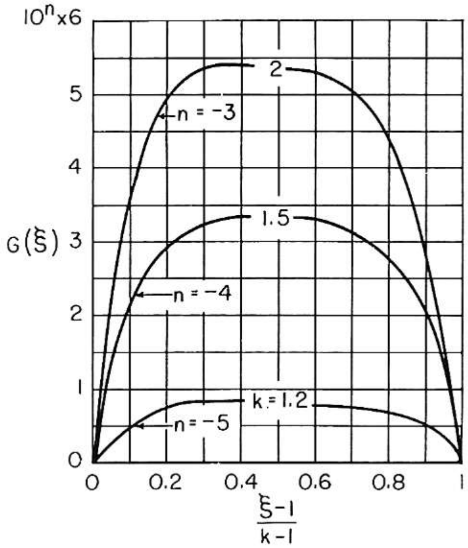

Fig. 6 Function for temperature rise owing to viscous shear for steady longitudinal flow

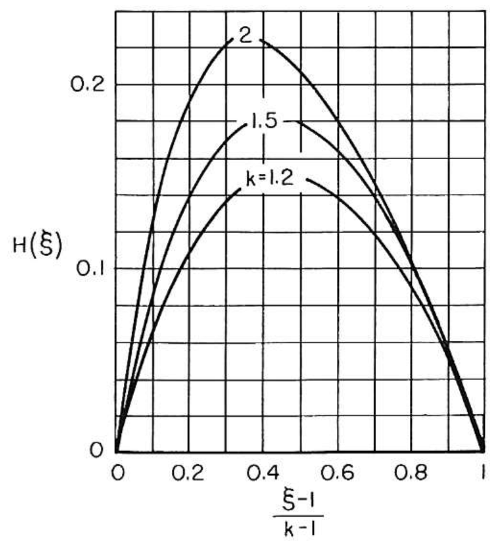

Fig. 7 Function for temperature rise owing to viscous shear for steady rotational motion, after Schlichting [7]

with two different constant speeds if one replaces ω by the difference between the two angular speeds. The function $G(\xi)$ is plotted in Fig. 6. The function $H(\xi)$ has already been given by Schlichting [7], but for the sake of completeness is reproduced in Fig. 7. In both Figs. 6 and 7, the parameter $(\xi - 1)/(k - 1)$ is used as the abscissa.

The temperature rise calculated in this section can be superposed on that calculated in the section, Unsteady Thermal Boundary Conditions.

General Method for Calculating Temperature Rise Due to Unsteady Viscous Shear. If the flow is unsteady the dissipation function represented by the bracket in (27) consists generally of infinitely many time-dependent terms, and an equation of the following type is encountered:

$$\frac{\partial \Omega}{\partial \tau} = \frac{\partial^2 \Omega}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \Omega}{\partial \xi} + \sum_{i=1}^{\infty} g_i(\xi) e^{-m_i \tau} \qquad (45)$$

By the methods developed in the two preceding sections, an equation of the type

$$\frac{\partial f_i}{\partial \tau} = \frac{\partial^2 f_i}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial f_i}{\partial \xi} + g_i(\xi) \qquad (46)$$

with the initial condition $f_i(\xi, 0) = 0$ can be solved, so that the functions f_i can be considered as known. Then the function

$$\Omega_i = \int_0^\tau -m_i e^{m_i \tau_1} f_i(\xi, \tau - \tau_1) d\tau_1 + f_i(\xi, \tau)$$

satisfies

$$\frac{\partial \Omega_i}{\partial \tau} = \frac{\partial^2 \Omega_i}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \Omega_i}{\partial \xi} + g_i(\xi) e^{-m_i \tau}$$

and the initial condition $\Omega_i\ (\xi, 0) = 0$, as can be shown by a straightforward calculation. This result illustrates still another application of the Duhamel method, which differs from the usual application in that it deals with a time-dependent term in the nonhomogeneous part of the diffusion equation instead of a varying boundary condition. With the Ω_i thus found, the final solution is

$$\Omega = \sum_{i=1}^{\infty} \Omega_i \qquad (47)$$

Varying boundary conditions can again be treated by the Duhamel method as in the foregoing.

Conclusions

1 Specific problems of laminar momentum or heat transfer in a fluid between two concentric cylinders have been solved by well-known methods. The solutions, expressible in terms of Bessel and associated functions, have been carried to the numerical stage.

2 Varying boundary conditions and nonhomogeneous terms depending on time in the equation of heat diffusion have been treated successfully with the Duhamel method.

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DISCUSSION

John A. Clark[3]

This is an important application of applied mathematics to an interesting problem. After reading the paper the following questions come to mind on which I would appreciate the author's comment. Does the author plan any extension of the work to include the influence of eccentricity? Would he have a comment on this in relation to the present results? Would he comment on the effect of thermal-dependent properties, especially viscosity, on the solutions presented?

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Author's Closure

Professor Clark's questions will be answered in the order in which they have been posed. The work reported here was done primarily in preparation for the study of the hydrodynamic stability of unsteady flows, which has recently been found to be amenable to analysis, but is published here because of its general bearing on the problems of lubrication. Solutions for steady flows (relative to the one and only cylinder that is eccentric) are already available elsewhere [5]. Those for unsteady flows between concentric cylinders are, as far as the author is aware, not known. Although they will be much more difficult to obtain, it is possible to obtain them for small eccentricities by expanding the solutions in ascending powers of the two parameters characterizing the two eccentricities. No such work is planned, because, as stated, this work has been done to facilitate future investigations of the hydrodynamic stability of unsteady flows. The present results will be useful for studying unsteady momentum and heat transfer between cylinders with small eccentricities in the sense that the solutions sought are in the neighborhood of the ones presented here or of others similar to them. As to the effects of thermal variation of fluid properties on the flow and on the temperature distribution, they can be determined by a process of successive approximation. Starting from the solutions presented here, one can correct the viscosity and diffusivity accordingly, and use the new (time- and space-dependent) values in the second approximation. Care must be taken in the second approximation to put the viscosity and the diffusivity in the correct places in the equations of motion and in the diffusion equation, since they are no longer constant. The resulting equations will be more complicated, but still linear, and of a type similar to (45). Consequently, they can be solved with some patience and effort.