arrives at the determination of greatest excursion in transient condition in function of characteristic data of system.

The functions-answer introduced in analytical development is studied by the author graphically, or with response curves.

Author's Closure

The author is grateful to Professor Tessarotto for bringing the problem so well into focus, and wishes to thank Dr. Capello for his interesting extension of the results of the paper to the problem of transients in rotating shafts. It would certainly be of great interest to further pursue the matter taking into account the internal damping of the material also in order to determine to what degree the results of the author's recent study[5] on the transient behavior of damped simple oscillators can be useful here.

Natural Frequencies of Two Nonlinear Systems Compared With the Pendulum[1]

R. W. SHREEVES.[2] The solution to Equation (1) of this interesting note may be found in terms of known functions, thus eliminating the need for a digital computer.[3]

Let

$$C_1 - 2C_2 = I/M\,a^2,\quad C_2 = R(R-a),\quad C_3 = g(R-a),\quad C_4 = C_1/2C_2$$

Substitute these relations into Equation (1) and let $p = \dot{\theta}$. This yields

$$(C_3/C_2)^{1/2}\tau/4 = \int_0^{\theta_0} \frac{d\theta}{\left(\dfrac{\cos\theta - \cos\theta_0}{C_4 - \cos\theta}\right)^{1/2}}$$

In this equation, let $u = \tan(\theta/2)$. The resulting expression immediately suggests the presence of elliptic integrals. With the aid of some algebraic manipulation and the substitution $u = \tan^2(\theta_0/2)\cos\phi$, the following expression for the square of the period is readily obtained:

$$\tau^2 = \frac{64C_2}{C_3(C_4 - 1)(1 + \cos\theta_0)(1 + p^2)}\{(1 + C_4)K(k) - (1 + \cos\theta_0)\Pi(\nu, k)\}^2$$

where K and Π are complete elliptic functions of the first and third kinds, respectively;

$$p^2 = \tan^2(\theta_0/2)/(C_4^2 - 1),\quad k^2 = [\tan^2(\theta_0/2)]/[\tan^2(\theta_0/2) + (C_4^2 - 1)],$$

and

$$\nu = (\cos\theta_0 - 1)/2$$

For convenience, the complete elliptic function of the third kind in the foregoing may be expressed in terms of elliptic functions of the first and second kinds.[4] The values of these functions are easily obtained.[5]

Therefore,

$$\tau^2 = \frac{64C_2}{C_3(C_4 - 1)(1 + \cos\theta_0)(1 + p^2)}\left\{(C_4 - \cos\theta_0)K(k) \left(\frac{2(1 - \cos^2\theta_0)}{(2k^2 - (1 - \cos\theta_0)}\right)^{1/2} [K(k)E(\alpha, k) - E(k)F(\alpha, k)]\right\}^2$$

where F and E are incomplete elliptic functions of the first and second kinds, respectively, and α varies from 0 to arc sin $\left(\dfrac{1}{k}\sin\dfrac{\theta_0}{2}\right)$.

K. KLOTTER.[6] E. W. Gaylord[1] recently treated the free oscillations of two systems, identified as "cylindrical profile rocker" and "bar rocking on a cylindrical surface," respectively. After having established the differential equations of the two systems, the author shows in a diagram (his Fig. 3) the amplitude-frequency relationships which he obtained by integrating the differential equations numerically (applying the Runge-Kutta method and using a digital computer). The results are given for one specific value of the parameter β of the first system and three specific values of the parameter λ of the second system.

Two remarks may be added to that note:

1 It seems worth mentioning that the differential equations allow exact integration. Using the author's nomenclature and (corrected) abbreviations,

$$\beta = \frac{R(R-a)}{k^2 + a^2}\quad,\quad \Omega^2 = \frac{g(R-a)}{k^2 + a^2} \qquad (1a)$$

and

$$\lambda = \frac{a^2}{l^2/12}\quad,\quad \Omega^2 = \frac{ga}{l^2/12} \qquad (1b)$$

respectively, we may write the differential equations as

$$\ddot{\theta}[1 + 2\beta(1 - \cos\theta)] + (\beta\dot{\theta}^2 + \Omega^2)\sin\theta = 0 \qquad (2a)$$

$$(1 + \lambda\theta^2)\ddot{\theta} + (\lambda\dot{\theta}^2 + \Omega^2\cos\theta)\theta = 0. \qquad (2b)$$

They can be transformed into the following first-order linear differential equations for $V = \dot{\theta}^2$ as a function of θ:

$$[1 + 2\beta(1 - \cos\theta)]V' + 2\beta\sin\theta V = -2\Omega^2\sin\theta \qquad (3a)$$

$$[1 + \lambda\theta^2]V' + 2\lambda\theta V = -2\Omega^2\theta\cos\theta. \qquad (3b)$$

The first step of integration can be carried out immediately; in the second step the variables can be separated, and the q-t relationships appear in the form of quadratures.

2 A rather accurate approximate amplitude-frequency relationship which contains the parameters in general form, not specified, can be obtained by applying the Ritz-Galerkin procedure[7] to Equations (2). Assuming

$$\theta = Q\cos\omega t \qquad (4)$$

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[5] A. Dornig, "Transitorio d'avviamento negli oscillatori semplici smorzati soggetti a perturbazione a caratteristica inerziale," *L'Energia Elettrica*, June, 1959.

[1] By Eber W. Gaylord, published in the March, 1959, issue of the Journal of Applied Mechanics, vol. 26, Trans. ASME, series E, vol. 81, pp. 145–146.

[2] Senior Scientist, AVCO Research and Advanced Development Division, Wilmington, Mass.

[3] R. W. Shreeves, "A Trochoidal 'Oscillator,'" *Avco RAD*, 1959.

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, "Higher Transcendental Functions," McGraw-Hill Book Company, Inc., New York, N. Y., 1955, p. 321.

[5] B. O. Peirce, "A Short Table of Integrals," Ginn and Company, New York, N. Y., 1929, pp. 121–123.

[6] Professor of Engineering Mechanics, Stanford University, Stanford, Calif.

[7] K. Klotter, "Nonlinear Vibration Problems Treated by the Averaging Method of W. Ritz," Proceedings of the First U. S. National Congress of Applied Mechanics, ASME, 1952, pp. 125–131.

and replacing the trigonometric functions in (2) by their power-series expansions, retaining terms up to the third order only, we find for the amplitude-frequency relationship

$$\eta^2 = \frac{1 - {}^1/_8 Q^2}{1 + {}^1/_2 \beta Q^2} \qquad (5a)$$

$$\eta^2 = \frac{1 - {}^3/_8 Q^2}{1 + {}^1/_2 \lambda Q^2} \qquad (5b)$$

respectively; η denotes the ratio ω/Ω. The result for the pendulum, to the same degree of approximation, reads

$$\eta^2 = 1 - {}^1/_8 Q^2; \text{ with } \Omega^2 = g/l. \qquad (5c)$$

Numerical values from Equations (5) agree very closely with the author's results for his specific values of the parameters. Equations (5), however, allow comparing the three systems for any other values of the parameters β and λ, without the need for subsequent integration.

The accuracy of the procedure may readily be increased, if need should be, by (*a*) keeping more terms in the expansions mentioned, and/or (*b*) replacing the approximation (4) either by one containing higher (odd) harmonics, or by an appropriate non-sinusoidal function.[8]

[8] K. Klotter and Phil R. Cobb, "The Use of Nonsinusoidal Approximating Functions for Nonlinear Oscillation Problems," to appear in the JOURNAL OF APPLIED MECHANICS.

Collapse Loads of Rings and Flanges Under Uniform Twisting Moment and Radial Force[1]

N. C. DAHL.[2] The author uses conventional plate-and-ring theory and this implies certain practical restrictions on the ratio of ring thickness to tube radius. The upper limit of this ratio certainly is of the order of $^1/_5$ and thus the upper limit of h/a which is practically of interest is about 0.1. Thus the area of Fig. 13 of the paper, which is of interest, is that region below $h/a = 0.1$. For all practical purposes the restriction on b/a may then be said to be independent of h/a with the result shown in the accompanying Fig. 1, where any combination of α and b/a in the shaded region is consistent with a statically admissible stress field. Consistent with the approximation represented by Fig. 1, it may also be stated that any value of h/a (which is less than 0.1) will result in a statically admissible stress field.

[1] By Burton Paul, published in the June, 1959, issue of the JOURNAL OF APPLIED MECHANICS, vol. 26, TRANS. ASME, vol. 81, series E, pp. 265–270.

[2] Professor of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Mass.

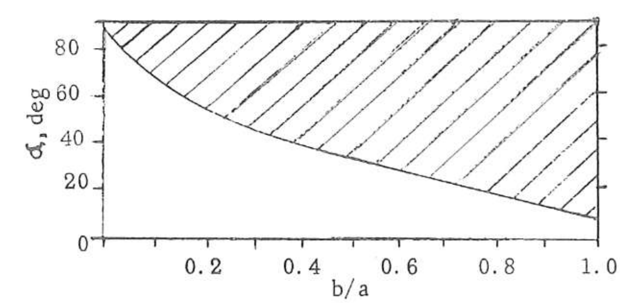

Fig. 1

Author's Closure

The problem of the axially compressed corrugated tube was treated merely to illustrate how a fairly complex problem can be simplified and solved *within the framework* of the given theory of rings. One of the stated assumptions of the theory is that the rings in question have a "small" ratio of depth to diameter. However, no attempt was made to delimit the meaning of the word "small." Professor Dahl suggests that, from a practical point of view, ratios of h/a less than 0.1 are *necessary* in order to apply the given solution. This value seems quite reasonable to the author and does lead, as Professor Dahl shows, to some simplification in the criteria for applicability of the solution.

In the same spirit, one should perhaps require that, in addition, the slant height L of the conical frustrum, depicted in Fig. 12 of the paper, should be somewhat greater (say five times) than the thickness $2h$.

Normal Vibrations of a Uniform Plate Carrying Any Number of Finite Masses[1]

W. H. HOPPMANN, II.[2] The theoretical treatment of the subject by the authors is very interesting and the results should be of value to those who have to analyze the vibrations of flat plates with attached masses. On the other hand, the experimental portion of the work may be criticized because the assumed boundary conditions are not satisfied. In order to obtain any reasonable approximation to simple support of a rectangular plate, positive provision must be made for maintaining zero deflection at each point on the boundary and especially at the vertexes. It is of course well known that, in order to prevent movement of the vertexes, substantial forces must be applied. The method used by the authors does not provide for these requirements. In the past the writer faced somewhat the same problem, and, as a result, introduced the so-called grooved plate for simulating simple support conditions on the boundary.[3,4] The authors are referred to the papers describing the method and it is hoped that they have an opportunity to do further experimental work in connection with the problem.

Authors' Closure

It is certainly true that to obtain the simply supported boundary condition provision must be made to insure zero deflection at all boundary points, and to restrain the corners. In some tests on a 10-in. × 20-in. plate, made after those described in the paper, each corner of the plate was held down with a 5-lb weight. The frequencies were found to be 3 to 7 per cent higher than when the corners were not restrained. Of course, even this provision did not insure that the desired boundary conditions were obtained completely, and the method devised by the discusser would be superior. The authors' interest in the work was primarily with large plates on which fairly large masses were mounted, specifi-

[1] By W. F. Stokey and C. F. Zorowski, published in the June, 1959, issue of the JOURNAL OF APPLIED MECHANICS, vol. 26, TRANS. ASME, vol. 81, series E, pp. 210–216.

[2] Professor of Mechanics, Department of Mechanics, Rensselaer Polytechnic Institute, Troy, N. Y. Mem. ASME.

[3] W. H. Hoppmann, II, and Joshua Greenspon, "An Experimental Device for Obtaining Elastic Rotational Constraint on Boundary of a Plate," Proceedings of the 2nd U. S. National Congress of Applied Mechanics, ASME, 1954, p. 187.

[4] W. H. Hoppmann, II, and W. G. Soper, "Note on the Boundary Conditions for Bending Experiments With Bars and Plates," JOURNAL OF APPLIED MECHANICS, vol. 23, TRANS. ASME, vol. 78, 1956, p. 646.