

A robust stage-discharge rating curve model based on critical flow from a reservoir

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Received 3 August 2005; accepted in revised form 28 February 2006

Abstract Hydrometric gauging stations are often sited in reservoirs such as lakes and river pools since they possess favourable features for streamflow determination. By applying the equations which govern critical flow from a reservoir along with a spline representation of the known geometric characteristics of the reservoir outlet, a new method is developed for estimating the stage-discharge relationship in reservoir-like situations. The proposed rating curve model is amenable to least squares estimation using the available stage-discharge measurements. The proposed model is shown by theory to be general, and includes both normal uniform flow from a reservoir and the classical power-law rating curve model as special cases. Using geometric data and stage-discharge measurements from two Norwegian gauging stations, it is demonstrated that (a) the proposed model is more appropriate than the classical power-law rating curve model when performance is measured by the AIC statistic and the sizes of the jackknife standard deviations of the predicted discharges, and (b) the proposed model displays a superior performance over the power-law model in both upwards and downwards extrapolation.

Keywords Critical depth; hydrometry; non-linear regression; rating curves; reservoir; stage-discharge

Introduction

The aim of a hydrometric gauging station is to produce an accurate and continuous record of discharge, which is essential data in almost all areas of water resources management. Typically, a gauging station continuously records stage at a cross-section where the relation between stage and discharge can be determined. The attainable accuracy of the stage-discharge relationship is governed to a large extent by the characteristics of the hydraulic control. Clearly, hydraulic controls are inadequate if they are affected by physical changes due to scour and fill and/or severe weed growth (Simons *et al.* 1973; Powell 1978) and/or ice build-up during cold periods (Tuthill *et al.* 1996; Hicks and Healy 2003), since a constantly shifting stage-discharge relationship requires a time-consuming and costly on-going programme of discharge measurements to define the instability. Hysteresis, or a looped stage-discharge relationship, due to unsteady flow (Chow 1959, part V), is another factor which can complicate rating curve modelling. If the stage-discharge relationship is significantly affected by unsteady flow, calculations then require both additional determinants and a more complex model (Fread 1975; Tawfik *et al.* 1997; Petersen-Øverleir 2005).

An outlet from a lake or larger pool often provides a useful stage-discharge relation since it usually comprises a hydraulic control that is resistant to erosion and hence very stable over a long period. Measuring flows at lake outlets has the added advantage that rating shifts due to sediment movement are much reduced as lakes are sediment traps. Furthermore, reservoirs possess considerable heat storage and their outlets are thus less susceptible to significant ice build-up during cold periods due to the thermal energy of the water leaving the reservoir. Another factor favouring a reservoir outlet as a site for stage gauging is that hysteresis due to

(1), other models exist and it is appropriate to list some of them. Herschy (1995) describes one where the discharge is represented as an m th-order polynomial function of the stage. Westphal *et al.* (1999) applied this method to stage-discharge data from gauging stations in the Mississippi River in the form of a second-order polynomial. Other versions of the polynomial method arise in models where a Box–Cox transformation is applied to the discharge such that the transformed discharge is a first degree polynomial function of the stage (Gawne and Simonovic 1994). Freeman *et al.* (1996) used a model very similar to the polynomial approach. They assumed that the stage is a sixth-order root polynomial of the discharge to produce the best fit line for stage-discharge data from 116 gauging stations located throughout the United States. Pinter and Heine (2005) fitted rating curves to stage-discharge data from stations in the Mississippi River using a model where the stage is a second degree polynomial of the log of the discharge. Recently, different approaches based on artificial neural networks have been applied for setting up rating curves (e.g. Jain and Chalisgaonkar 2000; Deka and Chandramouli 2003). Ratings in some countries are prepared on natural scale graph paper. Typically, one carefully draws a curve through the scatter of available stage-discharge measurements. This curve is subsequently used to derive and store estimated stage-discharge points in a computer-based archive or in the form of written rating tables.

It is difficult to find literature on the general physical background for the parameters p and q in Equation (1). One source is Henderson (1963) who showed that for steady uniform flow in a wide power-law channel, i.e. where the width of the channel can be well described by a single power function of the stage (see Stelkoff and Clemmens (2000) for details), p is related to the slope, the channel friction and the scale and shape of the channel, whereas q is related to both the friction law used and the shape of the channel. For example, as noted by Mosley and McKerchar (1993), if Manning's friction law is valid, then $q = 5/3$, $q = 13/6$ and $q = 8/3$ correspond to rectangular, parabolic and triangular channel shapes, respectively. However, if the flow is uniform and steady, and the channel is not sufficiently power-law and/or wide, the interpretation of p and q becomes unclear. More importantly for this study, it is uncertain whether Equation (1) yields an adequate stage-discharge model in the CFR situation.

Practical use of the model in Equation (1) often neglects all *a priori* knowledge about the geometric and hydraulic characteristics of the hydraulic control and relies on the stage-discharge measurements carrying sufficient information. Exceptions are: (a) the parameter h_o (the reference gauge height at zero flow) which can sometimes be ascertained in field surveys if the hydraulic control is stable and well defined; and (b) assigning a value to q using information on the main channel shape. However, if the channel is very irregular and deep and the wrong friction law is assumed, the latter action may be grossly misleading. It is known from open-channel hydraulics that in most flow situations the geometric characteristics of the hydraulic control to a large extent govern the stage-discharge relationship. Intuitively, providing both deterministic information on the geometry and the information in the stage-discharge measurements for an objective statistical assessment may give a more accurate stage-discharge relationship estimate, especially in the extrapolated area.

Extreme floods, and also low-flows, are often beyond the limits of the established stage-discharge relationship (Potter and Walker 1985; Parodi and Ferraris 2004). Typically this is due to the standard measurement techniques (current meter, ADCP, dilution, etc.) becoming impractical and/or dangerous to use in extreme flow situations. In addition, the hydrometric office responsible may not be well enough prepared or have access to accurate forecasts for catching extreme flow situations. It is readily evident that estimating beyond the established stage-discharge relationship can give rise to large errors if it is performed without hydraulic justification. Not surprisingly, it is generally acknowledged that extrapolation should be avoided as far as possible. Where it is desirable to extrapolate, the

application of indirect methods based on the physical conditions of the actual channel and hydraulic control is recommended. However, practical use of indirect methods typically relies on subjective information on parameters such as channel roughness instead of objective calibration using the available stage-discharge measurements. Objective and probably more accurate methods such as those found in Wasantha Lal (1995) and Ramesh *et al.* (2000) may be more appropriate. Another example on objective calibration is found in Leonard *et al.* (2000), who proved for one dataset from a river in France that upwards extrapolation using geometric knowledge of the control and objective calibration based on least squares yielded far more accurate prediction results than raw extrapolation using Equation (1) in a uniform flow situation. It is intuitive that this will also be the case in CFR situations.

This study primarily focuses on developing a hydraulic motivated stage-discharge model for CFR which applies known information about the geometry of the outlet and can be statistically and objectively calibrated by the stage-discharge measurements available. The first section develops the nonlinear CFR least squares regression model, and also relates both Equation (1) and uniform flow from a reservoir to the proposed model. The following section is mainly concerned with handling the geometric data and how to obtain the nonlinear least squares estimates while factoring in approximated uncertainty. In the last section, the proposed model is applied to data from two Norwegian gauging stations and compared to the conventional rating curve model of Equation (1).

A stage-discharge regression model based on CFR

Simplifying assumptions

The following simplifying hydraulic assumptions are introduced for a reservoir containing a gauging station, such as sketched in Figure 1: (i) the discharge flows from the reservoir into a steep and well defined natural channel at a point T ; (ii) the flow at T is critical for all discharges, i.e. the channel slope S_0 is always greater than the water surface slope S_T ; (iii) the point T acts as the hydraulic control for all discharges; (iv) the outlet is steady state; (v) the entrance loss coefficient k_c is constant for all discharges; (vi) the energy coefficient α which allows for non-uniform velocity distribution in T is constant for all discharges; (vii) the velocity head in the reservoir is negligible.

Developing the CFR stage-discharge regression model

The energy equation valid for Figure 1, following the notation in Jain (2001, chap. 4), yields

$$h = y_c + (1 + k_c) \frac{Q^2}{2gA^2(y_c - h_o)} \quad (2)$$

where h_o is the known point of zero flow on the reference gauge, $y_c - h_o$ the critical flow depth in the control section, $h - h_o$ the hydraulic head related to the gauging site in the reservoir and $A(\cdot)$ the flow area in control section T . The flow condition at T is governed by the well known critical flow equation, i.e.

$$\frac{Q^2}{gA^2(y_c - h_o)} = \frac{A(y_c - h_o) \cos \theta}{\alpha B(y_c - h_o)} \quad (3)$$

where θ is the angle of the channel and $B(\cdot)$ the flow-width in the control section. Furthermore, assuming that k_c , α and θ are unknown for all discharges, combining Equations (2) and (3) gives

$$h - y_c - \beta \frac{A(y_c - h_o)}{B(y_c - h_o)} = h - y_c - \beta B(y_c - h_o) = 0, \quad \beta = \frac{(1 + k_c) \cos \theta}{2\alpha} \quad (4)$$

Assuming that A and B are known continuous functions of y_c , and that $\partial H(\xi)/\partial \xi > 0$ for all ξ , which implies that the right-hand side of Equation (2) is monotonically increasing and a one-to-one relationship between H and y_c , i.e. Equation (4) has only one solution given the parameter β and the stage h , one gets

$$y_c - h_o = F(h; \beta) \quad (5)$$

Normally F in Equation (5) must be represented by a numerical scheme, e.g. a Newton–Raphson iteration procedure. Combining Equations (5) and (3) yields

$$Q = \omega \sqrt{\frac{A^3(F(h; \beta))}{B(F(h; \beta))}} = \omega G(H; \beta), \quad \omega = \sqrt{\frac{g \cos \theta}{\alpha}} \quad (6)$$

when the dependence on the known constant h_o is suppressed.

Hence, having n stage-discharge measurements available, and assuming log-normally distributed multiplicative measurement errors, one gets the nonlinear regression model

$$Q_i = \omega G(h_i; \beta)(1 + \varepsilon_i), \quad \log(\varepsilon_i) \sim N(0, \sigma^2) \quad (7)$$

Since, typically $|\varepsilon_i| \ll 1$ (Carter and Anderson 1963; Day 1975; Herschy 1999), Equation (7) can be written as

$$q_i = \log(\omega) + \log[G(h_i; \beta)] + \varepsilon_i \quad (8)$$

where $q_i = \log(Q_i)$. The corresponding projected residual sum of squares function (Golub and Pererya 2003) valid for Equation (8) is

$$s^2 = \sum_{i=1}^n \{q_i - W(h_i; \beta) - \log[G(h_i; \beta)]\}^2 \quad (9)$$

where $W(h_i; \beta)$ is the linear least squares (projected) solution of $\log(\hat{\omega})$ as a function of the nonlinear parameters c and β and the observed covariates, i.e.

$$W(h_i; \beta) = \frac{1}{n} \sum_{i=1}^n \{q_i - \log[G(h_i; \beta)]\} \quad (10)$$

Hence, the parameter $\hat{\beta}$ which minimises Equation (9) and the subsequently calculated $\hat{\omega}$ are the least squares estimated critical depth flow stage-discharge model parameters.

Generalisation to normal depth situation

If the slope of the outlet is so-called mild, i.e. $S_0 < S_T$, then the flow in point T is normal and uniform and Equation (3) must be replaced by a uniform flow equation. Equation (2) will, however, still be valid, except that the subscript c should be replaced by n in order to emphasise that the flow is normal in T . If the Chézy equation is applied, one gets

$$Q = CS_o^{1/2} \sqrt{\frac{A^3(y_n - h_o)}{P(y_n - h_o)}} \quad (11)$$

where P is the wetted-perimeter at point T and C is the Chézy coefficient of resistance. If the channel is wide, i.e. $B \gg (y_n - h_o)$ for all $(y_n - h_o)$, then P can be approximated by the width. Hence, Equation (11) becomes

$$Q \approx CS_o^{1/2} \sqrt{\frac{A^3(y_n - h_o)}{B(y_n - h_o)}} \quad (12)$$

implying that the CFR model is applicable for estimating the discharge for a known h . However, in this situation, the parameters are interpreted as

$$\beta = \frac{C^2 S_o (1 + k_n)}{2g} \quad (13a)$$

$$\omega = C S_o^{1/2}. \quad (13b)$$

Thus, the CFR stage-discharge regression model also includes situations where the flow out of the reservoir is normal and uniform from a wide outlet.

Connection to the classical power-law rating curve model

It is easy to show that Equations (2) and (3) represent a generalisation of the classical power-law rating curve model. If one assumes that the geometry of the control section is a so-called power law

$$B = a(y_c - h_o)^b \text{ implying } A = \frac{a}{b+1} (y_c - h_o)^{b+1} \quad (14)$$

where a and b are geometric parameters, one gets from Equation (4)

$$h = y_c + \beta \frac{y_c - h_o}{b+1} \text{ implying } y_c - h_o = \left(\frac{b+1}{\beta + b + 1} \right) (h - h_o) \quad (15)$$

Combining Equations (15) and (3) yields

$$Q = \omega \sqrt{\frac{A^3 (y_c - h_o)}{B (y_c - h_o)}} = \omega \sqrt{\frac{a^2 (b+1)^{2b}}{(\beta + b + 1)^{2b+3}} (h - h_o)^{2b+3}} = p (h - h_o)^q \quad (16)$$

Thus, the CFR stage-discharge model is a generalisation of the classical power-law rating curve model for CFR situations, including when the flow in T is uniform and normal and the outlet is wide.

Estimation

Estimating the geometric functions

The CFR model requires information about the geometry of the outlet. This means that the wetted perimeter of the outlet must be surveyed in accordance with the same datum used as a reference for the stage-discharge measurements. The latter is of some importance. If the wetted perimeter is inaccurately related to the reference datum, the model estimates can become significantly biased due to the lack of a location parameter to account for the datum error (see the appendix for more details on this issue).

A natural channel is typically highly irregular and it is impossible to pinpoint each and every inflection of the wetted perimeter. It is also doubtful whether an extremely rigorous survey would result in any gain in the accuracy of the final discharge estimates. Moreover, if G becomes an erratic and unsmooth function of the critical depth, locating the minimum of Equation (9) by a numerical searching procedure becomes unfeasible. However, a balance must be set, and a minimum requirement is that the main characteristics of the outlet are properly described.

The processing and formatting of the field data is essential for obtaining successful stage-discharge estimates. The collected geometric data are discrete, while the CFR model requires continuous representations of the critical flow area and width. Hence, an interpolation scheme for describing the wetted perimeter is needed. This study applies cubic spline fitting (de Boor 2001) for this purpose, of which the numbers of knots are chosen based on subjective visual analysis. Another method than splines can certainly be applied

(e.g. piecewise broken-line interpolation), as long as the main shapes are properly described. Once the interpolation is performed, calculating A and B is a simple programming task.

Intuitively, it is beneficial to finally represent H and also G by continuous functions in order to make the subsequent numerical evaluations of s^2 easier, provided that the main geometric features of the outlet are kept and no extrapolation is performed. In addition, H must be prepared in accordance with the restraint given by Equation (4), which is that only one critical depth exists for a given h . Hence, the function H may need to be approximated in accordance with this constraint, e.g. by smooth piecewise polynomial interpolation such as used in this study, in order to represent it as analytical and monotonically increasing. Clearly, if the flow situation for a given CFR situation admits several, say i , roots in Equation (4), the above-mentioned simplifying approach is appropriate and accurate only if all the i roots are relatively close to each other. If they are not, the situation becomes unclear.

The procedure used in this study to represent the geometric functions can be summed up as follows: (i) fit a cubic spline to the geometric data of the outlet; (ii) use the fitted spline function to calculate the width B and area A for a suitable range of critical depths with 0.02 m increments in order to obtain k pairs of $(y_c^{(j)} - h_o, H(y_c^{(j)}))$ and $(y_c^{(j)} - h_o, G(y_c^{(j)}))$; (iii) fit the k pairs of $(y_c^{(j)} - h_o, H(y_c^{(j)}))$ to a smooth piecewise polynomial function; (iv) fit the k pairs of $(y_c^{(j)} - h_o, G(y_c^{(j)}))$ to a continuous piecewise polynomial function.

Obtaining the non-linear least square estimate

Once the deterministic functions H and G are established, the right-hand side of Equation (9) must be minimised by numerical methods. Fortunately, since the projected s^2 takes values in a one-dimensional space, the RSS function can be evaluated visually in the domain of interest in order to: (1) assess whether a least square estimate actually exists; (2) spot any suboptimal minima that may trap the minimisation routine applied; and (3) obtain a proper starting value if an iterative scheme is applied for minimisation. There might be cases where Equation (9) admits no finite least squares solution. In general, this question is extremely difficult to answer theoretically (Mäkeläinen *et al.* 1981), and this study makes no attempt to do so since the problem has not been encountered by the author. The question is only of theoretical interest since it can be properly answered by a thorough visual inspection of the RSS function before launching a numerical minimisation scheme.

The problem of spurious minima can be tackled in the same manner. Demidenko (2000) demonstrated that there is a positive probability for a given intrinsically nonlinear regression with infinite tails to have more than one minimum. This result certainly holds true for the CFR regression model. Preliminary simulation experiments using plausible outlet shapes have shown the author that Equation (9) often admits one, occasionally two, suboptimal solutions. Figure 2 illustrates an example of one spurious minimum taken from one of the practical modelling cases presented later on in the study. Also note the presence of a local maximum, which can also trap some unfortunately chosen minimisation methods (e.g. routines based on solving the normal equations). Why the RSS function occasionally assumes a multimodal form will not be investigated more closely in this study, although the fact that compound channels can permit critical flow at several depths (Sturm and Sadiq 1996) appears to be the most likely reason.

This study uses the BFGS quasi-Newton routine (see Seber and Wild (1989) for references) for minimising Equation (9). Comprehensive testing prior to writing this study showed that this is an adequate approach if proper starting values are supplied. The scheme used in this study for obtaining the least square estimate is as follows: (i) after a visual inspection of s^2 , supply a starting value $\beta^{(k)}$; (ii) solve the n equations $h_i - y_c - \beta^{(k)} \frac{A(y_c - h_o)}{B(y_c - h_o)}$ by standard Newton–Raphson iterations in order to obtain the corresponding $\mathbf{y}^{(k)} = (y_{c,1}^{(k)}, y_{c,2}^{(k)}, \dots, y_{c,n}^{(k)})$; (iii) calculate Equation (10) using $\mathbf{y}^{(k)}$;

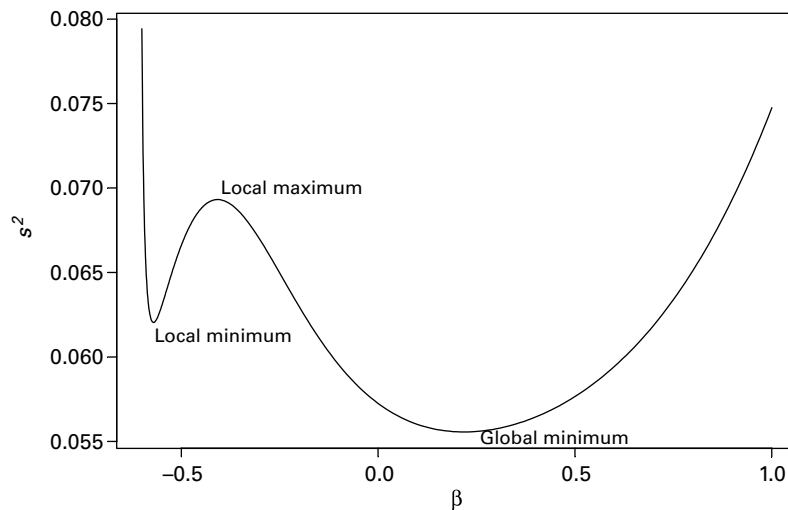


Figure 2 The projected residual sum of squares function when only the upper part of the stage-discharge measurements for Rovatn is used for estimation

(iv) calculate the Newton step $z_k = -\hat{H}(\beta^{(k)})\nabla s^2(\beta^{(k)})$, where $\hat{H}(\cdot)$ is an estimate of the Hessian matrix particular to the BFGS method and $\nabla s^2(\cdot)$ the gradient of s^2 evaluated at $\beta^{(k)}$ calculated by finite differences; (v) testing for convergence by checking whether $|s^2(\beta^{(k)} + z_k) - s^2(\beta^{(k)})| < 4 \cdot 10^{-11}$ and/or $|z_k| < 1 \cdot 5^{-08}$; (vi) if (v) is positive, stop and check by visual inspection of s^2 that the correct minimum is found; (vii) if (v) is negative, put $\beta^{(k+1)} = \beta^{(k)} + s^{(k)}$ and go to (ii). The statistical package S-PLUS (Venables and Ripley, 1997) was used to perform (iv) and (v). Typically, five to ten iterations are sufficient.

The above-mentioned scheme has been tested in situations where G is composed of several segmented polynomials, such that G is continuous but not smooth (second derivative does not exist at the knots). Even in these cases, which imply a continuous but unsmooth RSS function, the BFGS algorithm is able to locate the minimum without difficulty.

Inference and model comparison

At first sight, likelihood-based inference is available since normally distributed measurement errors are assumed. Unfortunately, the occurrence of compatible local maxima (corresponding directly to the least square minima) weakens the appropriateness of such an approach. According to Mäkeläinen *et al.* (1981), summarisation of data using a maximum likelihood estimate and its asymptotic variance could be meaningless when compatible maxima are present. Hence, in order to avoid misleading inference, re-sampling methods must be applied. This is not straightforward if one uses a local minimiser such as in this study. For each re-sample, one must ensure that the global minimum is found by visual inspection. This is a necessary but time-consuming approach. Consequently, applying bootstrap methods which require a large number of re-samples is extremely cumbersome. This study therefore applies the simpler, but still valid, jackknife estimator (Gray and Schucany 1972) for inference for the estimated model parameters and predicted discharges related to the highest, the mean and the lowest stages of the stage-discharge measurements. These predicted discharges are denoted as \hat{Q}_H^+ , \hat{Q}_M^+ and \hat{Q}_L^+ respectively.

An important aim of this study is to compare the CFR model to the classical power-law model. This study uses the well known Akaike information criterion (AIC) (Akaike 1974) for this purpose. The model which has the lowest AIC is assumed to be the most appropriate

working model. For comparing the performance in downwards and upwards extrapolation, measurements containing the stages higher/lower than the mean of the measured stages are applied for estimation.

Practical modelling

Data

Geometric data and stage-discharge measurements from two Norwegian gauging stations were used in order to explore the feasibility of the CFR rating curve model and to compare it with the conventional power-law rating curve model. Some details of the two stations are shown in [Table 1](#).

The gauging site of Rovatn is located in a lake of 8.1 km², approximately 1 km from the outlet which forms a hydraulic control characterised by large to smaller boulders. The control is assumed to have been stable throughout the period when the stage-discharge measurements were taken.

The Havelandselv site is located in a pool 10 m upstream of an outlet comprised of a smooth and stable sill of Archean bedrock. The pool is approximately 20 × 20 m across, with an average depth of about 2 m. Although the water approaches the pool via a river bend which slows down the water speed significantly, and the pool itself acts as an energy dispenser, it is believed there may be a significant velocity head when discharge is high.

The field surveys for obtaining the geometric data of the outlets were carried out during low flow situations using standard levelling equipment (measuring tape, rod and binoculars). Soundings were done at a sufficient number of verticals on the outlet to describe its main characteristics.

Numerical results with discussion

All data employed. The determinations of the height levels of the outlets and the cubic spline interpolations are shown in [Figure 3](#). Both controls are characterised by several smaller channels in the lower parts and a relatively abrupt transition over to the main channel perimeters where the cross-sections become almost rectangular in shape. [Figures 4\(a, c\)](#) show the relationship between the hydraulic radius H and the critical stage $y_c - h_o$ for Rovatn and Havelandselv, respectively; these were calculated using the cubic spline representations of the outlets. Both cases display a non-monotonically increasing critical stage- H relationship, with Havelandselv yielding by far the most complex one. In the case of Rovatn, a monotonically increasing and smooth polynomial interpolation fits the points relatively well. On the other hand, the H function for Havelandselv had to be rather crudely represented in order to make the geometry amenable to the CFR rating curve model. The calculated relationship between the G function and the critical stage $y_c - h_o$ is shown in [Figures 4\(b, d\)](#) for Rovatn and Havelandselv, respectively. Both cases exhibit noticeable shifts at approximately $y_c - h_o = 0.5$ m (Rovatn) and 0.8 m (Havelandselv). The piecewise polynomial interpolations of the G functions fit the calculated points well for both stations.

Using the approximated geometric functions, the CFR rating curve model was fitted to all the stage-discharge measurements available from the two stations. Additionally, the classical

Table 1 Details of gauging stations used in the practical modeling

Station	Area (km ²)	Regime	Number of measurements	Measurement methods
Rovatn	237	Regulated	54	Current meter, Flow Tracker*
Havelandselv	21	Natural	18	Current meter, Flow Tracker*

*A hydro-acoustic handheld meter

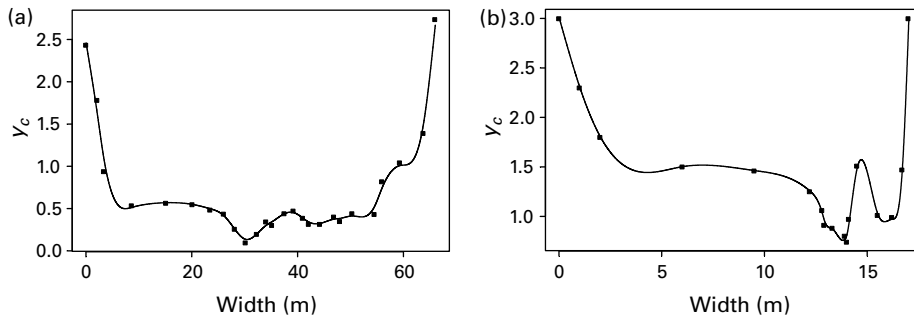


Figure 3 The levelling soundings for the cross-sections of Rovatn (a) and Havelandselv (b) along with the cubic spline representations

power-law rating curve model in Equation (1) was fitted to the datasets. Two different power-law model assumptions were applied: (1) known h_o ; and (2) unknown h_o . These models will be respectively referred to as the PL1 and PL2 models. The PL1 cases involved simple linear regression whilst non-linear regression methods were applied for the PL2 cases (see Reitan and Petersen-Øverleir (2005) for technical details). The numerical results are shown in Tables 2–4. In addition, the stage-discharge measurements and the estimated rating curves are shown in Figures 5(a) and 6(a).

It is readily evident from a visually inspection of Figure 5(a) that for the Rovatn data the fitted CFR rating curve yields a better overall fit than the two power-law models. Especially in the upper part ($h > 1.1$ m), the CFR model displays its superiority with a slight overestimation of less than 5%, whilst the PL1 and PL2 models overestimate by about 10% and 20%, respectively. The appropriateness of applying the CFR model to the Rovatn data is additionally reflected in Tables 2–4 where one sees that the CFR model has the lowest AIC in addition to the lowest variability of the predicted discharges.

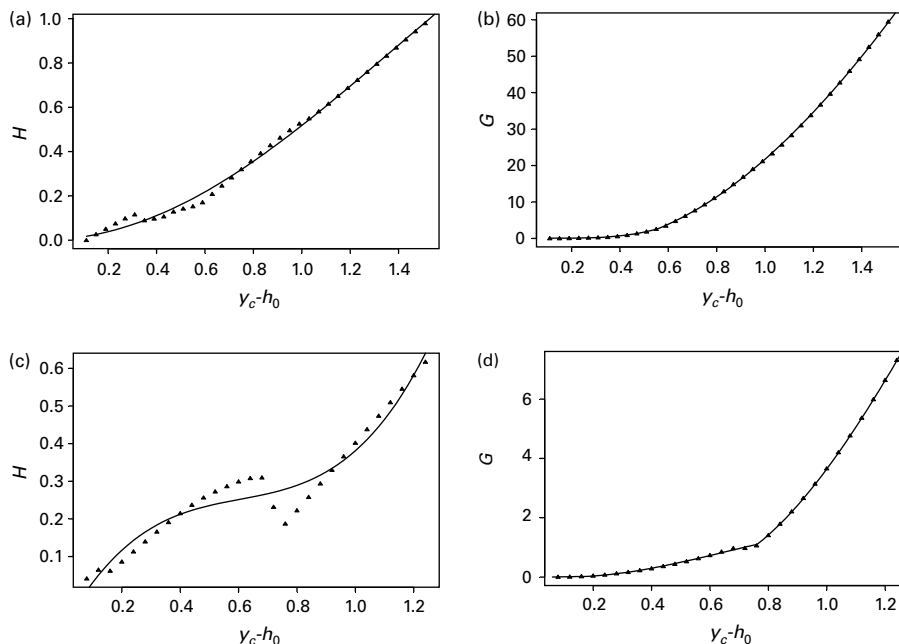


Figure 4 The calculated geometric functions (triangles) and their corresponding piecewise polynomial approximations for Rovatn (a, b) and Havelandselv (c, d)

Table 2 Estimated CFR rating curve parameters, AIC values and discharge predictions for Rovatn and Havelandselv

Station	Parameter estimates		AIC	Predicted discharges (m ³ /s)		
	$\hat{\beta}$	$\hat{\omega}$		\hat{Q}_L^+	\hat{Q}_M^+	\hat{Q}_H^+
Rovatn	-0.04 (0.13)	0.85 (0.17)	-48.8	0.43 (0.02)	14.82 (0.37)	48.91 (0.96)
Havelandselv	0.43 (0.15)	2.69 (0.39)	-3.8	0.06 (0.01)	1.38 (0.04)	7.80 (0.28)

Jackknife standard deviations are given in parantheses

Table 3 Estimated PL1 rating curve parameters, AIC values and discharge predictions for Rovatn and Havelandselv

Station	Parameter estimates		AIC	Predicted discharges (m ³ /s)		
	$\hat{\rho}$	\hat{q}		\hat{Q}_L^+	\hat{Q}_M^+	\hat{Q}_H^+
Rovatn	23.94 (0.54)	2.92 (0.08)	-19.0	0.56 (0.06)	12.94 (0.31)	58.32 (2.18)
Havelandselv	5.59 (0.34)	2.95 (0.16)	3.0	0.06 (0.01)	1.29 (0.06)	7.20 (0.53)

Jackknife standard deviations are given in parantheses

By considering [Figure 6\(a\)](#) one can conclude that both the CFR and the PL1 models fit the stage-discharge measurements from Havelandselv adequately, whilst the PL2 model is far off target for stages > 1.2 m. However, neither of the power-law models are capable of handling the phase shift occurring at $h = 1.62$ m. In order to do this, another segment would have to be applied to the power-law models, which would be highly questionable with only 18 measurements available. The CFR model also yields the lowest AIC and the smallest prediction variability for the Havelandselv data. Furthermore, one can see that the AIC favours the PL2 model over the PL1, as indicated by [Figure 6\(a\)](#) and the sizes of the jackknife standard deviations.

Hence, despite the ostensibly rough approximation for the hydraulic depth function, especially in the case of Havelandselv, the overall impression is of satisfactory practical application of the CFR model.

The estimated CFR model parameters merit a closer assessment. According to [Chow \(1959\)](#) and [Chaudhry \(1993\)](#) the typical value of the energy coefficient α in natural streams is between 1.15 and 1.50. [Chow \(1959\)](#) also states that upstream of weirs, values of α greater than 2 have been noted, and that a value of $\alpha = 3.87$ was obtained at the outlet section of a draft tube. Based on these figures, by considering Equation (6) one may assume that

Table 4 Estimated PL2 rating curve parameters, AIC values and discharge predictions for Rovatn and Havelandselv

Station	Parameter estimates			AIC	Predicted discharges (m ³ /s)		
	$\hat{\rho}$	\hat{q}	\hat{h}_o		\hat{Q}_L^+	\hat{Q}_M^+	\hat{Q}_H^+
Rovatn	33.32 (2.99)	2.32 (0.18)	0.22 (0.04)	-20.0	0.43 (0.10)	13.63 (0.36)	52.70 (1.84)
Havelandselv	6.47 (0.84)	2.09 (0.60)	0.86 (0.09)	-0.2	0.05 (0.02)	1.44 (0.12)	6.03 (0.90)

Jackknife standard deviations are given in parantheses

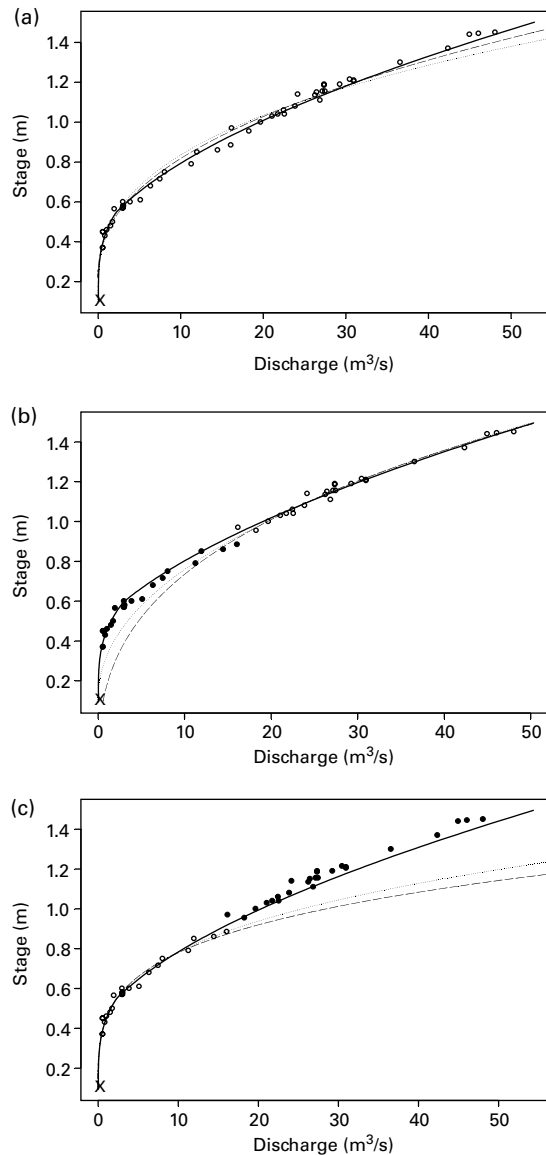


Figure 5 The estimated rating curves and stage-discharge measurements at Rovatn. CFR model (solid line), PL1 model (dotted line), PL2 model (dashed line), measurements used in estimation (transparent circles), measurements excluded from the estimation (filled circles) and point of zero flow (X). (a) All measurements used, (b) downwards extrapolation, and (c) upwards extrapolation

$\sqrt{\cos \theta} \leq \omega \leq 3\sqrt{\cos \theta}$ is a plausible range for this parameter in natural outlets. The slope angles θ for the Rovatn and Havelandselv outlet channels are approximately 1.5° and 15° , respectively. Hence, the estimate of θ for Rovatn appears somewhat low, but not significantly unrealistic if the standard deviation is taken into account. For Havelandselv the θ estimate seems reasonable. The coefficient of entrance loss k_c has, according to [Chow \(1959\)](#), an average value of 0.25 for well rounded outlets. It could then be plausible to assume that $0 \leq k_c \leq 0.5$ is a typical range of this quantity. Applying these values, by considering Equation (4) one can assume that $0.15 \cos \theta \leq \beta \leq 0.25 \cos \theta$ is a reasonable range for this parameter in conjunction with natural outlets. The estimate of θ for

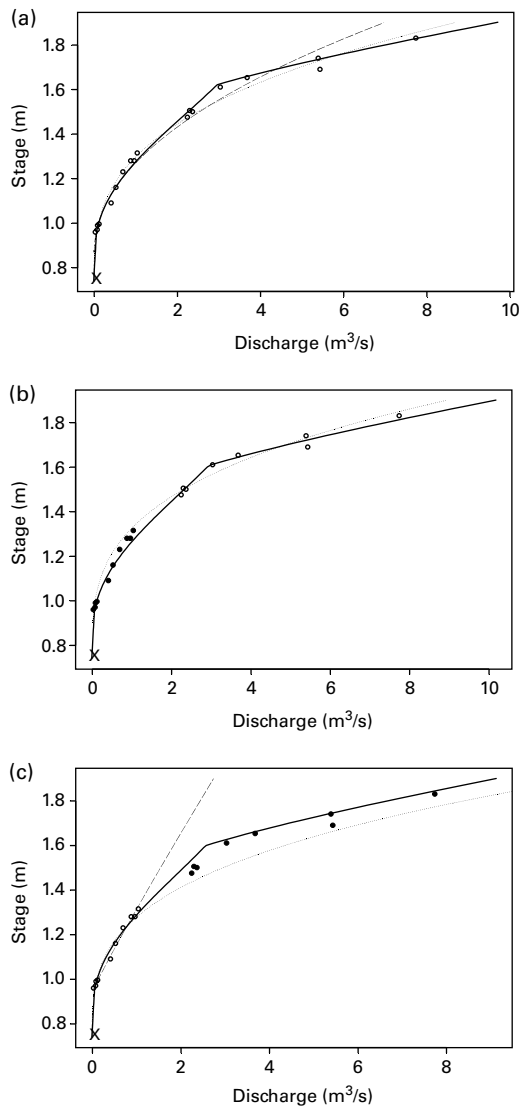


Figure 6 The estimated rating curves and stage-discharge measurements at Havelandelv. CFR model (solid line), PL1 model (dotted line), PL2 model (dashed line), measurements used in estimation (transparent circles), measurements excluded from the estimation (filled circles) and point of zero flow (X). (a) All measurements used, (b) downwards extrapolation, and (c) upwards extrapolation

Havelandselv is somewhat high, but not unrealistic when factoring in the approximated uncertainty. In the case of Rovatn, the estimated θ is almost zero, but still negative, which is impossible for the CFR flow situations. However, by considering the large standard deviation, the estimate does not lie far outside the above-mentioned range. Besides model variability, another plausible reason for obtaining unrealistic values for β is that the outlet was not accurately measured in relation to the reference datum. At Rovatn, the datum is located about 1 km from the outlet, which gives considerable scope for misspecification. The appendix shows that this can lead to unrealistic and even negative estimates of β .

Since the estimate of β is almost zero for Rovatn, it could be argued that β could have been neglected in the CFR model in the case of Rovatn, which would imply a simpler model yielding smaller prediction variability and a less computer-intensive estimation procedure.

The implications of this reasoning are not pursued in this paper, but one notes that the CFR rating curve model may justifiably be simplified for some cases.

Extrapolation study. Using only those stage-discharge pairs with stages larger than the mean of the measured stages at Rovatn (see [Figure 5\(b\)](#)), the CFR model fits almost as well when all the measurements available are used. The PL1 model behaves slightly better than the PL2 when extrapolating downwards, which most likely is due to the anchoring at h_o . Still, both the power-law models miss by up to 30% in the extrapolated area.

One sees from [Figure 6\(b\)](#) that also in the case of Havelandselv the CFR model displays superior performance in the downwards extrapolation when only the upper part of the measurements is used for estimation. In this situation the performance of the PL1 model is equal to the CFR model, which again indicates that anchoring the power-law rating curve at the known point of zero flow is appropriate when downwards extrapolation of the rating curve is needed. The PL2 model failed completely due to a non-existing least square estimate for the upper part of the Havelandselv dataset. A comprehensive treatment of the problem of existence for the PL2 model can be found in [Reitan and Petersen-Øverleir \(2005\)](#).

Censoring the upper part of the measurements produced devastating results for both power-law models in the case of Rovatn. One sees from [Figure 5\(c\)](#) that both the PL1 and the PL2 models missed by several hundred percent for the highest part of the upwards extrapolation, whereas the CFR model overestimated just slightly.

Despite some underestimation, one sees from [Figure 6\(c\)](#) that the CFR model yielded a fairly accurate upwards extrapolation for Havelandselv. The PL1 model significantly overestimates, while the PL2 model fails completely in the upwards extrapolation.

Concluding remarks

This paper proposes a method for estimating stage-discharge relationships based on the hydraulic equations for critical flow from a reservoir (CFR). These hydraulic equations are fitted to spline approximations of the geometric characteristics of the reservoir outlet and the stage-discharge measurements available by non-linear least squares.

The work reported in this paper shows one important and indisputable fact: the classical power-law rating curve model is only a special version of the more general CFR rating curve model in situations of critical or normal flows from a reservoir. This makes the CFR model theoretically more appropriate for stage-discharge modelling in conjunction with gauging stations placed in reservoirs such as lakes or pools. The applications to data from two Norwegian gauging stations illustrated that in practice the CFR model is also superior to the power-law models when the performances are measured by statistical quantities and extrapolation accuracy. There are, however, some questions to be put to the application of the CFR model. Firstly, the CFR model requires extra field information before it can be applied, while the stage-discharge measurements alone will suffice for the power-law model. Secondly, the CFR model brings with it a more computer-intensive analysis than the traditional approach. This criticism can be answered as follows: data used for water resources assessment is, as [Rodda \(1999\)](#) puts it, “crucial to the future of humanity, particularly in support of moves towards the desired goal of sustainable development.” Hence, if bias and uncertainty in discharge data can be reduced, the cost of additional field surveys and increased computation time is a small price to pay.

There is plenty of scope for further study on the work presented in this paper. In particular: (a) as implied in the study, it is possible to derive prior information about the CFR model parameters from hydraulic theory and at-site characteristics. Hence, a Bayesian

approach to the non-linear CFR rating curve regression model may prove to be a method for obtaining sounder estimates; (b) the practical applications implied that one of the two parameters may justly be ignored in the CFR model. Consequently, the regression problem will then be reduced to a simple linear regression. This needs to be further investigated by applying the CFR model to a large number of gauging stations; (c) due to the presence of several minima in the projected residual sum of squares function related to the CFR model, algorithms designed to search out global minima, for example the method of simulated annealing (Kirkpatrick *et al.* 1983; Otten and van Ginneken 1989), may better cope with the minimisation problem such that bootstrap methods can be safely applied for model and prediction inference.

Acknowledgements

The author would like to acknowledge the reviewers for their advice and in particular to thank Dr. A. McKerchar at the National Institute of Water and Atmospheric Research Ltd., New Zealand, whose comments significantly enhanced the quality of the paper.

References

- Akaike, H. (1974). A new look at the statistical model identification. *IEEE Trans. Auto. Control*, **19**, 716–723.
- Carter, R.W. and Anderson, I.E. (1963). Accuracy of current meter measurements. *J. Hydrol. Div., ASCE*, **89**(4), 105–115.
- Chaudhry, M.H. (1993). *Open Channel Hydraulics*, Prentice Hall, Englewood Cliffs, NJ.
- Chow, V.T. (1959). *Open-Channel Hydraulics*, McGraw-Hill, New York.
- Day, T.J. (1975). On the precision of salt dilution gauging. *Wat. Res. Res.*, **31**, 293–306.
- de Boor, C. (2001). *A Practical Guide to Splines* (revised edn.), Springer, New York.
- Deka, P. and Chandramouli, V. (2003). A fuzzy neural network model for deriving the river stage-discharge relationship. *Hydrol. Sci. J.*, **48**(2), 197–209.
- Demidenko, E. (2000). Is this the least square estimate? *Biometrika*, **87**, 437–452.
- Fenton, J.D. and Keller, R.J. (2001). *The Calculation of Streamflow from Measurements of Stage*, Tech. Rep. 01/6. Cooperative Research Centre for Catchment Hydrology, Melbourne, Australia.
- Fread, D.L. (1975). Computation of stage-discharge relationships affected by unsteady flow. *Wat. Res. Bull.*, **11**, 213–218.
- Freeman, G.E., Copeland, R.R. and Cowan, M.A. (1996). Uncertainty in stage-discharge relationships. In K.S. Tickle, I.C. Goulter, C. Xu, S. Wasimi and F. Bouchart (eds), *Proc. 7th IAHR Int. Symp. on Stochastic Hydraulics, Mackay, Queensland, Australia, Stochastic Hydraulics'96*, Balkema, Rotterdam, pp. 601–608.
- Gawne, K.D. and Simonovic, S.P. (1994). A computer-based system for modelling the stage-discharge relationship in steady state conditions. *Hydrol. Sci. J.*, **39**(5), 487–506.
- Golub, G.H. and Pererya, V. (2003). Separable nonlinear least squares: the variable projection method and its applications. *Inverse Problems*, **19**, 1–26.
- Gray, H.L. and Schucany, W.R. (1972). *The Generalized Jackknife Statistic*, Marcel Dekker, New York.
- Henderson, F.M. (1963). Flood waves in prismatic channels. *J. Hydrol. Div., ASCE*, **89**, 39–67.
- Hersch, R.W. (1995). *Streamflow Measurement* (2nd edn.), E & FN Spon, London.
- Hersch, R.W. (1999). Uncertainties in hydrometric measurements. In R.W. Hersch (ed.), *Hydrometry – Principles and Practices* (2nd edn.), John Wiley & Sons, Chichester, 355–370.
- Hicks, F.E. and Healy, D. (2003). Determining winter discharge based on hydraulic modeling. *Can. J. Civ. Engng.*, **30**, 101–112.
- ISO (International Standards Organisations) (1998). *ISO 1100/2. Determination of The Stage-discharge Relation*, International Standards Organisations, Geneva.
- Jain, S.C. (2001). *Open-channel Flow*, John Wiley & Sons, New York.
- Jain, S.K. and Chalisgaonkar, D. (2000). Setting up Stage-Discharge using ANN. *J. Hydrol. Engng.*, **5**(4), 428–433.
- Jónsson, P., Petersen-Øverleir, A., Nilson, E., Edstöm, M., Iversen, H.L. and Sirvio, H. (2002). Methodological and personal uncertainties in the establishment of rating curves. In Å. Killingtveit (ed.), *Proc. XXII Nordic*

- Hydrological Conference, Nordic Hydrological Programme, NHP Report no. 47* (vol. 1), Tapir Trykkeri, Trondheim, pp. 35–44.
- Kirkpatrick, S., Gelatt, C.D. and Vecchi, M.P. (1983). Optimization by simulated annealing. *Science*, **220**, 671–680.
- Lambie, J.C. (1978). Measurement of flow-velocity area methods. In R.W. Herschy (ed.), *Hydrometry – Principles and Practices* (1st edn.), John Wiley & Sons, New York, 1–52.
- Leonard, J., Mietton, M., Najib, H. and Gourbesville, O. (2000). Rating curve modelling with Manning's equation to manage instability and improve extrapolation. *Hydrol. Sci. J.*, **45**(5), 739–750.
- Mäkeläinen, T., Schmidt, K. and Styan, G.P.H. (1981). On the existence and uniqueness of the maximum likelihood estimate of a vector-valued parameter in fixed-sample space. *Ann. Statist.*, **9**, 758–767.
- Mosley, M.P. and McKerchar, A.I. (1993). Streamflow. In D.R. Maidment (ed.), *Handbook of Hydrology*, McGraw-Hill, New York.
- Moyeed, R.T. and Clarke, R.T. (2005). The use of Bayesian methods for fitting rating curves, with case studies. *Adv. Wat. Res.*, **28**, 807–818.
- Otten, R.H.J.M. and van Ginneken, L.P.P. (1989). *The Annealing Algorithm*, Kluwer, Dordrecht.
- Parodi, U. and Ferraris, L. (2004). Influence of stage discharge relationship on the annual maximum discharge statistics. *Natural Hazards*, **31**, 603–611.
- Petersen-Øverleir, A. (2004). Accounting for heteroscedasticity in rating curve estimates. *J. Hydrol.*, **292**, 173–181.
- Petersen-Øverleir, A. (2005). Modelling stage-discharge relationships affected by hysteresis using Jones formula and non-linear regression. *Hydrol. Sci. J.*, **51**(3), 365–388.
- Petersen-Øverleir, A. and Reitan, T. (2005). Objective segmentation in compound rating curves. *J. Hydrol.*, **311**, 188–201.
- Pinter, N. and Heine, R.A. (2005). Hydrodynamic and morphodynamic response to river engineering documented by fixed-discharge analysis, lower Missouri River, USA. *J. Hydrol.*, **302**, 70–91.
- Potter, K.W. and Walker, J.F. (1985). An empirical study of flood measurement error. *Wat. Res. Res.*, **21**(3), 403–406.
- Powell, K.E.C. (1978). Weed growth—a factor of channel roughness. In R.W. Herschy (ed.), *Hydrometry – Principles and Practices* (1st edn.), John Wiley & Sons, New York, pp. 327–352.
- Ramesh, R., Datta, B., Bhallamudi, S.M. and Narayana, A. (2000). Optimal estimation of roughness in open channel flows. *J. Hydraul. Engng., ASCE*, **126**(4), 299–303.
- Rantz, S.E., et al. (1982). *Measurement and Computation of Streamflow. Computation of Discharge*, vol. 2, US Geological Survey Water Supply Paper 2175, US Geological Survey, Reston, VA, USA.
- Reitan, T., Petersen-Øverleir, A. (2005). Estimating the discharge rating curve by nonlinear regression – the frequentist approach. *Statistical Research Report*, Preprint 2. University of Oslo.
- Rodda, J.C. (1999). Measuring up to water resources assessment. In R.W. Herschy (Ed.), *Hydrometry – Principles and Practices* (2nd edn.), John Wiley & Sons, Chichester, pp. 143–160.
- Seber, G.A.F. and Wild, C.J. (1989). *Nonlinear Regression*, John Wiley & Sons, New York.
- Simons, D.B., Stevens, M.A. and Duke, J.H. (1973). Predicting stages on sand-bed rivers. *J. Watrwy Harb. Coast. Engng., ASCE*, **99**(2), 231–243.
- Stelkoff, T.S. and Clemmens, A.J. (2000). Approximating wetted perimeter in power-law cross sections. *J. Irrig. Drain. Engng., ASCE*, **126**(2), 98–109.
- Sturm, T.W. and Sadiq, A. (1996). Water surface profiles in compound channels with multiple critical depths. *J. Hydraul. Engng., ASCE*, **122**(12), 703–709.
- Tawfik, M., Ibrahim, A. and Fahmy, H. (1997). Hysteresis sensitive Neural Network for modelling rating curves. *J. Comp. Civil Engng., ASCE*, **11**, 206–211.
- Tuthill, A.M., Wuebben, J.L., Daly, S.F. and White, K.D. (1996). Probability distributions for peak stage on rivers affected by ice jams. *J. Cold Region Engng., ASCE*, **10**(1), 36–57.
- Venables, W.N. and Ripley, B.D. (1997). *Modern Applied Statistics with S-PLUS* (2nd edn), Springer Verlag, New York.
- Venetis, C. (1970). A note on the estimation of the parameters in logarithmic stage-discharge relationships with estimation of their error. *Bull. IAHS*, **15**(2), 105–111.
- Wasantha Lal, A.M. (1995). Calibration of riverbed roughness. *J. Hydraul. Engng., ASCE*, **121**(9), 664–671.
- Westphal, J.A., Thompson, D.B., Stevens, G.T. and Strauser, C.N. (1999). Stage-discharge relations on the middle Mississippi River. *J. Wat. Res. Plann. Mngmnt, ASCE*, **121**(1), 48–53.

WMO (World Meteorological Organization) (1994). *Guide to Hydrological Practices* (2nd edn), WMO-No.168, World Meteorological Organization, Geneva.

Zarzer, E.A. (1987). Some considerations concerning the optimal calculation of stage-discharge functions. *Math. Meth. Oper. Res (ZOR)*, **31**(6), 193–212.

Appendix

The purpose of the following analysis is to exemplify the impact of referring the outlet to another datum than the stage-discharge measurements. First, in order to make the problem amenable to simple and illustrative analytical assessment, we assume that the outlet is rectangular with a width $B = a$. According to Equation (16), one gets

$$Q = \exp(\alpha)(h - h_0)^{3/2} \quad (\text{A1})$$

where $\exp(\alpha) = \omega a / (1 + \beta)^{3/2}$. For a given dataset with n pairs of stage-discharge measurements, and applying the same model assumptions as in the second section, the RSS valid for Equation (A1) becomes

$$s^2 = \sum_{i=1}^n \left[q_i - \alpha - \frac{3}{2} \log(h_i - h_0) \right]^2 \quad (\text{A2})$$

The least squares estimate of α is thus obtained by

$$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^n \left[q_i - \frac{3}{2} \log(h_i - h_0) \right] \quad (\text{A3})$$

Now, assume that the datum determination related to the stage measurement in the reservoir is erroneous by e m compared to the datum determination of the wetted perimeter of the outlet (see Figure 1). If this error is not accounted for, the least squares estimate of α will be biased. By considering the definition of α in conjunction with Equation (A3), one gets

$$\hat{\beta}_e = \exp \left[-\frac{1}{n} \sum_{i=1}^n \log \left(1 + \frac{e}{h_i - h_0} \right) \right] (1 + \hat{\beta}) - 1 \quad (\text{A4})$$

where $\hat{\beta}_e$ and $\hat{\beta}$ are the biased and the unbiased least squares estimate of β respectively. It is readily evident that the bias due to e can become significant if many of the stage values are small. If additionally $\hat{\beta}$ is relatively small, the biased estimate $\hat{\beta}_e$ may become negative. Although the theory leading to Equation (A4) yields absolutely correct if the outlet is rectangular by shape, simulation experiments indicate that other and more complex shapes are equally affected by a significant and unaccounted e .