error \( e(t) \) which is defined as \( e(t) = B(L, t) - (\tilde{B}(L, t) \) (see Figure 7) and the square error \( e^2(t) \).

Then, a quantitative analysis of the performance of the proposed observer can be carried out if one calculates the Mean Square Observation Error (MSOE) which is defined as \( \int_0^T e^2(t) dt \) (Bastin & Dochain 1990). For the present study case, the MSOE has a value of 0.58, obtained by applying the well-known Simpson’s 3/8 rule. As this value is relatively low, it can be concluded that proposed monitoring scheme reconstructs the estimated state satisfactorily at the output of the reactor even when the infinite dimensional model shows a discontinuity from the region influenced by the initial conditions to the region dominated by the boundary conditions (i.e. at \( t = \tau \)).

**Figure 6** Performance of the asymptotic observer-based scheme for biomass at \( z = L \).

The small circles (o) represent observed values, whilst the graph line represents simulated model values.

**Figure 7** | Dynamical behaviour of the observation error \((e(t))\).

**CONCLUSIONS AND PERSPECTIVES**

This paper deals with the design of a robust estimation scheme for a class of distributed parameter reactors whose model is described by a set of hyperbolic partial differential equations containing different convective terms. The structure of this scheme was obtained when the Method of Characteristics was applied to its infinite dimensional version that was derived from the design methodology of an asymptotic observer. Such structure can be seen as a couple of integral equations requiring measurements along the characteristic line of the estimated variable. The behaviour of the proposed scheme was simulated for a solid-waste anaerobic digestion process obtaining satisfactory results in the estimation of
methanogenic biomass concentration. This was possible because it was assumed that a large amount of measurements are available. Even when this observer performs adequately, it is difficult to accomplish with the aforementioned assumption in practice and, furthermore, it should be noted that its stability is not guaranteed. Thus, the main goal of future work now is to reduce the number of sensors that are necessary to carry out the monitoring by assuring the stability of the observer.

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