Journal of Hydroinformatics | 05.4 | 2003

Handling uncertainty in the hydroinformatic process

Jim W. Hall

ABSTRACT

Hydroinformatics combines topics of modelling and decision-making, both of which have attracted a great deal of attention outside hydroinformatics from the point of view of uncertainty. Epistemic uncertainties are due to the inevitably incomplete evidence about the dependability of a model or set of competing models. Inherent uncertainties are due to the varying information content inherent in measurements or model predictions, be they probabilistic or fuzzy. Decision-making in management of the aquatic environment is, more often than not, a complex, discursive, multi-player process. The requirement for hydroinformatics systems is to support rather than replace human judgment in this process, a requirement that has significant bearing on the treatment of uncertainty. Furthermore, a formal language is required to encode uncertainty in computer systems. We therefore review the modern mathematics of uncertainty, starting first with probability theory and then extending to fuzzy set theory and possibility theory, the theory of evidence (and its random set counterpart), which generalises probability and possibility theory, and higher-order generalisations. A simple example from coastal hydraulics illustrates how a range of types of uncertain information (including probability distributions, interval measurements and fuzzy sets) can be handled in the types of algebraic or numerical functions that form the kernel of most hydroinformatic systems.

Key words | decision-making, evidence, probability, uncertainty

Jim W. Hall
Department of Civil Engineering,
University of Bristol,
Queen's Building,
University Walk,
Bristol BS8 1TR,
UK
Tel.: +44 117 928 9763
Fax: +44 117 928 7783
E-mail: *Jim.Hall@bristol.ac.uk*

INTRODUCTION

In the same way as uncertainty has become a principal theme of research in Artificial Intelligence (Krause & Clark 1993), it is increasingly a concern for hydroinformaticians (Davis & Blockley 1996). Whilst early hydroinformaticians were concerned with the development and application of new modelling systems (Abbott 1991), the maturing of the discipline has seen more reflection on the quality of the information that hydroinformatics provides and how it contributes to decision-making processes (Abbott & Jonoski 1998, 2001; Abbott 2001). This increasing maturity coincides with the advent of the computational capacity to do a large number of simulations using fairly sophisticated models and to propagate more complex uncertainty structures in order to generate ensemble predictions and scrutinise the effects of uncertainties.

The maturing of hydroinformatics also coincides with the latest phase in a debate about the mathematisation of uncertainty that began about half a century ago. Within probability theory the neo-Bayesian school is in the ascendancy (Lindley 1987; Pearl 1988; Schum 1994; Wright & Ayton 1994). Meanwhile, after initial bewilderment and rejection by conventional probabilists, fuzzy methods and evidence theory are becoming regarded as legitimate extensions of classical probability. The rearguard action by conventional probabilists against non-standard methods is formidable and is being enacted most fiercely by the neo-Bayesians (Lindley 1982; French 1986, 1995; Cooke 1991). However, now that coherent generalisations of probability have emerged in the form of the mathematical theory of evidence (Shafer 1976), random set theory (Dubois & Prade 1991; Goutsias *et al.* 1997), the theory of imprecise probabilities (Walley 1991) and, retrospectively, in Choquet's theory of capacities (Choquet 1953) the argument is no longer one of mathematical coherence. The argument now hinges on the relative parsimony of the competing theories and practical issues of elicitation and accessibility.

The aim of this paper is to identify the challenges presented by uncertainty handling in hydroinformatics and indicate some potential responses to those challenges. The analysis is both discursive and mathematical. Understanding of uncertainty is grounded in epistemology, yet that understanding has to be formally encoded if it is to be enacted in computer systems. The title of this paper refers to 'the hydroinformatic process' in order to narrow the scope of the almost boundless topic of uncertainty to those aspects that have most bearing on the practice of hydroinformatics, i.e. the process of using electronically encapsulated knowledge to support management of the water environment. However, before turning to the practice of hydroinformatics we must begin with some philosophy.

THE MODELLING PROCESS

Philosophical preliminaries

Beven (2001), in his critical review of the prospects for distributed hydrological modelling, characterised modellers as pragmatic or heuristic realists. He justified this on the grounds that hydrological modellers reason and behave as if rainfall, runoff, streamflow and the other processes that concern them exist independently of their perceptions and empirical studies of them. This extends even to quantities that are not (yet) observable. Modelling is seen as a process of abstraction from objective reality. Uncertainty is associated with the mismatch between models and reality. It is a view with which we have a great deal of sympathy, particularly as it coincides with our own everyday experience of the conduct of hydraulic modelling activities, at least at an instinctive level. Beven's observation is of the everyday temperament and behaviour of practitioners rather than being an ontological assertion, and rightly so. The sceptical philosophers, Hume foremost amongst them, have demonstrated how little we can assert about reality and Kant went much further than any before him in defining the limits that circumscribe the assertions we can make about reality.

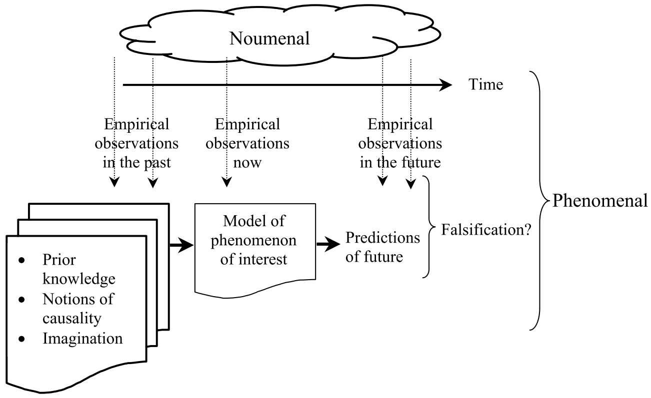

Figure 1 | An interpretation of Kant's distinction between the noumenal and the phenomenal.

Here we follow Kant in using the term 'noumenal' to refer to the underlying reality in the world and the term 'phenomenal' to refer to empirical observations and empirically derived knowledge (Figure 1). We cannot possibly know what the noumenal is like. All we have are empirical observations, which are profoundly influenced by the means of observing. The scientific process involves establishing shared standards of observation and measurement in order to establish a common (inter-subjective) vocabulary of the phenomenal. Scientific models are constructed in order to make sense of empirical observations and, in particular, to make predictions of the phenomena that may be observed in future. The motive for constructing models may be a purely scientific quest for knowledge but, in the context of hydroinformatics, it is driven by the more pragmatic concerns of management of the water environment, in which case the requirement is to predict how various management options, including the status quo, may behave in future.

Models begin their life as 'conjectures—highly informative guesses' (Popper 1963). They are externalised (increasingly in electronic media according to hydroinformaticians) so that they can be shared and tested. They come to inhabit what Popper & Eccles (1977) referred to as 'World 3'—the world of collective knowledge. Within that collective knowledge will be growing experience of how a model compares with empirical observations. However, comparison with observation will not be the only criterion that an experienced modeller takes into account when choosing to adopt a particular modelling strategy. Besides practical considerations of availability and cost, they will scrutinise the processes that a given model enacts and endeavour to reach a view on how those processes compare with alternative formulations of the phenomenon of interest. Process-based reasoning provides insights about the type of phenomena a model can be expected to predict and the range over which such predictions can be expected to be accurate.

Evidence-based reasoning about model dependability (epistemic uncertainty)

Beyond the first bold acts of imagination that initiate a model and accompany its evolution, the process of model testing and choice is one of evidence-based reasoning. It takes place in the 'open world' of our unbounded perceptions, whilst models themselves (at least once they are encoded digitally) are closed formal constructs. Because the domain of the mapping from 'open world' to model space is unbounded, there is a fundamental limit to the statements that can be made about the goodness of that mapping. Popper suggested that we can at least falsify a scientific model, by observation, even if we can never say that it is a 'true' model. The Bayesian notion of the 'probability of a model being true' is extremely problematic in this respect. In situations where there are populations of measurements then statistical constructs, such as significance testing, provide the pretence of objective model evaluation but, of course, depend on ultimately arbitrary significance levels. In data-poor situations, which are common in many of the most interesting aquatic problems (relating, for example, to extreme events or risk assessment), model users are obliged to assemble evidence from whatever source may be available, irrespective of the data format, be it measurements, analogues or expert judgements. The problem then becomes one of bringing that evidence together in a coherent way in order to make an evidence-based statement about the dependability of a model as a basis for decision-making. We use the term 'dependability' rather than 'truth' because we know from Popper that we can never say that a model is true. Dependability is a weaker term, which relates to the extent to which a model provides a sound basis for decision-making. The more well-tested a model is, the more (and the more diverse) the confirmatory instances that have been observed, and the better defined are the limits of applicability of a model, the greater is the justification for believing that it is dependable within those limits (Blockley 1980). For the time being, this evidence-based reasoning process, which is so central to responsible modelling, is scarcely supported in hydroinformatics systems, though Davis & Hall (1998, 2003) and Hall *et al.* (1999) have suggested how this may be done using graphical methods and an appropriate uncertainty calculus.

Evidence-based reasoning about the truth or dependability of a model is conventionally thought of as being a problem of 'epistemic uncertainty' (see Helton & Burmaster (1996) for a review). Epistemic uncertainty is associated with a lack of knowledge about a phenomenon of interest. Epistemic uncertainties change with ones' state of knowledge.

Information content (inherent uncertainty)

The information content in a model is a feature of the scale at which it represents a phenomenon of interest and the format in which the model results are presented. The most informative type of prediction is a precise deterministic prediction. Indeed, information theory suggests that such a prediction is infinitely informative (it has zero information entropy), so it is perhaps surprising that hydraulic modellers are so willing to present their results in this format. Less informative are probabilistic predictions, interval bounds (precise or fuzzy) on a (deterministic or probabilistic) prediction and predictions expressed in natural language. Generalised information theory (Klir 1991; Klir & Wierman 1999) provides a mechanism for measuring information content. The notion that information appears in a range of different degrees of fuzziness or vagueness (the term used by Russell (1923) and more recently by Williamson (1994)), from precise mathematical statements to vacuously vague statements, is as fundamental to hydroinformatics as it is to linguistics and human reasoning (Labov 1973). The more precise predictions are, the more useful they are, yet the easier they are to falsify (Blockley 1980).

Certain types of parameter naturally suggest a particular information format. For example, hydroinformaticians can treat the acceleration due to gravity as a deterministic value. Meanwhile time series measurements of weather-related phenomena (stream flows, wave heights, etc.) often conform well to a probabilistic model. Digital measurement devices provide interval measurements. It is a matter of controversy as to whether a fuzzy set is an appropriate way of representing a linguistic statement and, if so, how it should be constructed. A fuzzy set is certainly an appropriate way of representing a nested set of interval measurements.

The case when the observations lend themselves to a probabilistic description is often referred to as a situation of 'inherent' or 'aleatory' uncertainty, the term 'aleatory' coming from the Latin *aleator*, meaning 'dicer'. The existence of a special term (indeed, two terms that are commonly used more or less interchangeably) to refer to the situation where observations conform well to a probabilistic model betrays the prevailing prejudice towards probability amongst uncertainty theorists. Aleatory uncertainty is associated with variability in known (or observable) populations and is thought of as being irreducible (Ferson & Ginzberg 1996; Parry 1996; Winkler 1996). A step further, which in our opinion is ontologically insubstantiable, is to talk about aleatory uncertainties being a feature of reality. We prefer to proceed with an operational definition of aleatory uncertainties as being a feature of populations of measurements that conform well to a probabilistic model, thus avoiding insubstantiable assertions about the nature of reality (see Casti (1992) for a system-theoretic discussion). We use the term 'inherent uncertainty' to deal with any uncertainty inherent in the vocabulary of models. Thus in our definition probabilistic, fuzzy, set-based or linguistic information are all inherently uncertain.

Situations when observations obviously suggest a particular information format, be it deterministic, probabilistic or imprecise, are special cases. On the whole, observations of phenomena that interest hydroinformaticians will show some regularity, some random variability and some imprecision. Choosing an appropriate parameter format is a matter of skill and judgement.

Besides being related to the format in which model parameters or predictions appear, information content is also a feature of the structure of a model itself, for example the number of parameters a model employs and the scale at which it resolves the phenomenon of interest. A coarse-scale model contains less precise information than a fine-scale model. A model with more parameters is potentially more powerful than one with fewer because it may be able to resolve a more subtle set of potential changes, be they management options (changes at the discretion of a decision-maker, for example river maintenance strategies) or external changes (for example, climate change), though this power, if it is achievable at all, may be costly and perhaps also inelegant. However, due to lack of appropriate measurements, or lack of a dependable description of the processes at a fine scale, or lack of computational resources, or respect for Ockham's razor, a modeller may choose to adopt a coarser-scale model. Moreover, some system behaviours that are not evident at small scales become recognisable at larger scales (de Vriend 1991). Thus, whilst a finer-scale model will contain more information, that information may be neither meaningful nor dependable. This has motivated modern data mining activities within hydroinformatics which, by use of Artificial Neural Networks, Genetic Algorithms and other such devices, have often been able to generate accurate and quite parsimonious models, at least within the range of their training datasets (Babovic & Keijzer 2000; Lees 2000; Minns 2000).

General systems theory (von Bertalanffy 1969) provides a rationale for the notion of a hierarchy of models. However, in practice, the ordering of hydroinformatic models in a hierarchy according to their information content will be a matter of judgement. As suggested above, judgements of appropriate model scale and parametrisation should reflect the dependability of available information to identify parameters and of the routines used within the model to resolve the processes of interest. A modeller may wish to reduce information content (either by using a more abstract model or by presenting results in a non-deterministic format) until the information provided by the model is a reasonable reflection of the degree of dependability of the model. The criterion of reasonableness is an imprecise one and is an evidence-based judgement at the discretion of the modeller. A modeller should be disinclined to make deterministic predictions when observations of phenomena or prior knowledge suggest that they are likely to be falsified.

The modeller's task is not at all easy, as within the available formal vocabulary they have to address two related challenges. First, they have to do justice to the uncertain information about model parameters by representing them in an appropriate format. Second, they have to make an evidence-based judgement about the dependability of alternative models, the scale at which to represent the processes of interest and an appropriate format in which to present the model predictions. The relationship between these two aspects of uncertainty handling has been the subject of a great deal of debate, largely resting on whether it is feasible or useful to endeavour to try to separate these different aspects of uncertainty (a special edition of the *Journal of Reliability Engineering and Systems Safety* (Helton & Burmaster 1996) provides a good overview). There is no absolute criterion for separating the two: 'there is only one type of uncertainty stemming from our lack of knowledge concerning the truth of a proposition, regardless of whether this proposition involves the possible values of the hydraulic conductivity or the number of earthquakes in a period of time—distinctions between probabilities are merely for our convenience in investigating complex phenomena' (Apostolakis 1988, quoted in Winkler 1996). Moreover, there is the problem that the same mathematical vocabulary (conventionally probability) has to be used to accomplish two arguably different tasks. The suitability of probability for dealing with epistemic uncertainties has been particularly challenged (Shafer 1987; Hall *et al.* 1998). Evidence theory, which is described below, provides a richer mathematical vocabulary, but still does not provide a perfect solution to the problem of representing different aspects of uncertainty.

A number of authors (Hoffman & Hammonds 1994; Ferson & Ginzberg 1996; Hoffer 1996; Mosleh & Bier 1996) do identify situations where they argue that, from the point of view of decision-making, it is productive to try, as far as possible, to separate out uncertainties. In this difficult problem of separating uncertainties, as in other thorny problems of uncertainty, analysis of the decision context points towards some pragmatic solutions, so it is to decision-making in hydroinformatics that we now turn.

DECISION-MAKING CONTEXT

The inception of hydroinformatics has been traced to the development of fourth generation modelling tools in the mid-1980s (Abbott 1999). These were partially automated numerical simulation packages for modelling a range of behaviours in the hydraulic world (flow problems, sediment transport, pollution dispersal, etc.). They were being used to inform a diverse range of engineering and management decisions. However, these hydroinformatic tools provided insights about the water world that were meaningful, even in the absence of a specific decision context. Indeed, a given modelling tool could be used to inform a range of decisions, an approach that has found increasing favour in the intervening years, with water authorities establishing maintained models of the systems for which they are responsible so that they can be used on an *ad hoc* basis to inform decisions that may not even have been foreseen when the model was established.

Subsequent evolution in hydroinformatics can be seen as development towards systems that can be truly thought of as knowledge-based decision-support systems. They are being used in the conventional sense of decision-support systems, which aim to support rather than replace human judgement (Angehrn & Jelassi 1994). Together, the human and organisational systems that manage the water environment and their integrated computer-based tools constitute hydroinformatic systems in their true and broad sense. Here the full richness of decision problems is recognised. Decisions are not, on the whole, simple choices between precisely defined options that are predicted to deliver precise utilities under a set of mutually exclusive and collectively exhaustive states of nature. Rather, they are complex negotiated processes involving a range of stakeholders who make use of diverse evidence about options and their performance. Creative and communication processes play a vital role. Indeed, as more evidence is provided about the decision problem it may be reframed in entirely different terms (Tversky & Kahneman 1981), part of the process of collective learning. This deliberative level of communication and negotiation, that characterises most complex decision-making process, is referred to by Smets (1990) as the *credal level*. Smets used this term to distinguish the evidential reasoning about beliefs in a decision-making situation from the ultimate moment of choice, when uncertainty has to be temporarily suspended for an instant and a preferred option is selected. He referred to the moment of choice as the *pignistic level*. The word 'pignistic' is derived from the Latin *pignus*, which is a bet or wager. The term is used to signify that it is at the moment of choice that the conventional analogy of decision problems to gambling situations is, according to Smets, most applicable. Betting analogies are less meaningful at the credal level.

We have introduced Smet's distinction because it helps to explain our reservations concerning strictly decision-theoretic axiomisations of uncertainty. In clearly decision-theoretic situations that relate closely to the gambling analogies of subjective probability (for example, trading on the financial markets), the axiomatic arguments for subjective probability carry a great deal of weight, though, as has been shown by Walley (1991), it is possible to weaken those axioms and still avoid certain loss. Some hydroinformaticians may find themselves in analogous situations, for example if they are involved in pricing options for hydroelectric power supply. However, the majority are involved in less self-contained decision-making situations. They are involved in socially and politically negotiated processes where an individual decision-maker may not even be identifiable and where it will often be extremely difficult to attach a financial value to some, or indeed many, of the attributes of the water system. Hydroinformatics models bring evidence to bear on these complex social decision-making processes.

Knight (1921) referred to the decision-making situation where the predictions about future states of nature are expressed in probabilistic terms as 'decision-making under risk'. He identified another situation, where it is not possible to construct a probability distribution across future states of nature, and referred to that as 'decision-making under uncertainty'. Bayesians reject decision-making under uncertainty, arguing that in the hands of the right analyst (with their arm twisted sufficiently vigorously!) a decision-maker will always admit to some knowledge that can be expressed as a probability. Moreover, it can be shown that the common strategies for decision-making under uncertainty (minimax, maximin, least regret, etc.) are incoherent. Whilst accepting the latter problem and suggesting that the distinction presented by Knight between risk and uncertainty is a rather stark one, our analysis of the nature of uncertainty in hydroinformatic models indicates that it is by no means the case that knowledge is only encoded in deterministic or probabilistic terms. Indeed, we would argue that the majority of practical hydroinformatic decision-support problems lie somewhere between Knight's decision-making under risk and decision-making under uncertainty, a proposition to which we will return in the following section.

Recognition of the decision-making implications of uncertainty leads to the treatment of uncertainty being context-dependent and iterative (Hall 2002). It is context-dependent in the sense that the nature of the decision context informs the justification of the level of uncertainty analysis. It also influences the format in which model predictions are to be expressed. For routine decisions or decisions where the costs and benefits are well defined and agreed upon, it may be appropriate to reduce predictions to a pignistic value. For example, for routine flood warning decisions it can be argued that a probabilistic prediction of water level is not particularly useful and managers merely require a point-valued prediction (perhaps a mathematical expectation). Yet even apparently routine decisions can benefit from more subtle treatment of uncertainty, for example in staged warning systems where emergency services are brought to higher levels of readiness as the lead time to a flood, and the corresponding uncertainty in the prediction of water level, reduces (Eilts *et al.* 2000).

Consideration of the decision context enables analysis of decision robustness and of the value of obtaining further information (Raiffa 1968; Lindley 1990; Cox 2001). If a decision option is robust regardless of plausible variations in model predictions due to uncertainty, then no further analysis of uncertainty is required. Beyond this sensitivity-based approach more formal methods of robustness analysis are applicable (for example, Ben-Haim's info-gap theory (Ben-Haim 2001)). Considerable insight can be gained by inverting decision problems to identify the minimum information required in order to choose a robust course of action. Uncertainty is recognisable as an information gap between what is known and what is required in order to make a decision.

MATHEMATICS OF UNCERTAINTY

The conceptual appreciation of uncertainty and its role in decision-making is necessary but not sufficient to advance the treatment of uncertainty in hydroinformatics. To encode uncertainty in hydroinformatic tools requires a formal language. Fragments of that language are already commonplace, particularly in the field of stochastic hydraulics.

Of all the methods for handling uncertainty, probability theory has by far the longest tradition and is the best understood. That of course does not imply that it should be beyond criticism as a method of handling uncertainty. It does, however, mean that it is relatively well tested and well developed and can act as a standard against which other more recent approaches may be measured (Krause & Clark 1993).

The concept of probability may be defined and interpreted in several different ways (Lind 1996), the chief ones arising from the five approaches discussed in the following sections.

The classical approach

A game of chance has a finite number of different possible outcomes, which are assumed to be equally likely. The probability of an event (i.e. a particular outcome of interest) is then defined as the proportion of the total possible outcomes for which that event does occur. Evaluating probabilities in this framework involves counting methods (e.g. permutations and combinations).

Pioneering work on mathematising games of chance was undertaken in the late 16th and early 17th centuries by Cardano, Galileo, Pascal and others (Hacking 1975). However, the word *probability* was not mentioned by these early researchers. For medieval and Renaissance thinkers, probability belonged to the realm of opinion and argument, where the random was quite out of place.

The frequency approach

The frequency approach relates to the situation where an experiment can be repeated indefinitely under essentially identical conditions, but the observed outcome is random (not the same every time). Empirical evidence suggests that the proportion of times any particular event has occurred, i.e. its *relative frequency*, converges to a limit as the number of repetitions increases. This limit is the probability of the event. Proponents of the frequency approach, Fisher (1956) having been perhaps the most notable, take the view that probability is about countable events.

The propensity interpretation

The frequentist interpretation of a long sequence of events repeated under essentially identical conditions may be rather difficult to realise in practice. How is the probability of a breach in a specific dam to be interpreted when it has never been observed and may never be observed in the future? For unrepeatable phenomena the frequentist interpretation of probability ceases to have meaning. In this case probability is usually interpreted as a measure of the *propensity* or tendency for the specified event to occur (Popper 1959). It is possible to estimate the propensity of unrepeatable, yet nonetheless interesting, events by extending frequentist knowledge through causal models of physical phenomena.

The subjective approach

In this approach probability is used as a measure of belief in a statement. An event is a statement and the (*subjective*) *probability* of the event is a measure of the degree of belief which the subject has in the truth of the statement. If we imagine that a ‘prize’ is available if and only if the statement does turn out to be true, the subjective probability can be thought of as the proportion of the prize money which the subject is prepared to gamble in the hope of winning the prize. It is often by analogy to games of chance that subjective probabilities are constructed (de Finetti 1937/1980; Savage 1954).

According to the subjective view there is no such thing as a ‘true’ probability. Model parameters, for example, are all thought of as probability distributions that are expected to change with varying states of knowledge. Any assignment of subjective probabilities can be permitted in principle, provided they satisfy the requirements of coherence. To be coherent it should not be possible to construct a ‘Dutch Book’ against the individual assigning the probabilities. Probabilistic decision theory can be shown to be a logical consequence of adopting a set of coherency axioms (Savage 1954). Probabilities are merely a reflection of our current state of knowledge and should be updated in the light of new information. The conventional procedure for updating prior probabilities in the light of new information is usually attributed to Thomas Bayes (Bayes 1763), hence the customary label of ‘Bayesian’ probabilities.

The logical approach

Formal logic depends on relationships of the kind $A \rightarrow B$ (‘*A* implies *B*’) between propositions. The logical approach to probability generalises the concept of implication to *partial* implication; the conditional probability of *B* given *A* measures the extent to which *A* implies *B* (Keynes 1921; Jeffreys 1948). In the logical interpretation a conditional probability $P(A \mid B)$ is an assertion about the degree of ‘logical proximity’ of *A* and *B* or, more precisely, the degree to which statement *B* contains information that is expressed in *A*. If *B* says all that is said by *A*, so that *A* follows from *B*, then $P(A \mid B) = 1$ (Popper 1957). Carnap (1950) constructed a systematic version of this analysis, but found that in effect there is an infinite number of ways of assigning probabilities (Cohen 1989). The logical approach therefore differs from the subjective approach in which probabilities are assigned by recourse to some personal measure or judgement.

Probability and hydroinformatics

The frequency approach to probability is familiar in stochastic hydraulics and hydrology. However, hydroinformatics is not merely about numerical and statistical modelling. Decision-makers are often interested in unrepeatable phenomena, in which case a propensity interpretation of probability is more appropriate than a frequency interpretation. More profoundly, the hydroinformatic endeavour is to do with human and machine reasoning about the aquatic environment, and so requires a mechanism for manipulating uncertain beliefs and also for meta-level reasoning about the quality of models and the information they bring to bear on decision-making situations. Logical interpretations of probability have not been operationally demonstrated in these situations. Meanwhile, whilst the hydroinformatics’ requirement is for a mechanism for handling uncertain knowledge in its broadest sense, it is not at all clear that the Bayesian mathematisation of subjective probability can satisfy this requirement in practice. Let us therefore examine the axioms of probability theory to establish whether they are suitable for the task.

In the following axiomisation the universal set *X* contains all the relevant events. Probability is a function P: $X \rightarrow [0, 1]$, where [0,1] denotes the interval of real numbers from 0 to 1 inclusive, such that for any set $A \subset X$

Axiom 1: $P(A) \geq 0$

Axiom 2: $P(X) = 1$

Axiom 3: For any sequence of disjoint sets $A_i \in X$:

$$P(\bigcup_{i \in \mathrm{N}} A_i) = \sum_{i \in \mathrm{N}} P(A_i)$$

where N is the set of positive integers. The third axiom is the additivity axiom of probability. For a single $A \subset X$,

$$P(A)+P(\overline{A})=1 \quad (1)$$

i.e. the probability of an event and its negation must sum to unity. Although in some circumstance beliefs about an event and its negation may be described by the additivity axiom there are empirical and theoretical arguments why this should not be universally true (Shafer 1978). Descriptive studies of human reasoning and decision-making provide little evidence that human subjects process information in a manner which corresponds to probability theory (Kahneman *et al.* 1982). On the contrary, humans adopt heuristics and biases which, whilst not conforming to probability theory, may be an intelligent way of coping with complexity and the limited information processing powers which humans can apply to decision-making situations (March 1988). Now hydroinformatics systems are rarely intended to be descriptive models of human decision-making, in which case it could be argued that these empirical observations are irrelevant. However, if the intention of a hydroinformatics system is intelligent processing of uncertain information then it is surely wise to reflect on the reasons why intelligent human behaviour may depart from the axioms of conventional probability theory.

Probing the nature of human reasoning about aquatic systems it is found that much of that reasoning is *possibilistic* rather than strictly probabilistic. We reason about whether a given scenario *could* happen, without necessarily endeavouring to attach probabilities to the likelihood of it happening, particularly in situations of very scarce information. Shackle (1961) addressed the same concept when he referred to the 'degree of surprise' as a (non-probabilistic) measure of uncertainty. This type of reasoning is a natural response to uncertainty, but is much looser than probabilistic reasoning where, as we have seen above, one has to subscribe to a strong set of axioms. Ben-Haim (2001) has argued that 'modern engineering, management and social decision-making faces degrees and qualities of uncertainty which are unparalleled in the physical sciences.' Just as statistical mechanics and then quantum mechanics brought statistics into areas previously thought of as being the domain of exact sciences, so have problems of decision-making in unstructured and information-scarce situations challenged the probabilistic treatment of uncertainty that has been inherited from statistics and games of chance.

The additivity axiom is particularly problematic if one is reasoning about the dependability of a particular model or set of alternative models, because it suggests that the evidence for and against a given model is complete and known. As we have already argued, evidence about model dependability is sparse and incomplete and possibly inconsistent or conflicting. Aspects of incompleteness or inconsistency have to be engineered out of the problem domain in order to apply the additivity axiom.

Bayesians can be legitimately challenged to explain where prior probability distributions originate (Levi 1982). In practice, populations of data are finite and there are practical limits to the quantity of information that can be elicited from experts. In both cases some further assumptions, on grounds that may be unclear or questionable, are required in order to construct a precise distribution. The problem becomes particularly acute in situations of very scarce data and where experts express indecision about an uncertain quantity and cannot be induced to provide further information in the appropriate probabilistic format. For example, climate scientists have resisted attaching probabilities to climate change scenarios, primarily because of the tremendous uncertainty associated with long-term global patterns of development and greenhouse gas emissions. A recourse of the probabilistic analyst in the face of indeterminacy is to adopt a uniform probability distribution. However, a uniform distribution (or indeed any other unique distribution) represents a definite statement about the relative likelihoods of different states, which in cases of legitimate indeterminacy will overstate the available knowledge. Moreover, Keynes (1921) showed that incautious adoption of the uniform distribution can lead to contradictions.

Ben-Haim (1996) cites several practical cases in which engineering practice reflects the practical impossibility of identifying a specific model from a set of alternatives. For example, standard spectra are used to represent wave load spectra for offshore platforms because, in practice, it is impossible to precisely determine the loading spectrum at a given site. Conventions such as standard loading spectra

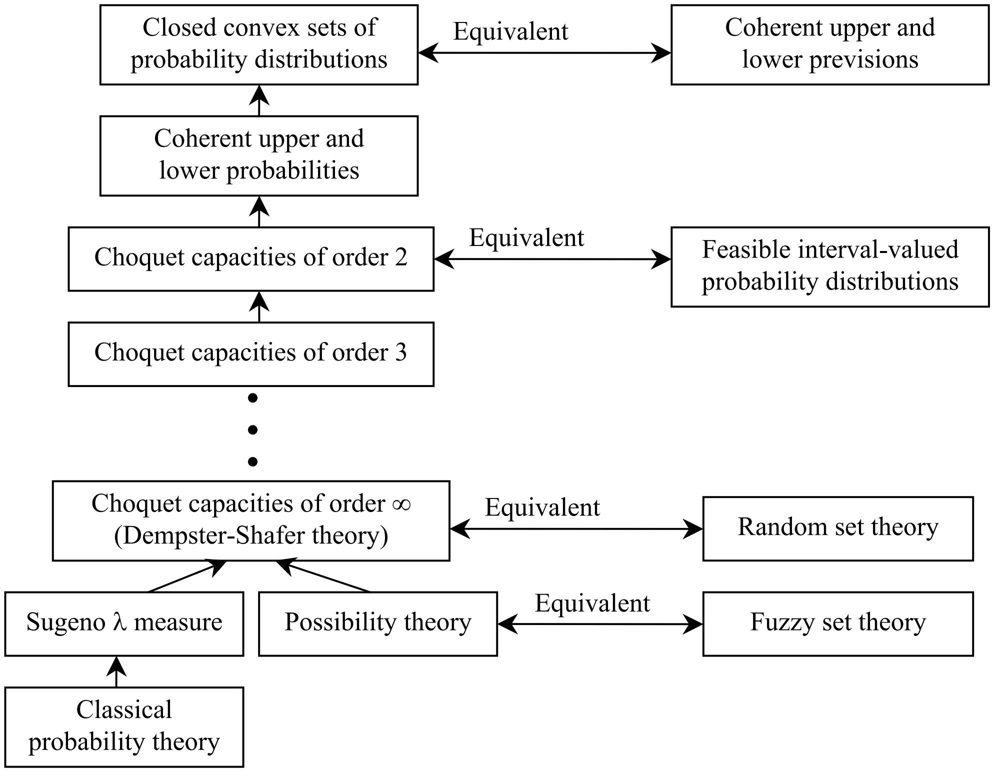

Figure 2 | Hierarchical relationship among theories of uncertainty (after Klir & Smith 2001).

have been essential to enable practical application of probabilistic methods in engineering design and regulation. Yet consequently the probability numbers generated can be highly misleading and their status quite unclear.

The forgoing leads us to conclude that conventional probability is not a rich enough vocabulary to handle uncertainty in hydroinformatics. We have already noted some counter-arguments to this view, summed up by Lindley (1987) who suggests that 'probability is a rich and subtle concept capable of dealing with beautifully delicate and important problems'. We do not subscribe to this view, believing that the modern mathematics of uncertainty provides a richer vocabulary. The existence of this vocabulary cannot be denied. At issue merely is the question of whether to ignore this vocabulary and, for reasons of parsimony and coherence, confine ourselves to conventional probability, or alternatively critically scrutinise this emerging paradigm to establish whether it may be helpful, particularly in situations where conventional probability has been most challenged. We have adopted the latter approach, and in the following section provide a brief introduction, an outline of the relationship between different theoretical branches and a very simple example.

GENERALISED MATHEMATISATION OF UNCERTAINTY

It is possible to generalise classical probability theory by weakening the third axiom from one of additivity to one of monotonicity, i.e. for all $A, B \in X$, if $A \subseteq B$ then $P(A) \leq P(B)$. Indeed a monotone measure does not necessarily have to be normalised (Axiom 2), and so can be defined as the function g: $A \rightarrow [0, \infty]$ provided it satisfies the above monotonicity condition and $g(\emptyset) = 0$. Figure 2, which has been adapted from Klir & Smith (2001) and Oberkampf *et al.* (2001), shows how various theories of uncertainty can be related to one another from a hierarchical point of view, i.e. the theory shown above subsumes the theory shown below. All of the theories shown in Figure 2, other than classical probability, are theories of nonadditive monotone measures.

The split at the bottom of the tree, at Choquet capacities order ∞, reflects the two sides to uncertainty that can be represented by using measure theory: randomness (or ambiguity) and fuzziness (or imprecision). Probability deals with the random aspects of uncertainty. For example, where there are A_i: $i=1, \ldots, n$, precisely defined mutually exclusive and collectively exhaustive possible outcomes of an experiment, but the outcome that will materialise is, *a priori*, unknown. A unit probability can be distributed across the A_i in proportion to their relative chance of occurrence.

Imprecision in the set definition of the events A_i is dealt with in the right branch of Figure 2, addressing the uncertainty associated with the scale at which events are defined. Supposing some basic population of events can be identified, for example decimal digits from 1 to 6, then the degree of imprecision of sets A_i defined on this space could be measured by their cardinality, i.e. the set {1,2,3,4} (cardinality 4) is less precise than the set {1,2,3} (cardinality 3). This says nothing about the relative likelihood of the result of an experiment (for example, the toss of a dice) being located in either of these sets, but relates to the way in which the events of interest are defined. Fuzzy set theory, which stresses the vagueness with which set-based concepts are defined, in particular in natural language, is equivalent to possibility theory (Zadeh 1978). A fuzzy set is usually defined by a membership function $\mu_A: X \rightarrow [0,1]$. μ_A is the *degree of membership* of any element of X in A. Classical (crisp) set theory corresponds to the situation where μ_A can take one of only two values, 0 or 1. The membership function generalises crisp set theory to include partial set membership.

Above classical probability in Figure 2 lie Sugeno λ-measures (Sugeno 1977), which are a mixture of classical probability theory and evidence theory, depending on the value of λ. Sugeno l-measures have not been developed in practice.

The Dempster–Shafer theory of evidence (Shafer 1976), referred to here as 'evidence theory', includes both possibility theory and probability theory as special cases. It therefore includes aspects of imprecision, in as much as it is structured on sets of varying granularity (known as 'focal elements'). A normalised measure (referred to by Shafer as a 'basic probability assignment' but here called a 'mass assignment') is distributed over these focal elements, thereby also capturing aspects of ambiguity/randomness. Evidence theory is outlined in more formal terms below, where it will be seen to be equivalent to the theory of random sets, which originated in the field of stochastic geometry (Kendall 1974; Matheron 1975).

As indicated in Figure 2, evidence theory is equivalent to the simplest of the infinite sequence of theories of uncertainty identified by Choquet (Klir & Smith 2001). Choquet referred to this simplest theory as a capacity of 'order infinity'. The higher-order theories in Figure 2 can be thought of as relating to families of probability distributions that are monotonic to a progressively lesser degree than the Dempster–Shafer theory. These higher-order theories are in the early stages of mathematical development and the interested reader is directed towards the literature of coherent lower and upper previsions (Walley 1991, 2000), closed convex sets of probability distributions (Kyburg 1987), coherent lower and upper probabilities (Walley 1991, 1996) and feasible interval-valued probability distributions (Weichselberger & Pohlmann 1990; Pan & Klir 1997). For the time being, very few practical applications of these theories have been published, so it is too early to judge their potential use for uncertainty handling in hydroinformatics.

Not included in Figure 2 are convex methods (Ben-Haim & Elishokoff 1990) of uncertainty handling, which have recently been developed by Ben-Haim into so-called info-gap decision theory (Ben-Haim 2001). In info-gap theory the uncertainty in a system model is parametrised with an uncertainty parameter α (a positive real number), which defines a family of nested sets that bound regions or clusters of system behaviour. When $\alpha=0$ the prediction from the system model converges to a point, which is the anticipated system behaviour, given current available information. However, it is recognised that the system model is incomplete so there will be a range of variation around the nominal behaviour. Uncertainty, as defined by the parameter α, is therefore a range of variation of the actual around the nominal. No further commitment is made to the structure of uncertainty. α is not a normalised measure, so info-gap theory is fundamentally different to the family of methods based on measure theory shown in Figure 2.

Evidence theory

Evidence theory is the simplest and most established of the theories that combine probability and possibility in the same theoretical framework. It has therefore been adopted here to provide a simple illustrative example of the application of generalised uncertainty handling to a water-related problem. Evidence theory is distinct from probability theory in the sense that the mass assignment m is a mapping from non-empty members $\mathfrak{I}$ of the power set $P(X)$, i.e. the set of all the subsets of X, rather from just the singleton subsets of X, which are the domain of classical probability theory. Thus m is a mapping $\mathfrak{I}\rightarrow[0, 1]$ such that

$$\sum_{A\in\mathfrak{I}} m(A)=1 \quad (2)$$

where A are not necessarily disjoint or singleton sub-sets of X. Here m is the amount of probability mass assigned to *A but not to any specific subset of A*. A random set is the pair $(\mathfrak{I}, m)$. A belief function Bel (Shafer 1976) can be defined as the following set function:

$$\forall A\in X, Bel(A)=\sum_{B\subseteq A} m(B) \quad (3)$$

and its dual plausibility function $Pl(A)$ is defined by

$$\forall A\in X, Pl(A)=\sum_{B\cap A\neq\varnothing} m(B)=1-Bel(\overline{A}). \quad (4)$$

$Bel(A)$ can be viewed as the lower bound on a family of probability measures and $Pl(A)$ as the upper bound, although the converse is not true, i.e. lower and upper probability functions are more general than belief and plausibility functions (Figure 2). When $\mathfrak{I}$ contains only singletons $Bel=Pl$ is a probability measure.

When $\mathfrak{I}$ is a nested family $A_1\subset A_2\subset \ldots \subset A_n$ then Bel and Pl satisfy the decomposability properties:

$$Bel(A\cap B)=\min(Bel(A), Bel(B)) \quad (6)$$

$$Pl(A\cap B)=\max(Pl(A), Pl(B)) \quad (7)$$

and Pl is a possibility measure in the sense of Zadeh (1978). In that case $(\mathfrak{I}, m)$ is called a consonant random set. A fuzzy set A can be defined from any random set $(\mathfrak{I}, m)$ as follows:

$$\forall u, \mu_F(u)=\sum_{u\in A} m(A)=Pl(\{u\}) \quad (8)$$

where μ_F is the fuzzy membership function. When $\mathfrak{I}$ is nested, μ_F is normalized, i.e. $\exists u,\ \mu_F(u)=1$.

Use of evidence theory in hydroinformatics models

Many hydroinformatics models are of the form $\mathbf{y}=f(\mathbf{x})$, where $\mathbf{x}$ is a vector $(x_1, x_2, \ldots, x_n)$ of input parameters and $\mathbf{y}$ is a vector $(y_1, y_2, \ldots, y_n)$ of output parameters. The function f usually constitutes a numerical model, perhaps implementing in some explicit or implicit fashion a set of differential equations. Uncertainty analysis is required to analyse the uncertainty in $\mathbf{y}$ that is a consequence of uncertainty in $\mathbf{x}$ and f. The uncertain dependency between $x_1, x_2, \ldots, x_n$ can be expressed in terms of a random relation, which is a random set $(\mathfrak{R}, \rho)$ on the Cartesian product $X_1\times\ldots\times X_n$, i.e. a mass assignment over the sets defined on this joint space. The range of y is the random set $(\mathfrak{I}, m)$ such that (Dubois & Prade 1991)

$$\mathfrak{I}=\{y(R_i)\ R_i\in z\mathfrak{R}\},\ y(R_i)=\{y(\mathbf{x})\ |\ \mathbf{x}\in zR_i\} \quad (9a)$$

$$\forall A\in\mathfrak{I}, m(A)=\sum_{A=y(R_i)} \rho(R_i) \quad (9b)$$

This is the extension principle for random sets. It provides a mechanism for projecting uncertain information through a function f, provided it can be represented as a random relation. In general Equations (9) involve calculating the image of each focal element $R_i\in\mathfrak{R}$, by applying

Table 1 | Joint measurements of wave height and water level

Significant wave height H_s (*m*)	Still water level h_w (*mAD*)										
	6.0–6.5	6.5–7.0	7.0–7.5	7.5–8.0	8.0–8.5	8.5–9.0	9.0–9.5	9.5–10.0	10.0–10.5	10.5–11.0	Total in row
4.5–5.0	0	0	0	0	0	0	1	2	0	0	3
4.0–4.5	0	0	0	0	0	0	1	2	1	0	4
3.5–4.0	0	0	0	5	3	6	7	8	0	1	30
3.0–3.5	1	0	2	12	21	19	17	16	5	2	95
2.5–3.0	0	2	8	28	27	34	38	25	5	1	168
2.0–2.5	0	7	26	48	81	106	75	58	10	0	411
1.5–2.0	0	20	56	158	164	166	177	79	11	2	833
1.0–1.5	2	34	155	305	344	437	348	168	28	2	1,823
0.5–1.0	3	49	254	512	644	744	649	309	53	3	3,220
0.0–0.5	1	53	234	438	545	628	504	225	43	0	2,671
Total in column	7	165	735	1,506	1,829	2,140	1,817	892	156	11	9,258

twice the techniques of global optimisation. However, special cases exist when the function *f* is monotonic (Dong & Shah 1987; Tonon *et al.* 2000), when the random relation ($\Re$, ρ) is consonant, i.e. a fuzzy set, and when it is a probability distribution (Dubois & Prade 1991).

To illustrate the operation of the extension principle we consider a simple overtopping problem from coastal hydraulics. Owen's equation (HR Wallingford 1980) for the time-averaged overtopping discharge Q ($m^3/s/m$) at a smooth sloping seawall can be written as

$$Q = gH_sT_m a exp\left[\frac{-b(h_c-h_w)}{T_m\sqrt{gH_s}}\right] \quad (10)$$

where
H_s is the significant wave height (m)
T_m is the mean wave period (m)
h_c is the seawall crest level (mAD)
h_w is the still water level (mAD)

and a and b are coefficients. In this example statistical data exists for wave height (H_s), period (T_m) and water level (h_w). Joint measurements of wave height and water level and wave height and period are listed in Tables 1 and 2, respectively. Random relations were constructed at the granularity shown, with the mass assigned to each sub-set corresponding to the relative frequency in the dataset. Beyond their joint dependency on wave height, wave period and water level were assumed to be stochastically independent. The crest level (h_0) has been surveyed to an accuracy of ± 0.01 m and is represented by the interval [13.72, 13.74]. The coefficients a and b are mean values extracted from physical model studies. There was scatter in this experimental data, which could be reflected in a probability distribution. However, more significant are the uncertainties related to transferring a parametric model developed from experimental data to a specific site. To reflect these uncertainties, the coefficients a and b (corresponding to a 1:2 seawall slope)

Table 2 | Joint measurements of wave height and period

Significant wave height H_s (m)	Mean wave period T_m (s) 2.0–3.0	3.0–4.0	4.0–5.0	5.0–6.0	6.0–7.0	7.0–8.0	8.0–9.0	9.0–10.0	10–11.0	11–12.0	Total in row
4.5–5.0	0	0	0	0	0	0	1	2	0	0	3
4.0–4.5	0	0	0	0	0	1	1	2	0	0	4
3.5–4.0	0	0	0	0	3	10	8	5	3	1	30
3.0–3.5	0	0	0	4	14	41	27	8	1	0	95
2.5–3.0	0	0	0	6	40	64	26	23	6	3	168
2.0–2.5	0	0	0	6	151	189	46	16	3	0	411
1.5–2.0	0	0	6	56	480	254	28	6	3	0	833
1.0–1.5	0	17	170	636	534	305	111	42	8	0	1,823
0.5–1.0	61	153	552	981	920	460	61	31	1	0	3,220
0.0–0.5	0	141	422	562	843	562	141	0	0	0	2,671
Total in column	61	311	1,150	2,251	2,985	1,886	450	135	25	4	9,258

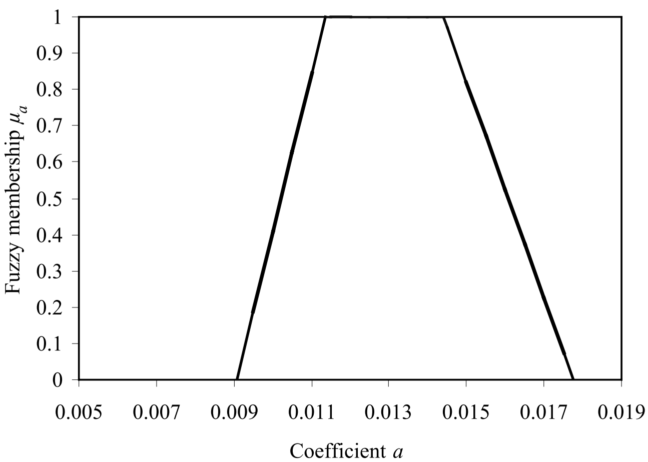

Figure 3 | Fuzzy set on coefficient a.

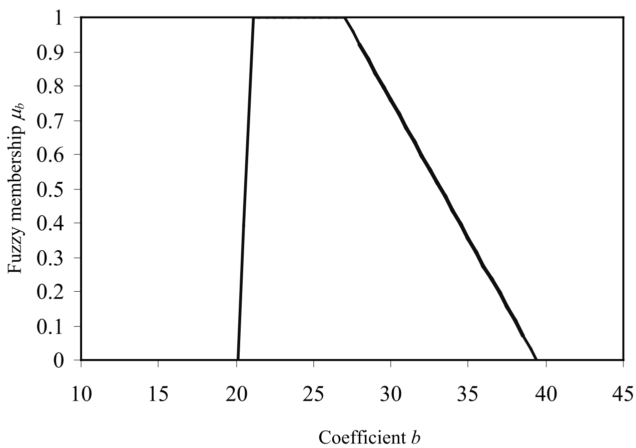

Figure 4 | Fuzzy set on coefficient b.

have been assigned the fuzzy sets shown in Figures 3 and 4, respectively. These fuzzy sets were represented as consonant random sets at five different membership levels.

The random set obtained by applying the extension principle (Equations (9)) to Owen's equation (Equation (10)) contained 12,525 focal elements. One of the problems of random set analysis is visualising the rather

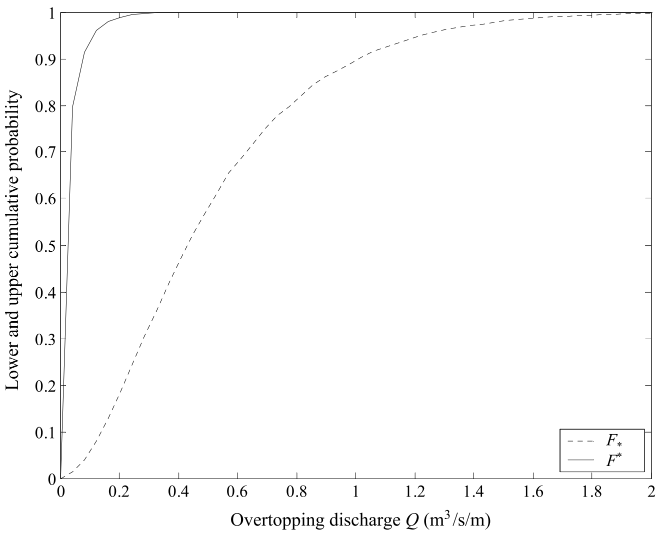

Figure 5 | Cumulative belief and plausibility distributions on overtopping discharge *Q*.

complex uncertainty structures that are generated. A convenient way of doing so is to plot lower and upper cumulative probabilities (Figure 5), defined as follows. Suppose that a closed interval $[x_1, x_{s+1}]$ is partitioned into disjoint sub-intervals $[x_1, x_2]$, $(x_2, x_3]$, . . ., $(x_{s-1}, x_s]$, $(x_s, x_{s+1}]$ labelled A_1, A_2, . . ., A_s, respectively. A set of intervals $\{A_i, A_j, A_k\}$ $i<j<k$ is labelled $\{A_{i,k+1}\}$, i.e. according to its extreme lower and upper limits. The lower and upper cumulative probability distribution functions, $F_*(x)$ and $F^*(x)$, respectively, at some point x can be obtained as follows:

$$F_*(x)=\sum_{x\geq_j} m(\{A_{i,j}\}) \tag{13}$$

and

$$F^*(x)=\sum_{x\geq x_i} m(\{A_{i,j}\}). \tag{14}$$

The bounds shown in Figure 5 are upper and lower probability bounds on the overtopping discharge. They have been obtained using only the information listed above, making no further assumptions. They provide a risk analyst with a probabilistic estimate of overtopping discharges (for use, for example, in flood risk analysis) and an impression of the uncertainty in that estimate (represented as the probability bounds) due to field measurement errors and uncertainty in the parametric overtopping model.

This simple example has illustrated how evidence theory can be used to project uncertain information in a range or formats through a function relation. It is easy to see how this could be extended to more complex numerical models, though the computational implications of so doing may be significant. The approach has illustrated important features of the type of generalised uncertainty handling promoted in this paper. Information is, as far as possible, preserved in the format in which it appears, be it populations of measurements that conform to a frequentist probability distribution, interval estimates or fuzzy sets. Well known frequentist information is represented as probability distributions, whereas parameters whose values are not precisely known, due to ignorance about the underlying processes or the applicability of the model to the site in question, are represented in an imprecise format. In this way, statements about frequencies or propensities of empirical events are distinguished from measures of belief in the dependability of those statements. Probabilities based on extensive data can be immediately distinguished from probabilities based on ignorance.

Furthermore, in evidence theory, situations of total ignorance can be represented by making vacuously imprecise statements. This is in stark contrast to the conventional probabilistic approach in which the least informative (maximum entropy) statement is a uniform distribution across possible states, confusing indeterminacy with equiprobability (Ben-Haim 1997).

As argued towards the beginning of this paper, the decision as to whether an uncertain quantity or process should be modelled using probability or imprecision is not a clear-cut one. The vocabulary of the resulting prediction will have to be explained to the decision-maker and wider stakeholder groups. Fortunately, empirical evidence, although limited, suggests that methods based on fuzzy sets and intervals are no less accessible than probabilistic methods (Curley & Golden 1994) and our experience is that they can be much more so (Davis and Hall 2003).

CONCLUSIONS

Uncertainty is one of the outstanding challenges in hydroinformatics. If its treatment is to be advanced, a conceptual structure and practical methods are required for handling uncertainty in the hydroinformatic process. This paper has endeavoured to contribute in both conceptual and practical terms. We anticipate that, due to a combination of increasing appreciation of the mathematics of uncertainty, ever-improving computational resources and demands from policy and decision-makers, the analysis of uncertainty will in the future become a much more routine aspect of hydroinformatics. The success of simple spreadsheet add-ins for Monte Carlo simulation illustrates the demand for uncertainty analysis, albeit often fairly naive. In hydroinformatics there is the opportunity to develop a much more subtle appreciation of uncertainty and decision-support tools.

Uncertainty is an inevitable consequence of abstracting from perceptions of the phenomenal world into closed formal constructs (models). A model will be a selective abstraction, usually constructed with a specific purpose or purposes in mind. One fundamental source of uncertainty, which is customarily referred to as 'epistemic' uncertainty, is the nature of that abstraction and the incomplete and sometimes conflicting evidence about whether a model is fit for its purpose. Meanwhile, any measurements or predictions will be expressed in a vocabulary that will be more or less uncertain. Thus, for example, probabilistic statements about populations of measurements are said to contain 'aleatory' uncertainty. In this paper 'inherent' uncertainties have been thought of in more general terms as the uncertainty associated with the non-deterministic format of measurements or predictions, be that format probabilistic, fuzzy or both. The choice of scale at which to model a phenomenon of interest is influenced by considerations of both epistemic and inherent uncertainties.

A formal language is required in order to encode uncertainty in hydroinformatics systems. Beginning with probability theory, which is the standard against which alternative theories will inevitably be compared, various mathematisations of uncertainty and their interpretations have been reviewed. The choice of uncertainty mathematisation and the semantics applied to it will always, to some extent, be a matter of taste. However, we have argued that purely probabilistic treatment of uncertainty does not lend itself to some of the more subtle aspects of uncertainty handling in hydroinformatics, such as reasoning about model dependability, representation of very sparse knowledge or informing deliberative decision processes. Although some important issues remain to be resolved, we believe that the generalised mathematics of uncertainty provides a more subtle and expressive vocabulary than conventional probability. Since the inception of fuzzy set theory in the 1970s there has been widespread confusion about the relationship between fuzzy sets and probability. It is now clear that these two theories relate to different aspects of uncertainty and that there are coherent generalisations. We have illustrated how evidence theory (and its counterpart, interpretation as a random set) is the best developed of these generalisations and can be applied in practice to the types of algebraic or numerical functions that form the kernel of hydroinformatic systems.

REFERENCES

Abbott, M. B. 1991 *Hydroinformatics: Information Technology and the Aquatic Environment*. Avebury Technical, Aldershot.

Abbott, M. B. 1999 Introducing hydroinformatics. *J. Hydroinformatics* **1**(1), 3–19.

Abbott, M. B. 2001 The democratisation of decision-making processes in the water sector I. *J. Hydroinformatics* **3**(1), 11–22.

Abbott, M. B. & Jonoski, A. 1998 Promoting collaborative decision-making through electronic networking. In: *Hydroinformatics '98* (ed. C. Babovic & L. C. Larsen), pp. 485–491. Balkema, Rotterdam.

Abbott, M. B. & Jonoski, A. 2001 The democratisation of decision-making processes in the water sector II. *J. Hydroinformatics* **3**(1), 23–34.

Angehrn, A. A. & Jelassi, T. 1994 DSS research and practice in perspective. *Decision Support Systems* **12**, 267–275.

Apostolakis, G. E. 1988 The concept of probability in safety assessments of technological systems. *Science* **250**, 1359–1364.

Babovic, V. & Keijzer, M. 2000 Genetic programming as a model induction engine. *J. Hydroinformatics* **2**(1), 35–60.

Bayes, T. 1763 An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society London* **53**(A), 370–418. (In *Studies in the History of Statistics and Probability* (ed. E. S. Pearson & M. G. Kendall), pp. 131–153. Griffin, London.)

Ben-Haim, Y. 1996 *Robust Reliability in the Mechanical Sciences*. Springer, Berlin.

Ben-Haim, Y. 1997 Beyond maximum entropy: exploring the structure of uncertainty. In *Uncertainty: Models and Measures* (ed. H. G. Natke & Y. Ben-Haim), pp. 11–26. Akademie Verlag, Berlin.

Ben-Haim, Y. 2001 *Information-Gap Decision Theory: Decisions Under Severe Uncertainty*. Academic Press, San Diego.

Ben-Haim, Y. & Elishakoff, I. 1990 *Convex Methods of Uncertainty in Applied Mechanics*. Elsevier, Amsterdam.

von Bertalanffy, L. 1969 *General Systems Theory: Foundations, Development, Applications*. Revised edn. George Brazillier, New York.

Beven, K. 2001 How far can we go in distributed hydrological modelling? *Hydrology and Earth System Science* **5**(1), 1–12.

Blockley, D. I. 1980 *The Nature of Structural Design and Safety*. Ellis Horwood, Chichester.

Carnap, R. 1950 *Logical Foundations of Probability*. University of Chicago Press, Chicago.

Casti, J. L. 1992 *Reality Rules: Picturing the World in Mathematics*. Wiley, New York.

Choquet, G. 1953 Theory of capacities. *Annales de L'Institut Fourier* **5**, 131–295.

Cohen, L. J. 1989 *An Introduction to the Philosophy of Induction and Probability*. Clarendon Press, Oxford.

Cooke, R. M. 1991 *Experts in Uncertainty: Opinion and Subjective Probability in Science*. Oxford University Press, New York.

Cox, L. A. 2001 *Risk Analysis: Foundations, Models, and Methods*. Kluwer, Dortrecht.

Curley, S. P. & Golden, J. I. 1994 Using belief functions to represent degrees of belief, *Organisational Behaviour and Human Decision Processes* **58**(2), 271.

Davis, J. P. & Blockley, D. I. 1996 On modelling uncertainty. In *Hydroinformatics '96* (ed. A. Mueller), pp. 485–491. Balkema, Rotterdam.

Davis, J. P. & Hall, J. W. 1998 Assembling uncertain evidence for decision-making. In *Hydroinformatics '98* (ed. V. M. Babovic & L. C. Larsen), pp. 1089–1094. Balkema, Rotterdam.

Davis, J. P. & Hall, J. W. 2003 A software supported process for assembling evidence and handling uncertainty in decision-making. *Decision Support Systems* **35**(3), 415–433.

Dong, W. & Shah, H. C. 1987 Vertex method for computing functions of fuzzy variables. *Fuzzy Sets & Systems* **24**, 65–78.

Dubois, D. & Prade, H. 1991 Random sets and fuzzy interval analysis. *Fuzzy Sets and Systems* **42**, 87–101.

Eilts, M. D., Goodman, W., Johnson, J. T., Rothfusz, L. & Ruth, D. 2000 Warning operations in support of the 1996 centennial Olympic Games. *Bulletin of the Meteorological Soc.* **81**(3), 543–554.

Ferson, S. & Ginzburg, L. R. 1996 The characterization of uncertainty in probabilistic risk assessments of complex systems. *Reliability Engineering and Systems Safety* **54**, 133–144.

de Finetti, B. 1937/1980 La prévision: ses lois logiques, ses sources subjectives. In *Studies in Subjective Probability* (ed. H. E. Kyburg & H. E. Smolker), pp. 53–118. Wiley, New York.

Fisher, R. A. 1956 *Statistical Methods and Scientific Inference*. Oliver & Boyd, Edinburgh.

French, S. 1986 *Decision Theory: An Introduction to the Mathematics of Rationality*. Ellis Horwood, Chichester.

French, S. 1995 Uncertainty and imprecision: modelling and analysis. *Journal of the Operational Research Society* **46**, 70–79.

Goutsias, J., Mahler, R. P. S. & Nguyen, H. T. (eds.) 1997 *Random Sets: Theory and Applications*. Springer, New York.

Hacking, I. 1975 *The Emergence of Probability: A Philosophical Study of Early Ideas About Probability, Induction and Statistical Inference*. Cambridge University Press, Cambridge.

Hall, J. W., Blockley, D. I. & Davis, J. P. 1998 Non-additive probabilities for representing uncertain knowledge: theoretical and practical implications. In *Hydroinformatics '98* (ed. V. M. Babovic & L. C. Larsen), pp. 1101–1108. Balkema, Rotterdam.

Hall, J. W. 2002 A contingency approach to choice. *Civil Engineering and Environmental Systems* **19**(2), 87–118.

Hall, J. W., Davis, J. P. & Blockley, D. I. 1999 Uncertainty analysis of coastal projects. In *Coastal Engineering 1998* (ed. B. L. Edge), pp. 1461–1474. ASCE, New York.

Helton, J. C. & Burmaster, D. E. (eds.) 1996 Treatment of aleatory and epistemic uncertainty in performance assessments for complex systems. *Reliability Engineering and System Safety* **54**, 91–258.

Hoffer, E. 1996 When to separate uncertainties and when not to separate. *Reliability Engineering and System Safety* **54**, 113–118.

Hoffman, F. O. & Hammonds, J. S. 1994 Propagation of uncertainty in risk assessments: the need to distinguish between uncertainty due to lack of knowledge and uncertainty due to variability. *Risk Analysis* **14**, 707–712.

HR Wallingford 1980 *Design of Seawalls for Wave Overtopping*. Report EX924.

Jeffreys, H. 1948 *Theory of Probability*. Clarendon Press, Oxford.

Kahneman, D., Slovic, P. & Tversky, A. 1982 *Judgement Under Uncertainty, Heuristics and Biases.* Cambridge University Press, Cambridge.

Kendall, D. G. 1974 Foundations of a theory of random sets, In *Stochastic Geometry* (ed. E. F. Harding & D. G. Kendall), pp. 322–376. Wiley, Chichester.

Keynes, J. M. 1921 *A Treatise on Probability*. Macmillan, London.

Klir, G. J. 1991 Generalised information theory. *Fuzzy Sets and Systems* **40**, 127–142.

Klir, G. J. & Smith, R. M. 2001 On measuring uncertainty and uncertainty-based information: recent developments. *Annals of Mathematics and Artificial Intelligence* **32**, 5–33.

Klir, G. J. & Wierman, M. J. 1999 *Uncertainty-Based Information: Elements of Generalised Information Theory*. Physical-Verlag, New York.

Knight, F. H. 1921 *Risk, Uncertainty and Profit*. Houghton Mifflin, Boston.

Krause, P. J. & Clark, D. A. 1993 *Representing Uncertain Knowledge: An Artificial Intelligence Approach*. Intellect Books, Oxford.

Kyburg, H. E. 1987 Bayesian and non-Bayesian evidential updating. *Artificial Intelligence* **31**, 271–293.

Labov, W. 1973 The boundaries of words and their meanings. In: *New Ways of Analysing Variations in English* (ed. C.-J. N. Bailey & R. W. Shuy), pp. 340–373. Georgetown University Press, Washington, DC.

Lees, M. J. 2000 Data-based mechanistic modelling and forecasting of hydrological systems. *J. Hydroinformatics* **2**(1), 15–34.

Levi, I. 1982 Ignorance, probability and rational choice. *Synthese* **53**, 387–417.

Lind, N. C. 1996 Validation of probabilistic models. *Civil Engineering Systems* **13**, 175–183.

Lindley, D. V. 1982 Scoring rules and the inevitability of probability. *International Statistical Review* **50**, 1–26.

Lindley, D. V. 1987 The probability approach to treatment of uncertainty in artificial intelligence and expert systems. *Statistical Science* **2**(1), 3–44.

Lindley, D. V. 1990 *Making Decisions*. 2nd edn. Wiley, Chichester.

March, J. G. 1988 Bounded rationality, ambiguity and the engineering of choice. In: *Decision Making: Descriptive, Normative and Prescriptive Interactions* (ed. D. E. Bell, H. Raiffa & A. Tversky), pp. 33–57. Cambridge University Press, Cambridge.

Matheron, G. 1975 *Random Sets and Integral Geometry*. Wiley, New York.

Minns, A. W. 2000 Subsymbolic methods for data mining in hydraulic engineering. *J. Hydroinformatics* **2**(1), 3–14.

Mosleh, A. & Bier, V. M. 1996 Uncertainty about probability: a reconciliation with the subjectivist viewpoint. *IEEE Trans. on Systems, Man and Cybernetics—Part A: Systems and Humans* **26**(3), 303–310.

Oberkampf, W. L., Helton, J. C. & Sentz, K. 2001 Mathematical representation of uncertainty. *42nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conf., 16–19 April, 2001, Seattle*, American Institute of Aeronautics and Astronautics, pp. 1–23.

Pan, Y. & Klir, G. J. 1997. Bayesian inference based on interval-valued prior distributions and likelihoods. *J. Intelligent and Fuzzy Systems* **5**(3), 193–203.

Parry, G. W. 1996 The characterisation of uncertainty in probabilistic risk assessments of complex systems. *Reliability Engineering and Systems Safety* **54**, 119–126.

Pearl, J. 1988 *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Kaufmann, San Mateo.

Popper, K. R. 1957 Probability magic or knowledge out of ignorance. *Dialectica* **11**, 354–373.

Popper, K. R. 1959 The propensity interpretation of probability. *British Journal of Philosophical Science* **10**, 25–42.

Popper, K. R. 1963 *Conjectures and Refutations*. Routledge, London.

Popper, K. R. & Eccles, J. C. 1977 *The Self and Its Brain*. Springer, Berlin.

Raiffa, H. 1968 *Decision Analysis: Introductory Lectures on Choices Under Uncertainty*. Addison-Wesley, Reading, MA.

Russell, B. A. W. 1923 Vagueness. *Australian Journal of Philosophy and Psychology* **1**, 84–92.

Savage, L. J. 1954 *The Foundations of Statistics*. Wiley, New York.

Schum, D. A. 1994 *The Evidential Foundations of Probabilistic Reasoning*. Wiley, New York.

Shackle, G. L. S. 1961 *Decision, Order and Time in Human Affairs*. Cambridge University Press, Cambridge.

Shafer, G. 1976 *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, NJ.

Shafer, G. 1978 Non-additive probabilities in the work of Bernoulli and Lambert. *Archive for History of Exact Sciences* **20**, 309–370.

Shafer, G. 1987 Probability judgements in artificial intelligence and expert systems. *Statistical Science* **2**(1), 3–44.

Smets, P. 1990 Constructing the pignistic probability function in a context of uncertainty. In: *Uncertainty in Artificial Intelligence 5* (ed. M. Henrion, R. D. Schachter, L. N. Kanal & J. F. Lemmer), pp. 29–39. North-Holland, New York.

Sugeno, M. 1977. Fuzzy measures and fuzzy integrals: a survey. In: *Fuzzy Automota and Decision Processes* (ed. M. M. Gupta, G. N. Saridis & B. R. Gaines), pp. 89–102. North-Holland, New York.

Tonon, F., Bernardini, A. & Mammino, A. 2000 Reliability analysis of rock mass response by means of random set theory. *Reliability Engineering and System Safety* **70**(3), 263–282.

Tversky, A. & Kahneman, D. 1981 The framing of decisions and the psychology of choice. *Science* **211**(D), 453–458.

de Vriend, H. J. 1991 Mathematical modelling of large-scale coastal behaviour, part 2: predictive models. *Journal of Hydraulic Research* **29**(6), 741–753.

Walley, P. 1991 *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.

Walley, P. 1996 Measures of uncertainty in expert systems. *Artificial Intelligence* **83**(1), 1–58.

Walley, P. 2000 Towards a unified theory of imprecise probability. *Int. J. Approximate Reasoning* **24**(2–3), 125–148.

Weichselberger, K. & Pohlmann, S. 1990 *A Methodology for Uncertainty in Knowledge-Based Systems*. Springer-Verlag, New York.

Williamson, T. 1994 *Vagueness*. Routledge, London.

Winkler, R. L. 1996 Uncertainty in probabilistic risk assessment. *Reliability Engineering and Systems Safety* **54**, 127–132.

Wright, G. & Ayton, P. 1994 *Subjective Probability*. Wiley, New York.

Zadeh, L. A. 1978. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems* **1**, 2–28.