Pulse Propagation in a Laterally Heterogeneous Solid Elastic Sphere

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Summary

The motion of a solid elastic sphere caused by an impulsive point source has been calculated by a finite difference scheme. Results are obtained for a radially and laterally heterogeneous sphere. Both discontinuous changes in elastic parameters and density, and their continuous variation inside the sphere are treated. The results show reflection and diffraction effects. A stability analysis for the finite difference scheme in spherical co-ordinates is given. The method of solution presented here is general enough to include the actual variations in density and elastic parameters in the Earth. This includes the well-known models of radial distribution and the recently observed lateral inhomogeneity in the mantle.

Introduction

The propagation of a pulse in an elastic sphere was studied previously in cases which can be solved by methods relying on separation of variables (Alterman & Aboudi 1969a, 1969b, and references cited in these papers).

An example of a laterally heterogeneous elastic sphere, for which the equations of motion are not separable, is investigated in the present paper applying a finite difference method. In a previous analysis of pulse propagation in a fluid sphere the wave equation for the velocity potential was discretized in cylindrical co-ordinates, and the motion of a laterally heterogeneous fluid was obtained (Alterman & Aboudi 1968). For the present case of a solid elastic sphere it is found to be more practical to treat the equations of motion for the displacement components directly (and not the wave equations for the potentials) and to write the finite difference scheme in a grid in spherical co-ordinates (Section 1). The stability of the scheme is investigated and a special finite difference stencil is applied near the origin (Sections 2 and 3).

To check the accuracy of results, the case of a source at the centre of a homogeneous sphere is solved by the present scheme and compared with previous results. When the source is at an arbitrary location in the sphere, an estimate of the accuracy is provided by a comparison of different grid sizes.

Propagation in a heterogeneous sphere is considered in the case when the sphere consists of several homogeneous regions and in the case of continuous variation in density and elastic parameters. In the first case boundary conditions are applied at the interface between adjacent homogeneous regions. In the second case the complete equations of motion for a heterogeneous medium are solved. Similarly the interface between two homogeneous regions can be treated as a discontinuity or it can be obtained as the limit of a continuous transition. The change of amplitude
and dispersion in reflected and transmitted pulses with width of transition zone is found. A transition over two grid intervals is found to approximate sufficiently well the results for adjacent homogeneous regions and is found to be computationally simpler than application of boundary conditions (Sections 4 and 8). A similar analysis is applied at the free surface (Section 7).

As a simple example of lateral heterogeneity the configuration depicted in Fig. 1 is considered. A conical wedge has density and elastic parameters which differ from the density and parameters of the remaining part of the sphere. The effect of a high density and high velocity wedge, of a low density and low velocity wedge and of an empty wedge are compared. Section 8 gives a comparison of amplitudes and phases of reflected and diffracted pulses for varying density and wave velocities in the wedge.

The method of solution is general enough to include the actual variations in density and elastic parameters in the Earth. This includes the well-known models of radial distribution and the recently observed lateral inhomogeneity in the mantle (Toksoz, Chinnery & Anderson 1967; Greenfield & Sheppard 1969; Toksoz, Arkani-Hamed & Knight 1969, Kanamori—in press).

1. The equations of motion in a heterogeneous medium and their finite difference approximation in spherical co-ordinates

The equations of motion for a heterogeneous elastic medium can be written in the form

\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \text{grad div } \mathbf{u} - \mu \text{ rot rot } \mathbf{u} + \text{div } \mathbf{u} \cdot \text{grad } \lambda + 2(\text{grad } \mu \cdot E). \]  

(1)

Here \( \mathbf{u} \) is the displacement vector, \( \lambda \) and \( \mu \) are the Lamé elastic parameters, \( \rho \) the density and \( E \) is the strain–tensor. In the system of spherical co-ordinates \( (R, \theta, \phi) \) and for motions having cylindrical symmetry for which also \( u_\phi = 0 \), let us denote \( \mathbf{u} = (u_R, u_\theta) \) and

\[ E = \begin{pmatrix} e_{RR} & e_{R\theta} \\ e_{\theta R} & e_{\theta\theta} \end{pmatrix} \]  

(2)

introducing the vector \( \mathbf{v} = R \mathbf{u} \)

(3)

of components \( F = Ru_R, \ G = Ru_\theta \).  

(4)
Pulse propagation in a solid elastic sphere

Equation (1) can be written as:

$$\frac{1}{c_p^2} \frac{\partial^2 \mathbf{v}}{\partial t^2} = \mathbf{A} \frac{\partial^2 \mathbf{v}}{\partial R^2} + \frac{\mathbf{B}}{R} \frac{\partial \mathbf{v}}{\partial R} - \frac{\mathbf{C}}{R^2} \frac{\partial^2 \mathbf{v}}{\partial \theta^2} + \frac{\mathbf{D}}{R^2} \frac{\partial \mathbf{v}}{\partial \theta} + \frac{\mathbf{E}}{R^2} \frac{\partial \mathbf{v}}{\partial \phi} + \frac{\mathbf{F}}{R} \frac{\partial \mathbf{v}}{\partial \phi} + \left[M \frac{\partial \mathbf{v}}{\partial R} + N \frac{\partial \mathbf{v}}{\partial \theta} + Q \frac{\partial \mathbf{v}}{\partial \phi}\right].$$

Here:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad B = \begin{pmatrix} 0 & (1-b) \cot \theta \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 2 \cot \theta \\ 0 & \frac{1}{\sin \theta} \end{pmatrix}, \quad D = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix},$$

$$E = \begin{pmatrix} b \cot \theta & -2 \\ 1+b & \cot \theta \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1-b \\ 1-b & 0 \end{pmatrix},$$

$$M = \frac{1}{\rho c_p^2} \left(\frac{\partial \lambda}{\partial R} + 2\frac{\partial \mu}{\partial R} \frac{\partial \mu}{\partial \theta} \right) \frac{\partial \lambda}{\partial R},$$

$$N = \frac{1}{\rho c_p^2} \left(\frac{\partial \mu}{\partial R} \frac{\partial \lambda}{\partial \theta} + \frac{\partial \mu}{\partial \theta} + 2\frac{\partial \mu}{\partial R} \frac{\partial \mu}{\partial \theta} \right),$$

$$Q = \frac{1}{\rho c_p^2} \left(\frac{\partial \lambda}{\partial \theta} + 2\frac{\partial \mu}{\partial R} \frac{\partial \mu}{\partial \theta} \frac{\partial \lambda}{\partial \theta} + \frac{\partial \mu}{\partial \theta} + 2\frac{\partial \mu}{\partial R} \frac{\partial \mu}{\partial \theta} \frac{\partial \lambda}{\partial \theta} \right).$$

$$b = \frac{c_p^2}{c_p^2} = \frac{\lambda + 2\mu}{\rho}, \quad c_p^2 = \frac{\mu}{\rho}. \quad (7)$$

In a homogeneous medium

$$M = N = Q = 0. \quad (8)$$

A finite difference approximation to equation (5) is obtained by replacing all derivatives by central finite differences. The finite difference equation can then be solved, expressing $v$ at the time $t + \Delta t$ in terms of $v$ at the time $t$ and $t - \Delta t$. The result is:

$$v(R, \theta, t + \Delta t) = \varepsilon_1^2 \left[ A + B \frac{\Delta R}{2R} + M \frac{\Delta R}{2} \right] v(R + \Delta R, \theta, t) + \varepsilon_1^2 \left[ A - B \frac{\Delta R}{2R} - M \frac{\Delta R}{2} \right] v(R - \Delta R, \theta, t) + 2 \left[ 1 - \varepsilon_1 e_2^2 \left( \frac{\Delta R}{R} \right)^2 - D e_2^2 + \frac{Q}{2} \varepsilon_2^2 \left( \frac{\Delta R}{R} \right)^2 \right] v(R, \theta, t) + \varepsilon_2^2 \left[ D + E \frac{\Delta \theta}{2} + N \frac{R \Delta \theta}{2} \right] v(R, \theta + \Delta \theta, t) + \varepsilon_2^2 \left[ D - E \frac{\Delta \theta}{2} - N \frac{R \Delta \theta}{2} \right] v(R, \theta - \Delta \theta, t) + \frac{1}{2} e_2 \varepsilon_2 \left[ H[v(R + \Delta R, \theta + \Delta \theta, t) + v(R - \Delta R, \theta - \Delta \theta, t) - v(R + \Delta R, \theta - \Delta \theta, t) - v(R - \Delta R, \theta + \Delta \theta, t)] - v(R, \theta, t - \Delta t) \right]. \quad (9)$$
Here

\[ \begin{align*}
\varepsilon_1 &= \frac{c_p \Delta t}{AR} \\
\varepsilon_2 &= \frac{c_p \Delta t}{R \Delta \theta}
\end{align*} \]

and \( c_p \) is the compressional wave velocity

\[ c_p = \sqrt{\left( \frac{\lambda + 2\mu}{\rho} \right)} . \]

Equation (9) is valid for \( 0 < \theta < \pi \). For \( \theta = 0 \) and \( \theta = 180^\circ \) the coefficients \( B, C, E \) and \( Q \) in equations (6) become infinite. In this case the corresponding terms were evaluated taking into account that due to symmetry

\[ u_\theta = 0; \quad \frac{\partial u_\theta}{\partial \theta} = 0 \] at \( \theta = 0 \) and \( \pi \)

and also

\[ \frac{\partial u_\theta}{\partial R} = \frac{\partial^2 u_\theta}{\partial R^2} = 0 \] at \( \theta = 0 \) and \( \theta = \pi \).

Following L'Hopital's rule \((\partial u_\theta/\partial \theta) \cot \theta \) is replaced by \( \partial^2 u_\theta/\partial \theta^2 \), and \( u_\theta/\sin^2 \theta \) by \([\partial^2 u_\theta/2\partial \theta^2]\). The finite difference expression is then modified to

\[ \begin{align*}
F(R, 0, t + \Delta t) &= \varepsilon_1^2 \left[ 1 + \frac{2\Delta R}{\rho c_p^2} \left( \frac{\partial \lambda}{\partial R} + \frac{\partial \mu}{\partial R} \right) \right] \left[ F(R + \Delta R, 0, t) - F(R - \Delta R, 0, t) \right] \\
&+ 4\varepsilon_1^2 bF(R, \Delta \theta, t) \\
&+ \left[ 2 - 2\varepsilon_1^2 - 2\varepsilon_2^2 \left( \frac{\Delta R}{R} \right)^2 - 4e_2^2 b + \frac{\varepsilon_1^2 (\Delta R)^2}{\rho c_p^2 R} \left( \frac{\partial \lambda}{\partial R} - \frac{\partial \mu}{\partial R} \right) \right] F(R, 0, t) \\
&+ \varepsilon_1 \varepsilon_2 (1-b) [G(R + \Delta R, \Delta \theta, t) - G(R - \Delta R, \Delta \theta, t)] \\
&+ \Delta \theta \varepsilon_2^2 \left[ 4 + \frac{2\partial \lambda/\partial R}{\rho c_p^2} R \right] G(R, \Delta \theta, t) - G(R, 0, t - \Delta t)
\end{align*} \]  

and a similar expression holds for \( G(R, 0, t + \Delta t) \) and for \( F(R, \pi, t + \Delta t) \). In the latter case the signs of \( \Delta \theta \) are interchanged. Equations (9) and (14) have a singularity also at \( R = 0 \). However, as explained in the sequel, the finite difference equations are not applied at this point.

2. Stability of the finite difference scheme

In order to estimate the stability of the vector finite difference equation (9), let us consider a homogeneous medium in which \( M = N = Q = 0 \) and let \( \Delta R = 0(1), \Delta \theta = 0(1) \), \( R \gg \Delta R \) while \( \varepsilon_1 = 0(1) \) and \( \varepsilon_2 = 0(1) \). The terms in \( B, C \) and \( E \), in equation (9) are then small compared with the other non-zero terms, and the equation reduces to

\[ \begin{align*}
v(R, \theta, t + \Delta t) - 2v(R, \theta, t) + v(R, \theta, t - \Delta t) \\
&= A\varepsilon_1^2 [v(R + \Delta R, \theta, t) - 2v(R, \theta, t) + v(R - \Delta R, \theta, t)] \\
&+ D\varepsilon_2^2 [v(R, \theta + \Delta \theta, t) - 2v(R, \theta, t) + v(R, \theta - \Delta \theta, t)] \\
&+ \frac{H}{4} \varepsilon_1 \varepsilon_2 [v(R + \Delta R, \theta + \Delta \theta, t) + v(R - \Delta R, \theta - \Delta \theta, t)] \\
&- v(R + \Delta R, \theta - \Delta \theta, t) - v(R - \Delta R, \theta + \Delta \theta, t).
\end{align*} \]
Equation (15) has constant coefficients, so that a standard stability analysis applies. Setting

$$v(R, \theta, t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{i(m \Delta R + \beta n \Delta \theta) \xi p}$$

with

$$R = m \Delta R$$
$$\theta = n \Delta \theta.$$

(16)

It follows from equation (15) that:

$$\left[ -4Ae_2^2 \sin^2 \frac{\alpha \Delta R}{2} - 4De_2^2 \sin^2 \frac{\beta \Delta \theta}{2} - \frac{\beta \Delta \theta}{2} \sin \alpha \Delta R \sin \beta \Delta \theta 
+ I(-\xi + 2 - \xi^{-1}) \right] v(m \Delta R, n \Delta \theta, p \Delta t) = 0$$

(18)

where $I$ is the unit matrix.

Equation (18) is satisfied if the determinant of coefficients vanishes, or

$$\begin{vmatrix} 2D_1 - \xi - \xi^{-1} & -\sqrt{D_3} \\ -\sqrt{D_3} & 2D_2 - \xi - \xi^{-1} \end{vmatrix} = 0$$

(19)

where

$$D_1 = 1 - 2e_1^2 \sin^2 \frac{\alpha \Delta R}{2} - 2e_2^2 \sin^2 \frac{\beta \Delta \theta}{2}$$

$$D_2 = 1 - 2e_1^2 \sin^2 \frac{\alpha \Delta R}{2} - 2e_2^2 \sin^2 \frac{\beta \Delta \theta}{2}$$

$$D_3 = (1 - b)^2 e_1^2 e_2^2 \sin^2 \alpha \Delta R \sin^2 \beta \Delta \theta.$$

Multiplying equation (19) by $\xi^2$

$$(\xi^2 - 2D_1 \xi + 1)(\xi^2 - 2D_2 \xi + 1) - D_3 \xi^2 = 0$$

(21)

or

$$(\xi^2 - 2A_1 \xi + 1)(\xi^2 - 2B_1 \xi + 1) = 0$$

(22)

where

$$A_1 = \frac{1}{2} \left\{ D_1 + D_2 \pm \sqrt{[(D_1 - D_2)^2 + D_3]} \right\}$$

$$B_1 = \frac{1}{2} \left\{ D_1 + D_2 \mp \sqrt{[(D_1 - D_2)^2 + D_3]} \right\}$$

(23)

For stability $|\xi| \leq 1$, and according to equation (22) this implies that

$$-1 \leq A_1 \leq 1$$

or

$$-1 \leq \frac{D_1 + D_2}{2} + \sqrt{[(D_1 - D_2)^2 + D_3]} \leq 1$$

(24)

and

$$-1 \leq B_1 \leq 1$$

or

$$-1 \leq \frac{D_1 + D_2}{2} - \sqrt{[(D_1 - D_2)^2 + D_3]} \leq 1.$$

(25)

The square roots in (24) and (25) are positive and the inequalities in (24), (25) are satisfied if

$$\sqrt{[(D_1 - D_2)^2 + D_3]} \leq 2 \pm (D_1 + D_2)$$

(26)

or

$$D_1 D_2 \pm (D_1 + D_2) + 1 - \frac{1}{4} D_3 \geq 0.$$
With the minus sign (27) implies that

\[ 4b \left( \varepsilon_1^2 \sin^2 \frac{\alpha \Delta R}{2} + \varepsilon_2^2 \sin^2 \frac{\beta \Delta \theta}{2} \right)^2 \]

or

\[ b \left( \varepsilon_1^2 \sin^2 \frac{\alpha \Delta R}{2} + \varepsilon_2^2 \sin^2 \frac{\beta \Delta \theta}{2} \right)^2 \]

\[ + (1-b)^2 \varepsilon_1^2 \varepsilon_2^2 \sin^2 \frac{\beta \Delta \theta}{2} \sin^2 \frac{\alpha \Delta R}{2} \left( 1 - \cos^2 \frac{\alpha \Delta R}{2} \cos^2 \frac{\beta \Delta \theta}{2} \right) \geq 0 \]  

(28)

as both terms are positive, this inequality is always satisfied. With the plus sign in (27) we find

\[ \left[ b \left( \varepsilon_1^2 \sin^2 \frac{\alpha \Delta R}{2} + \varepsilon_2^2 \sin^2 \frac{\beta \Delta \theta}{2} \right) - 1 \right] \left[ \varepsilon_1^2 \sin^2 \frac{\alpha \Delta R}{2} + \varepsilon_2^2 \sin^2 \frac{\beta \Delta \theta}{2} - 1 \right] \]

\[ + (1-b)^2 \varepsilon_1^2 \varepsilon_2^2 \sin^2 \frac{\beta \Delta \theta}{2} \sin^2 \frac{\alpha \Delta R}{2} \left( 1 - \cos^2 \frac{\alpha \Delta R}{2} \cos^2 \frac{\beta \Delta \theta}{2} \right) \geq 0 \]  

(30)

as \( b \leq 1 \) and the second term is always non-negative, the inequality (30) is satisfied if

\[ \varepsilon_1^2 \sin^2 \frac{\alpha \Delta R}{2} + \varepsilon_2^2 \sin^2 \frac{\beta \Delta \theta}{2} \leq 1 \]

(31)

or if

\[ \varepsilon_1^2 + \varepsilon_2^2 \leq 1 \]

(32)

which implies finally

\[ \left( \frac{\Delta t}{\Delta R} \right)^2 \leq \frac{1}{1 + (\Delta R/R \Delta \theta)^2}. \]  

(33)

In the special case of \( \varepsilon_1 = \varepsilon_2 = \varepsilon \) and disturbances such that \( \alpha \Delta R = \beta \Delta \theta \) (33) reduces to \( \varepsilon^2 \leq 1 \), while one finds easily that (26) is satisfied in this case even if

\[ \varepsilon^2 \leq \frac{1}{1+b}. \]  

(34)

The condition (34) for equal disturbances in the \( R \) and \( \theta \) directions is less restrictive than condition (32) which reduces to \( \varepsilon^2 \leq 1/2 \) when \( \varepsilon_1 = \varepsilon_2 \). The example of \( \varepsilon_2 = 0 \) shows that in the general case the condition (32) is indeed necessary. When \( \varepsilon_2 = 0 \), i.e., in the one-dimensional case, it is clearly necessary that \( \varepsilon_1^2 \leq 1 \). The same follows immediately from (30) as for \( \varepsilon_2 = 0 \) the last term is zero, and choosing

\[ 1 \leq \varepsilon_1^2 \sin^2 \frac{\alpha \Delta R}{2} \leq \frac{1}{b} \]

the inequality (30) is not satisfied, so that we obtain again that \( \varepsilon_1^2 \leq 1 \) is necessary in this case.
3. Stability near the origin

Near the centre of the sphere when \( R \) is of the order of \( \Delta R \), the dominant term in equation (9) is the term in \( D \). In this case, an indication of stability is obtained by an analysis of the following equation:

\[
\nu(r, \theta, t + \Delta t) - 2\nu(R, \theta, t) + \nu(R, \theta, t - \Delta t) = \varepsilon^2 D [\nu(R, \theta + \Delta \theta, t) - 2\nu(R, \theta, t) + \nu(R, \theta - \Delta \theta, t)].
\] (35)

Expressing \( \nu \) by (16), equation (35) reduces to

\[
\left\{ -4D \varepsilon^2 \sin^2 \frac{\beta \Delta \theta}{2} + I \left[ -\xi + \frac{2}{1} - \frac{1}{\xi} \right] \right\} \nu(R, \theta, t) = 0.
\] (36)

Equation (36) can be satisfied if the determinant of coefficients vanishes, or if

\[
\left[ \xi^2 - 2\xi \left( 1 - 2\varepsilon^2 \sin^2 \frac{\beta \Delta \theta}{2} \right) + 1 \right] \left[ \xi^2 - 2\xi \left( 1 - 2\varepsilon^2 \sin^2 \frac{\beta \Delta \theta}{2} \right) + 1 \right] = 0
\] (37)

and the condition for \( |\xi| \leq 1 \) becomes

\[
-1 \leq 1 - 2\varepsilon^2 \sin^2 \frac{\beta \Delta \theta}{2} \leq 1
\] (38)

\[-1 \leq 1 - 2\varepsilon^2 \sin^2 \frac{\beta \Delta \theta}{2} \leq 1.
\]

It implies

\[
\varepsilon^2 \frac{c}{c_p} \leq 1 \quad \text{and} \quad \varepsilon^2 \leq 1.
\] (39)

Finally, the time step is determined by

\[
\Delta t \leq \frac{\Delta R \Delta \theta}{c_p}.
\] (40)

The stability condition in equation (40) requires a small \( \Delta t \) near the origin. In order to avoid extremely small time steps, the first interval \( \Delta R_1 \) in \( R \) was chosen larger than the following intervals \( \Delta R \). Fig. 2 shows the grid near \( R = 0 \). Here

\[
\Delta R_1 = j\Delta R \quad (j = 2, 3, 4, \ldots).
\] (41)

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![Diagram](https://example.com/diagram.png)

**Fig 2.** (a) Grid points near the origin. (b) Grid points at the interface \( R = d \).
In most calculations we choose $j = 2$ or $3$. The appropriate time step is $\Delta t = 0.003 \alpha/c_p$ for $j = 2$ as against $\Delta t = 0.0045 \alpha/c_p$ in a grid of $R_1 = 3 \Delta R$. Fig. 3 shows the radial displacement $u_R$ of a homogeneous sphere caused by a source at the centre $b = 0$. The solid curve is the result obtained in a grid with $j = 2$, the dashed curve is obtained with $j = 3$. The results are almost identical in both cases. The finite difference stencil for points near the origin has unequal lengths in the $R$ direction and the finite difference scheme for such points is modified accordingly. The point at the origin is not included in the finite difference scheme. At $R = 0$ it follows from equation (4) that $F = G = 0$ as long as the displacement $u$ is bounded, or as long as no source is located at $R = 0$. When the source is at $R = 0$, its contribution in the neighbourhood of $R = 0$ is known analytically and the reflected field only is calculated by the finite difference scheme. (See Alterman & Karal 1968.)

4. Boundary conditions

In the case of a free boundary, the vanishing of the normal and tangential component of stress is required.

\[ P_{RR} = \lambda \text{div}u + 2\mu \frac{\partial u_R}{\partial R} = 0 \tag{42} \]

at $R = a$.

\[ P_{R\theta} = \mu \left( \frac{\partial u_\theta}{\partial R} + \frac{1}{R} \frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R} \right) = 0 \tag{43} \]
At the interface of two different elastic media the boundary conditions express the fact that the displacement vector and the normal and tangential stress components are continuous across the interface, \( R = d \).

\[
\begin{align*}
(u_R)_1 &= (u_R)_2 \\
(u_\theta)_1 &= (u_\theta)_2 \\
(P_{RR})_1 &= (P_{RR})_2 \\
(P_{R\theta})_1 &= (P_{R\theta})_2.
\end{align*}
\]

The finite difference expressions for (42)–(47) are readily obtained. Whenever normal derivatives occur, a fictitious line is added to the grid, extending one of the media. As an example, let us assume that an interface between medium 1 of constants \( \mu_1, \lambda_1 \) and medium 2 of constants \( \mu_2, \lambda_2 \) occurs at \( R = d \). Fig. 2 shows the grid at and near the interface. The regular grid points are denoted by \( x \). Additional fictitious grid points, denoted by circles, extend medium (1) from \( R = d \) to \( R < d \). The equations (46) and (47) of continuity of stresses then determine \( u_{R1} \) and \( u_{\theta1} \) on the fictitious line \( R = d - \Delta R \). They are denoted \( u_{R} \) and \( u_{\theta} \) and are given by

\[
\begin{align*}
(\lambda_1 + 2\mu_1) u_{R}(d - \Delta R) &= (\lambda_1 + 2\mu_1) u_{R}(d) + \frac{2\Delta R}{d} (\lambda_1 - \lambda_2) u_{R}(d) \\
&\quad + (\lambda_1 - \lambda_2) u_{\theta}(d) \cot \theta \frac{\Delta R}{d} + \frac{\Delta R}{\Delta \theta} \frac{\lambda_1 - \lambda_2}{2d} [u_{\theta}(\theta + \Delta \theta) - u_{\theta}(\theta - \Delta \theta)] \\
&\quad + (\lambda_2 + 2\mu_2)[u_{R}(d) - u_{R}(d - \Delta R)] \quad (48)
\end{align*}
\]

\[
\begin{align*}
\mu_1 u_{\theta}(d - \Delta R) &= \mu_1 u_{\theta}(d) + \frac{\mu_1 - \mu_2}{2d} \frac{\Delta R}{\Delta \theta} [u_{R}(\theta + \Delta \theta) - u_{R}(\theta - \Delta \theta)] \\
&\quad + \frac{\mu_2 - \mu_1}{d} \Delta R u_{\theta} - \mu_2 [u_{\theta}(d) - u_{\theta}(d - \Delta R)]. \quad (49)
\end{align*}
\]

A more detailed description of boundary conditions at the interface between two adjacent elastic media is given by Alterman & Karal (1968).

5. The source

A point source is located in the elastic medium. The displacement potential is defined by:

\[
\phi_s = -\Delta_3 \left[ \frac{A}{6R_s} \left( t - \frac{R_s}{c_p} \right)^3 H \left( \frac{t - R_s}{c_p} \right) \right] \quad (50)
\]

where \( R_s \) is the distance of the observer from the source:

\[
R_s = \sqrt{(R^2 + b^2 - 2Rb \cos \theta)} \quad (51)
\]

and \( b \) is the distance of the source from the origin of co-ordinates 0 (Fig. 1). \( \Delta_3 \) is the third finite difference defined by

\[
\Delta_3 f(t, R) = \{ f(t, R) - 3 f(t - \Delta, R) + 3 f(t - 2\Delta, R) - f(t - 3\Delta, R) \} / \Delta^3. \quad (52)
\]
FIG. 4. Time variation of the potential $\phi_s$ and of the displacement $u_R$ for a source in a homogeneous medium.

$H$ is the unit step function. The constant $\Delta$ determines the duration of the variable part of the pulse. The displacement of the solid elastic medium as derived from equation (50) is given by

$$u = \text{grad} \phi_s \quad (53)$$

or

$$u_R = \frac{\partial \phi_s}{\partial R} = \frac{\partial \phi_s}{\partial R} \frac{R - b \cos \theta}{R_s}, \quad (54)$$

$$u_\theta = \frac{1}{R} \frac{\partial \phi_s}{\partial \theta} = \frac{\partial \phi_s}{\partial R_s} \frac{b \sin \theta}{R_s}. \quad (55)$$

The variation in time of the potential $\phi_s$ and of the displacement $u$ in a homogeneous medium is depicted in Fig. 4.

FIG. 5. The radial displacement in a homogeneous sphere with the source at the centre $b = 0$. The solid line shows $u_R$ obtained by a grid in one space dimension. The dashed line shows $u_R$ calculated by the finite difference scheme in two space dimensions $R, \theta$. 
6. Results for a source in a homogeneous sphere

When the source is located at the centre of a homogeneous sphere, the motion has spherical symmetry. This problem has been solved by equations in a single space-dimension and in time (Alterman & Kornfeld 1968) and serves as a check on the accuracy of the results of the finite difference scheme in two space dimensions and time which is determined in equations (9)—(14). Fig. 5 shows a comparison between the previously obtained one-dimensional results (solid curve) and the results of equations (9)—(14) (dashed curves) for a grid of $\Delta R = 0.02a$ and $\Delta \theta = \pi/60$. The figure depicts the variation in time of $R^2 u_R$ at a distance $R = 0.7a$ from a source which is located at the origin so that $b = 0$. The agreement between the curves in the direct $P$-pulse and in the once and several times reflected pulses is good. However, we notice some spurious oscillation after the twice reflected $PPP$ and the second once-reflected $PP$ pulse. They can be reduced by filtering or by applying a finer grid.

When the source is at any arbitrary location in the sphere, an estimate of the accuracy of results is provided by a comparison of different grid sizes. Figs 6 and 7 show a comparison between a grid of $\Delta R = 0.02a$ and $\Delta \theta = \pi/60$ (dashed curve) and $\Delta R = 0.02a$; $\Delta \theta = \pi/80$ (solid curve). The source is at a distance of half a radius from the centre of the sphere, $b = 0.5a$. Fig. 7 shows the radial displacement $u_R$ at an angular distance $\theta = 135^\circ$ from the source and at $R = 0.8a$ and $R = a$. There is good agreement between the curves, except at $R = a$ after arrival of the $PP(D)$ pulse where the dashed curve for the coarse grid shows spurious oscillations which do not occur in the finer grid. In Fig. 6 $\theta = 45^\circ$ and $R = 0.6a$, $R = a$.

![Fig. 6. Radial displacement at $\theta = 45^\circ$ and $R/a = 0.6$; $1.0$ for a source at $b = a/2$. The dashed line is the solution for a coarse grid of $\Delta R = 0.02a$, $\Delta \theta = \pi/60$ the solid line for a finer grid of $\Delta R = 0.02a$ and $\Delta \theta = \pi/80.$](https://doi.org/10.1093/gji/ggi263.610280)
curves for the two grid sizes are almost indistinguishable. The arrows indicate the arrival-time of the maximum amplitudes of several reflected and diffracted pulses. The notation PP(D) indicates the diffracted pulse which is connected with the once reflected PP-pulse discussed by Alterman & Kornfeld (1963).

An analytic solution to the problem of an impulsive source at an arbitrary location in a homogeneous elastic sphere has been derived previously (Alterman & Abramovici 1965). Our calculations show that even in the case of a homogeneous sphere, when an analytic solution exists, the finite difference solution has the advantage that it needs much less computer time than the numerical evaluation of the analytic solution while revealing all the pulses which are identified in the analytic solution. However, the main purpose of applying the finite difference scheme is for obtaining solutions for the heterogeneous sphere.

7. Boundary conditions and results for a radially heterogeneous sphere

At every given time-step equations (9)–(14) and (42)–(43) constitute a boundary value problem which can be solved by two different methods. One is solving the equations for a homogeneous sphere and applying boundary conditions at the free surface. The second method relies on solving the equations for a heterogeneous sphere in which the elastic parameters decrease continuously to zero for $R > a$. In this case no boundary conditions are applied, thus simplifying the calculations. The sphere is considered homogeneous for all $0 \leq R < a - \Delta R$. Only in the interval $a - \Delta R \leq R \leq a + \Delta R$ \(\partial \mu/\partial R\) and \(\partial \lambda/\partial R\) assume non-zero values, such that at $R = a$
Pulse propagation in a solid elastic sphere

255

the velocities \( c \) and \( c_p \) have half their original values, and at \( R = a + \Delta R \) finally \( c = c_p = 0 \). The heterogeneity is thus distributed over two grid intervals in the \( R \) direction. Fig. 8 shows a comparison of results obtained by the two methods. \( u_R \) is drawn at \( R = 0.8a \) and at \( R = 0.4a \). A slight difference in amplitudes is found and the maximum of reflected pulses in the heterogeneous sphere without boundary conditions arrives by \( 0.02a/c_p \) to \( 0.05a/c_p \) later than in the homogeneous sphere with boundary conditions at \( R = a \) (dashed curve). This delay in arrival-time occurs due to the decrease in propagation velocities near \( R = a \). Although the present model is still very crude, the decrease in velocities near the free surface simulates well the situation in the real Earth.

Changes in the elastic properties in the interior of the sphere may be included in the computational scheme in the same way as described for the free surface. They may either be treated as interface conditions or as a gradual change in parameters over a few grid points. As an example, let us consider a sphere consisting of two concentric layers for which \( c_{p1} = 1.1c_p, \ c_2 = 1.1c_1, \) and \( \rho_2 = 2\rho_1; \) the index 1 refers to the layer, index 2 to the core, the constants were chosen to agree with previous numerical experiments (Alterman & Aboudi 1968), they are easily replaced by averages of a real Earth model. The transition between the layers is considered:

(a) as a discontinuity at \( R = 0.7a \),
(b) as a gradual transition over two-grid intervals adjacent to \( R = 0.7a \), and
(c) as a transition over four-grid intervals between \( 0.7a - 2\Delta R \) and \( 0.7a + 2\Delta R \).

Fig. 9 shows the results in the three cases. The solid line depicts the solution (a) with a discontinuity at \( R = 0.7a \). The dashed curve shows the effect of (b), a gradual transition over two-grid intervals. The dotted curve shows the result of (c)—a transition over four-grid intervals. The source is at \( b = 0 \), and the time-variation of the

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![Fig. 8](https://example.com/fig8.png)

Fig. 8. Radial displacement in a sphere with a source at the centre \( b = 0 \). The dashed line represents the solution by applying the boundary conditions equations (42)-(43) at \( R = a \). The solid line is for a radially heterogeneous sphere, with no boundary conditions applied at \( R = a \).
256 Z. S. Alterman, J. Aboudi and F. C. Karal

radial displacement $u_R$ is shown at the locations $R = 0.4a$ and $R = 0.8a$. The direct $P$-pulse arrives at $R = 0.4a$ without crossing $R = d = 0.7a$ and is the same in all cases (a), (b) and (c).

The next arriving pulse is $PP$, which is reflected from the transition region. Its amplitude decreases with increasing width of the transition region, and it is dispersed (broadened) at the same time. The $PP$ $P$-pulse, which crosses from the core into the outer region, is reflected at the surface, and then returns, slightly increases in amplitude with increasing width of transition region. Similar effects are found in the multiply-reflected pulses which arrive later at $R = 0.4a$, upper curve in Fig. 9, and in all pulses which arrive at $R = 0.8a$ (lower curve in Fig. 9). The figure shows clearly that the differences in amplitude increase with increasing number of reflections at the transition region.

8. A laterally heterogeneous sphere

Recently several authors discussed the occurrence of lateral heterogeneity in the Earth’s upper mantle (Toksoz et al. 1967; Greenfield & Sheppard 1969; Toksoz et al. 1969). The present numerical scheme can be applied not only to spherically
Pulse propagation in a solid elastic sphere

symmetric models, for which the equations of motion are separable, but also to such a laterally heterogeneous sphere.

As a simple example of lateral heterogeneity, let us consider the configuration depicted in Fig. 1. A conical wedge is defined by \( d \leq R \leq a \) and \( 0 \leq \theta \leq \theta_1 \).

The elastic material inside the wedge has parameters \( c_1, \rho_1 \), while in the remaining part of the sphere the parameters are \( c_2, \rho_2 \). Instead of imposing boundary conditions on the surfaces of the wedge and on its edges, the complete equations (9)-(14) are applied. They allow for variation of the elastic parameters both in radial and in angular direction. The transition at the boundary of the wedge is performed over two-grid points near the boundaries of the wedge. The wedge causes reflection and diffraction which is exhibited in the computed results. Let us consider the specific values of parameters \( d = 0.7a \) and \( \theta_1 = 36^\circ \). A comparison is

![Fig. 10. Radial displacement at \( \theta = 60^\circ \) distance \( R = 0.4a \) from the centre of the heterogeneous sphere. The curves (1) show the solution for (a) \( c_p = 1.1c_1, c_s = 1.1c_2, \rho_s = 2\rho_2 \) (solid curve) and (b) for the above velocities and \( \rho_1 = \rho_2 \) (dashed curve). The curves (2) show the solution for (c) \( c_p = c_2, c_s = c_1, \rho_1 = 2\rho_2 \) (solid curve) and \( \rho_1 = \rho_2 \) (dashed curve). The curve (3) shows the solution for (d) \( c_p = c_p/1.1, c_s = 1.1c_1, \rho_1 = 0.5\rho_2 \). The curve (4) is the solution for (e) \( c_p = 0, c_s = 0, \rho_1 = 0 \).](https://academic.oup.com/gji/article-abstract/21/3/243/610280)
made between:

(a) a high-density, high-velocity wedge in which \( \rho_1 = 2\rho_2, \ c_1 = 1.1c_2, \)
(b) a high-velocity wedge in which \( \rho_1 = \rho_2 \) and \( c_1 = 1.1c_2, \)
(c) a high-density wedge in which \( \rho_1 = 2\rho_2, \ c_1 = c_2, \)
(d) a low-velocity low-density wedge in which \( \rho_1 = 0.5\rho_2, \ c_1 = c_2/1.1, \) and
(e) an empty wedge \( \rho_1 = 0, \ c_1 = 0. \)

In all cases \( c_p = \sqrt{(3)c}. \)

The motion of the sphere due to a source at \( b = 0 \) in these five cases is compared with results for the homogeneous sphere. Figs 10, 11 and 12 show \( u_R \) at the points \( \theta = 60^\circ, \ R = 0.4a; \ \theta = 10^\circ, \ R = 0.8a; \) and \( \theta = 180^\circ, \ R = 0.6a, \) respectively. The point \( R = 0.8a, \ \theta = 10^\circ \) in Fig. 11 is inside the wedge. The first arriving pulse is the transmitted \( PP_1. \) The index 1 indicates a ray path inside the wedge. The next
Pulse propagation in a solid elastic sphere

Arriving pulse is $P_e P_1$. Here $c$ indicates that the pulse arrives at an edge of the wedge and diffraction takes place. An index $i$ indicates reflection at a boundary surface of the wedge. Examples of ray-paths of pulses $P_e P$ and $P_i P$ are indicated in Fig. 1. As another example, $PP_1 P_1 P_1$ is transmitted into the wedge, reflected at the surface and then reflected back into the wedge from the interface at $R = d$. Fig. 11 shows that the amplitudes of reflected pulses in the high-density wedge are smaller than in the homogeneous sphere, while in the low-density wedge they are larger. The amplitude of pulses which are reflected at the interface are smaller in cases (a) and (c), of a high-density wedge than in (d) for a low-density wedge. Case (b) differs only slightly from the homogeneous case. Two diffracted pulses are indicated: $P_e P_1$ and $PP_1 P$. Their amplitudes are small at the given location.

Fig. 10 shows the variation in time of $u_R$ at $\theta = 60^\circ$, $R = 0.4a$ outside the wedge. Some well separated diffracted pulses as $P_e P$, $P_e S$, $PP_e P$, $PP_e S$ are clearly exhibited. The direction of the diffracted pulses changes with changing density. The amplitude of the diffracted pulses is largest in case (e) of the empty wedge.

Fig. 12. Radial displacement at $\theta = 180^\circ$ at distance $R/a = 0.6$ from the centre of the heterogeneous sphere. The curves (1) show the solution for (a) $c_{e1} = 1.1c_{p1}$, $c_1 = 1.1c_2$, $\rho_1 = 2\rho_2$ (solid curve) and (b) for the above velocities and $\rho_1 = \rho_2$ (dashed curve). The curves (2) show the solution for (c) $c_{p1} = c_{p2}$, $c_1 = c_2$, $\rho_1 = 2\rho_2$ (solid curve) and $\rho_1 = \rho_2$ (dashed curve). The curve (3) shows the solution for (d) $c_{e1} = c_{p1}/1.1$, $c_2 = 1.1c_1$, $\rho_1 = 0.5\rho_2$. 

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Fig. 12 shows the influence of the wedge at a distant point \( \theta = 180^\circ, R = 0.6a \). As expected, the direct pulse \( P \) and the first once reflected \( PP \) are the same in all cases, unchanged by the wedge. The \( P_2P \) and \( P_1P \) change sign with changing density. The \( PP_1 P_1 P \) pulse has the largest amplitude for the high-velocity wedges and is small for the sphere with the low-velocity wedge. Several other pulses are indicated in the figure.

These examples show that the motion of a heterogeneous sphere can be easily derived and analysed. Further models of a heterogeneous sphere where \( \rho, \lambda, \mu \) are arbitrary functions of the co-ordinates \( R \) and \( \theta \) can be treated by the same method.

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