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Dynamics of an Ice Covered Lake with Through-Flow

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The dynamics of different regimes in an ice covered inland lake with through-flow are discussed. A new theory for selective withdrawal is tested with success. The mixing of the entering river with lake water is estimated and it is found that the net volume flux increases with about 60% through the mixing. The flow in the main part of the lake is discussed and some measurements are presented.

Introduction

The dynamics of hydroelectrical power reservoirs have in the past been discussed mainly by hydraulic engineers. Wunderlich and Elder (1973) have reviewed this subject. The dynamics of flushed inland lakes are in many ways similar to those of hydroelectrical power reservoirs. The main difference between the two systems is the form and location of the outlet. In a lake the outlet is a river and the transition between lake and river is more or less funnel shaped. In a hydroelectrical power reservoir, however, the outlet is often situated, submerged, at the more or less vertical dam wall. Two recent works have treated the dynamics of inland lakes with through-flow. Stigebrandt (1978b) discussed an ice-covered lake and treated the whole lake from inlet to outlet. The work presented here is mainly coincident with that work. Hamblin and Carmack (1978) discussed the summer situation and they concentrated upon the region nearest the inlet region.

There seem to be several good reasons to study an ice-covered inland lake with through-flow. First of all it contains several interesting regimes such as the entrance mixing zone where entering river water mixes with lake water, the outlet zone where the process of selective withdrawal operates and the zone with thin baroclinic currents along the lake which are influenced by the Coriolis force. Secondly it is important to know the dynamics of a system before one is intending to make a change, e.g. to increase the river discharge in winter. A knowledge of the dynamics of the whole system permits calculation of the effects of an intended change and thereby one can at best minimize any negative consequences that may arise.

In early winter when an ice cover develops the vertical temperature distribution in the underlying water is to a great extent determined by the weather conditions from the autumnal vertical overturning to the appearance of ice. A lake that has no sources or sinks of water will retain this vertical temperature stratification more or less unchanged until late winter when solar radiation begins to penetrate the (then snow-free) ice. At this time the water below the ice is heated, thermal convection is set up, and the ice starts to melt. In this paper we will deal with the period between the development of an ice cover and the onset of solar heating. If the lake has sources and/or sinks of water the stratification is continually changing. The temperature can reach a steady state during the winter in lakes with sufficiently small volume/flowrate ratio. Then all flowing water in the lake has the same temperature as the inflowing water (the source).

Two different physical processes are responsible for the way in which the stratification changes. The inflowing river water mixes with lake water upon entering the lake, the amount of mixing depends in some way on the velocity and temperature of river water, the geometry of the river mouth and the stratification in the lake. The temperature distribution in the outflowing water is determined by the process of selective withdrawal. The governing parameters for this process are the flowrate, the geometry of the outlet area and the vertical stratification in the lake. The temperature distribution of the outflowing water can be quite different from that produced by the mixing in the inlet area. Indeed the distributions are different as long as the flowing water in the lake has temperatures different from that of the source water (incoming river). This difference is the reason for the change in the vertical temperature distribution in the lake.

The main part of the lake, outside the inlet and outlet areas, can be looked upon as a storage and transportation region. The necessary transports in the lake, longitudinal currents from the inlet area and to the outlet area, seem to be modified by the Coriolis force. Because of the Coriolis force and friction against the rigid top (the ice sheet) there is an upwelling of warmer water along the right bank which leads to a decreased ice thickness there. This phenomenon is known to exist in several flushed Norwegian lakes, the most famous of them in this connection is perhaps the Sperillen which is described in Appendix A.

The vertical mixing in an ice-covered lake seems to be restricted to the inlet region. The mixing is there often so strong that an ice-free area is maintained even during extremely cold periods. On the other hand high resolution velocity measurements conducted outside the inlet mixing region in the Sperillen did not show fluctuations with shorter periods than T_N (= $2\pi/N$), where N is the buoyancy frequency defined later which means that turbulence is very weak. This indicates that mixing in the lake proper, if it occurs, at least is very rare. T_N is typically on the order of ten minutes.

Accordingly we can, for dynamical reasons, divide the lake into three parts:

i) *The lake proper.* This part is a storage and transportation volume.

ii) *The inlet region* where the initial mixing of the inflowing river water occurs. Almost all vertical mixing in the ice-covered lake is supposed to take place here.

iii) *The outlet region* where the process of selective withdrawal dominates the dynamics.

A schematic map of the lake with its different regions is given in Fig. 1. This paper is written in two parts. In the main body the different dynamical regimes of a general ice covered and flushed lake are discussed. In the Appendices the theoretical findings are exemplified with examples from lake Sperillen where a lot of measurements have been conducted.

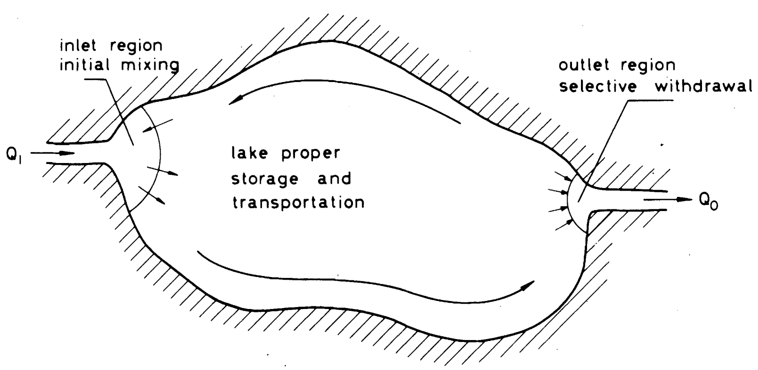


Fig. 1. Sketch of a typical ice covered lake with the different dynamical regimes indicated.

The Inlet Region

During the period when the lake is covered by ice the temperature of the inflowing river water is, in many cases, near 0°C. The lake, however, is vertically stratified and the entering river water is less dense than the lake water. This is especially true near the beginning of the ice-covered period. It is upon this case we concentrate. Hamblin and Carmack (1978) have studied the summer case when the river water is heavier than the surface water in the lake.

The way the river water enters the lake depends on the geometry of the river mouth, the volume flux, Q_i, in the river and the density difference, $\Delta\rho$, between river and lake water. If the transition between the river and the lake is geometrically abrupt, like a canal sitting on a vertical lake wall, two cases are possible depending on the magnitude of a densimetric Froude number, F_d, defined by

$$F_d \equiv \frac{u_i}{(g \frac{\Delta\rho}{\rho} H)^{\frac{1}{2}}} \quad (1)$$

where u_i is the velocity in the river, H is the depth of the river, which for simplicity is assumed to be rectangular in the vertical cross section with width B, and g is the acceleration of gravity. The velocity u_i is defined by $Q_i \equiv u_i x B x H$. For a two-layer system it can be shown that if $F_d<1$ the lake water will penetrate up into the river. A phenomenon quite analogous to the salt wedge in an estuary will occur, which we call the temperature wedge. If $F_d>1$ lake water can not penetrate into the river and the river will discharge into the lake like a jet. The mixing is known to be small in the temperature wedge case (subcritical flow). In the jet case (supercritical flow) the mixing can possibly be quite strong, depending on the magnitude of the Froude number and geometrical conditions in the lake. The mixing caused by the energy of the entering river is called the initial mixing and it is discussed in more detail later in this section.

If the transition from the river to the lake is geometrically smooth the river water can possibly follow the (diverging) boundaries. Stigebrandt (1978b) discussed the separation of the river water from the bottom for this case which he called the external temperature wedge. He believed the flow in this case to be subcritical and expected the mixing to be weak.

The three mentioned cases are outlined in Fig. 2. As a rule the stratification in the lake is not two-layered but merely continuous and often nearly linear. This is especially true during the first period with ice cover. A linear stratification is described by the buoyancy frequency

$$N^2 = -\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}$$

where ρ_0 is a reference density and z is the vertical coordinate (positive upwards). Many results from two-layer dynamics can readily be translated to cases with linear stratification by simply substituting $g\frac{\Delta\rho}{\rho}$ with $N^2 d$, where d is

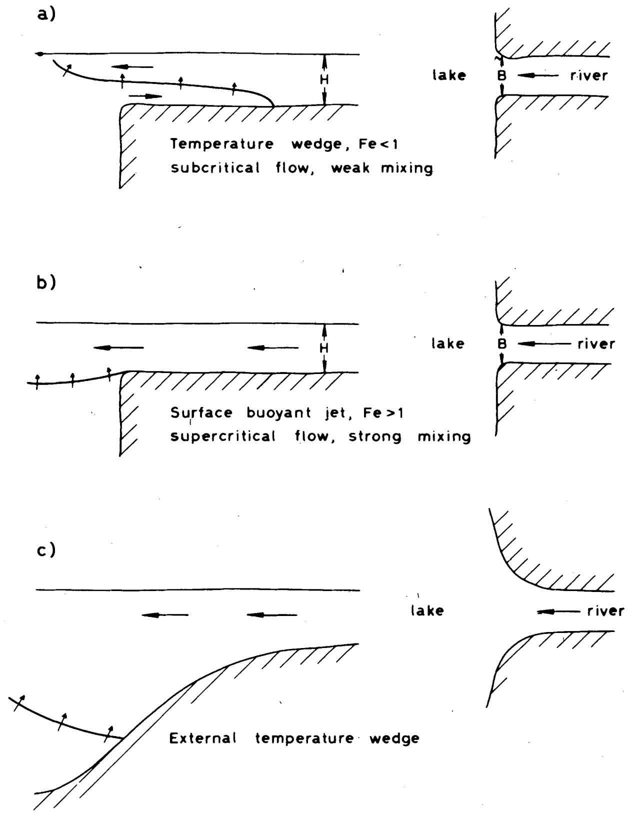


Fig. 2. Outline of different dynamical possibilities when a river meets a lake. Density of river water is less than density of lake water.

an appropriate vertical length. We believe that the classification of the ways a river can enter a homogenous lake, discussed above, can be translated simply to the case when the lake is linearly stratified.

For the purpose of constructing a predictive temperature model for a specific lake it is obviously important to have an adequate model for the interaction between the inflowing river water and the lake water. This is especially true for an ice covered lake where almost all vertical mixing in the lake is believed to occur in the inlet mixing region.

An adequate model for the mixing between river water and lake water should give the net amount of volume fluxes through the border between the mixing zone and the lake proper. Given the length of the border at any depth one thus wants to have the vertical distribution of the mean horizontal current velocity normal to the border, $u_{in}(z, t)$, or equivalently the distribution of $q_i\ (T, t)$ (for the definition of q_i, see later). However there is no method to calculate u_{in} (or q_i) for a lake for either the temperature wedge cases or the jet case. It is to be noted that even the most sophisticated jet theory is not applicable to this problem because of the backcoupling between the jet and the lake. The backcoupling is often termed recirculation and methods to calculate it are lacking. Huber, Harleman and Ryan (1972) recognized this lack of models for the inlet mixing and they proposed a very crude one. In other subsequent work, however, results from theories on a buoyant surface jet discharging in an infinite recipient have been used for inland lakes, thus completely neglecting recirculation.

Sometimes weak mixing in the inlet region can be explained by downstream conditions. The baroclinic transport capacity can be chooked in a narrow and/or shallow sound downstream the inlet mixing region. The concept of baroclinic transport capacity is discussed in Stigebrandt (1978a). For a given buoyancy and volume discharge from the river and given outside density stratification there exist a maximal baroclinic transport capacity for the sound. This maximal transport capacity can not be increased by increased input of mixing energy in the inlet region. The hydrographic state in the inlet region is then said to be overmixed. The state of overmixing was first explored for estuaries by Stommel and Farmer (1953). In a rotating system another kind of hydraulic control is possible and a state of overmixing could possibly be reached also in this case, see Whitehead, Leetmaa and Knox (1974).

Shallow water in the inlet mixing region can obviously be the reason for weak mixing when the river discharges like a buoyant surface jet. The jet can flow in contact with the bottom and vertical entrainment of lake water into the jet will then be hampered.

No reliable estimates of inlet mixing of buoyant rivers in stratified lakes exist in the literature. In Appendix B the inlet mixing in lake Sperillen is calculated from field temperature data. The net amount of flow out of the mixing zone was found to be about $1.6 \mathrm{x} Q_i$ which is a surprisingly small increase in view of the relatively high densimetric Froude number (F_d 5, if it is assumed that no outer temperature wedge exists). The explanation of this surprising result can probably be ascribed to the backcoupling. The weak initial mixing in the Sperillen can probably neither be explained as a result of overmixing, caused by downstream choking of the baroclinic flow, nor can it be explained by shallow water in the inlet mixing region because the lake is more than 25 meter deep just outside the river mouth.

Looking closer at the results on inlet mixing in the Sperillen, presented in Appendix B, one finds that the water giving a net increase in volume flux of the buoyant surface discharge comes from the depth interval 8-16 m and the mixture is intruding into the lake proper at the depth interval 0-8 m. Thus only water lifted vertically from depths greater than 8 m contributes to the net mixing. Water entrained from the sides (from depths less than 8 m) is just recirculating. The work per unit time against the bouyancy forces carried out by the net mixing process in the inlet mixing region is estimated to be about 5% of the kinetic energy released by the inflowing river pe unit time. Another way to express this finding is to say that the flux Richardson number, *Rf*, for the mixing process in the inlet region is 0.05. *Rf* has been found to be of the same order of magnitude in other situations with mixing in stably statified fluids, see e.g. entering river int he inlet mixing region seems to be quite small, the efficiency of turbulance with respect to working against bouyancy forces seems to be normal.

In the case of a surface discharge of a heavy fluid recirculation should not be the same serious problem as in the case of a buoyant surface discharge. The reason for this is that the sinking plume is mainly mixed with water from levels higher than the final plume level. The lower the final plume level is situated the smaller are the problems with recirculation. Hamblin and Carmack (1978), who treated the sinking plume case, did not report any problems with recirculation. Chooking because of insufficient downstream transport capacity can possibly occur in their lake because of the huge transports/unit width involved. The most actual cases with respect to chooking are their May 31 and June 20 cases.

The Outlet Region – Selective Withdrawal

Normally only water from the upper layers in the lake can leave the lake and flow into the river. The reason for this is obviously the vertical stratification in the lake. This phenomenon, that only water from the levels nearest the outlet level can leave a stratified system, is called selective withdrawal. It is an important problem from both practical and theoretical points of view. The three-dimensional process has been explored by Stigebrandt (1978a). He showed that the thickness of the flowing layer in a linearly stratified system could be calculated by using a generalized momentum conservation equation from two-layer dynamics. The theory ignores frictional effects and is developed for an outlet sitting on a vertical wall. For small outlets the depth of the flowing layer, δ , in the reservoir is approximately given by

$$\delta \equiv \ell_3 \left(\frac{Q^2}{N^2 B_0^2 D_0} \right)^{\frac{1}{3}} \qquad \left(\frac{B_0}{B_r} << 1, \ \frac{D_0}{a_3} << 1 \right) \tag{2}$$

Where Q is the discharge, N is the buoyancy frequency in the reservoir away from the outlet, B_0 and D_0 are width and height respectively of the rectangular outlet and ℓ_3 is a universal constant which has a value of 0.74 for outlets at the free surface or bottom and a value of 1.18 for outlets at middepth. B_r is the width of the reservoir and a_3 is a length defined by

$$a_3^2 = \frac{Q}{B_0 N} \tag{3}$$

For larger outlets, in the sense that one or both of the ratios B_0/B_r and D_0/a_3 are not much smaller than one, the depth of the flowing layer is approximately given by the equation

$$\left(\frac{\tilde{\delta}}{a_3}\right)^4 - \left(\frac{D_0}{a_3}\right)^2 \left(\frac{\tilde{\delta}}{a_3}\right)^2 - \ell_3^3 \left(\frac{a_3}{D_0}\right)\left(\frac{\delta}{a_3}\right) + \ell_3^3 \frac{B_0}{B_r} \quad 0 \tag{4}$$

Thus if the outlet opening is sitting on a vertical lake wall the situation is clearcut and it is easy to define B_0 and D_0. Natural outlets, however, are not very often canals sitting on vertial lake walls. The transition between lake and river is often funnel shaped. If this is the case it is impossible to tell in advance where the (dynamical) border between lake and river is situated. We assume that the current is laterally uniform, thus filling up the whole width of the transition. For simplicity we assume that the funnel can be described by linear changes in depth and width and accordingly we write

$$B = B_0 (1 + k_1 r) \tag{5}$$

$$D = D_0 (1 + k_2 r) \tag{6}$$

where r is the distance from the geometrical border between the funnel and the channel. The dynamical border between the lake and the river must be situated in that section where the current hits the bottom. Upstream of this section baroclinic adjustments are possible but downstream of it the flow can be considered to be essentially barotropic. The recognition of this dynamical border is perhaps not of much immediate help. We know, however, that Eq. (4) can be used upstream of this border. We now have to use a minimalization principle in order to determine the thickness of the flowing layer, δ, in the lake. We choose that solution of Eq.(4) which, together with Eqs.(5) and (6), gives the thinnest flowing layer. This minimizes the momentum flux in the lake. This choice is quite analogous to the usual ciritcal depth assumption in hydraulics. The most direct way of calculating δ_{min} is to calculate δ for different values of r and plot δ versus r and from the plot determine δ_{min}. It is not necessary to use just one value of k_1 resp.k_2 for the whole transition. B and D can be treated as piecewise continuous and linear in r. Even higher order polynomials in r can be used.

In order to calculate the selective withdrawal from a lake or reservoir it is of course important to know the vertical velocity distribution, $u(z)$, over the flowing layer. Several empirical functional forms have been used. Huber, Harleman and Ryan (1972) used a Gaussian velocity distribution, others have used a linear or a cosine distribution for a linear density profile (N = const.). The Gaussian profile is theoretically unsound because it only reaches zero velocity asymptotically but, of course, this does not matter for practical calculations. All of the distributions of course have to satisfy the following integral condition

$$\int_{-\delta}^{0} u(z)dz = \frac{Q_0}{B}$$

where $z=0$ is the approximate coordinate for the lake surface and B is the local width. In Appendix C we have calculated the depth of the flowing layer towards river Begna in lake Sperillen. The calculated depth, 13 meters for the case treated, is very near the measured depth, 15 meters. It is of course not possible to draw any definite conclusions from just one test but one can at least say that the method seems to be promising.

Temperature Budget for the Lake Proper

We shall now discuss the stratification in an ice covered inland lake which has one source and one sink (a river flowing through the lake). When the lake is covered by a rigid, land-fast ice sheet the wind can not transmit any stress or energy to the water below the ice. The ice cover also insulates the water and heat losses from non-freezing water through the ice are believed to be small. These factors both contribute to create a relatively clean source-sink circulation in the lake in the sense that disturbing velocity fields are small.

Under the assumption that all vertical mixing in the lake occurs in the inlet region, the lake proper must behave in a purely advective manner. If the volume fluxes into and out of the lake proper are unequal at some levels this will lead to vertical advection which in turn will result in changes in the vertical temperature stratification. Baines and Turner (1969) called such an advective model a »filling box model«. In their case, however, the vertical advection was driven by a buoyant plume. Huber, Harleman and Ryan (1972) developed a modified filling box model for hydroelectrical power reservoirs. The vertical advection was in their case driven by entering rivers and outflow from a submerged outlet. Their model also included vertical diffusion in the lake proper and heat exhange through the water-air interface. Both of the above mentioned models are onedimensional and thus retain only the vertical z-dependence.

For a pure (without diffusion) filling box the temperature (heat) equation is simply

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial z} \equiv 0 \qquad (7)$$

The one-dimensionality of the model means that horizontal inflows and outflows affect the temperature solely by vertical advection. The vertical velocity, w, can be obtained from the integrated continuity equation for volume

$$w(z_1,t) = \frac{1}{A(z_1)} \int_{z=b}^{z=z_1} \{q_i(z,t) + q_0(z,t)\} dz \qquad (8)$$

where $A(z)$ is the horizontal area of the lake and $z = b$ is the vertical coordinate for the bottom. $q_i(z,t)$ and $q_0(z,t)$ are the transports into respectively out of the lake proper per unit vertical height. The signs of $q_i(z,t)$ and $q_0(z,t)$ are chosen in such a way that transports into the lake proper are positive and transports out of the lake proper are negative. Because of mixing in the inlet region $q_i(z,t)$ has both positive and negative values but $q_0(z,t)$ has just negative. Thus from the vertical distribution of inflows to and outflows from the lake proper, temperature changes in the lake can be calculated. As discussed earlier it should be possible to use Eqs. (7) and (8) to find the source function $q_i(z,t)$ from the described theory for selective withdrawal. The vertical distribution of the inflow, $q_i(z,t)$, however, can not be calculated yet with any degree of certainty for a buoyant surface discharge problem. It should nevertheless be possible to use Eqs. (7) and (8) to explore the source function $q_i(z,t)$ when $T(z,t)$ and $q_0(z,t)$ are known. However this is perhaps better done with the help of another kind of filling box model which is described below. Beforehand we note that from Eqs. (7) and (8) it follows that the stratification is stationary ($\partial T/\partial t = 0$) only when all flowing water is homogeneous or when $q_i(z,t) + q_0(z,t) = 0$ at all levels.

Another kind of description of a diffusive filling box was given by Walin (1977). Instead of dividing a water body into horizontal slices of equal vertical thickness, as in the model above, he divides the water body into different water masses of equal thickness in the temperature (or salt) space. In that way he does not have to rely upon the assumption of horizontal density surfaces. Inflows, outflows and diffusion have to be specified in the temperature space instead of in conventional geometrical space. Below we shall follow this procedure and develop a diffusionless filling box model for the lake proper.

For simplicity we choose to study the lake during a period when the thickness of the ice is approximately constant. The relation between lake volume, $V(t)$, and fluxes in the incoming and outgoing rivers is then

$$Q_i + Q_0 = \frac{\partial V}{\partial t} \qquad (9)$$

where the source is taken positive and the sink is negative. Although homogeneous in the rivers, the source and sink that affects the lake proper are nonhomogeneous with respect to temperature as long as the flowing water in the lake is vertically stratified. The fluxes in the temperature interval $(T, T+\Delta T)$ are $q_i(T,t)$ and $q_0(T,t)$ for source and sink respectively. We have the following integral conditions

$$Q_i = \int_0^4 q_i(T,t)\, dT \tag{10}$$

$$Q_0 = \int_0^4 q_0(T,t)\, dT \tag{11}$$

As limits for the integration have been chosen the temperature for freezing (0°C) and the temperature of maximum density (4°C) for fresh water. Water with temperature outside this interval does not normally exist in a lake covered by ice.

We divide the water in the lake into parts, and define $v(T,t)$ as the volume of water in the temperature interval $(T,T+\Delta T)$. The following integral condition must of course be satisfied

$$V(t) = \int_0^4 v(T,t)\, dT \tag{12}$$

If the isothermal surfaces are horizontal, which is a good approximation in the lake proper, we have that

$$v(T,t) = A(T,t)\, h(T,t) \tag{13}$$

where $A(T,t)$ is the area of the isothermal surface with temperature $T+\Delta T/2$ and $h(T,t)$ is the vertical distance between isothermal surfaces with temperatures T and $T+\Delta T$. In that way $h(T,t)$ gives the vertical stratification.

As mentioned earlier we assume that all vertical mixing in the lake is localized to the inlet area where small scale turbulence is generated by the entering river water. With $v(T,t)$ as a pure filling box we then have

$$\frac{\partial v(T,t)}{\partial t} \equiv q_i(T,t) + q_0(T,t) \tag{14}$$

With sufficient temperature data from the lake proper it is possible to estimate $\partial v(T,t/\partial t$. It is also possible to estimate the sink function $q_0(z,t)$, see page 225, and thereby also $q_0(T,t)$. One is then able to estimate the source function with the help of Eq.(14). It is thus possible to study the entrance mixing in an ice covered lake from just temperature changes in the lake. This can, as earlier mentioned, also be done with the conventional filling box (in z,t-space) but it is more easily done in the T,t-space. We have done this for lake Sperillen, see Appendix B.

If the lake does not exchange heat with the atmosphere or with bottom sediments, conservation of heat gives, for the lake proper, the inlet region and the outlet region respectively

$$\int_0^4 (q_i+q_0)T\, dT = \int_0^4 \frac{\partial v}{\partial t} T\, dT \tag{15}$$

$$\int_0^4 q_i T\, dT = Q_i T_i \tag{16}$$

$$\int_0^4 q_0 T\, dT = Q_0 T_0 \tag{17}$$

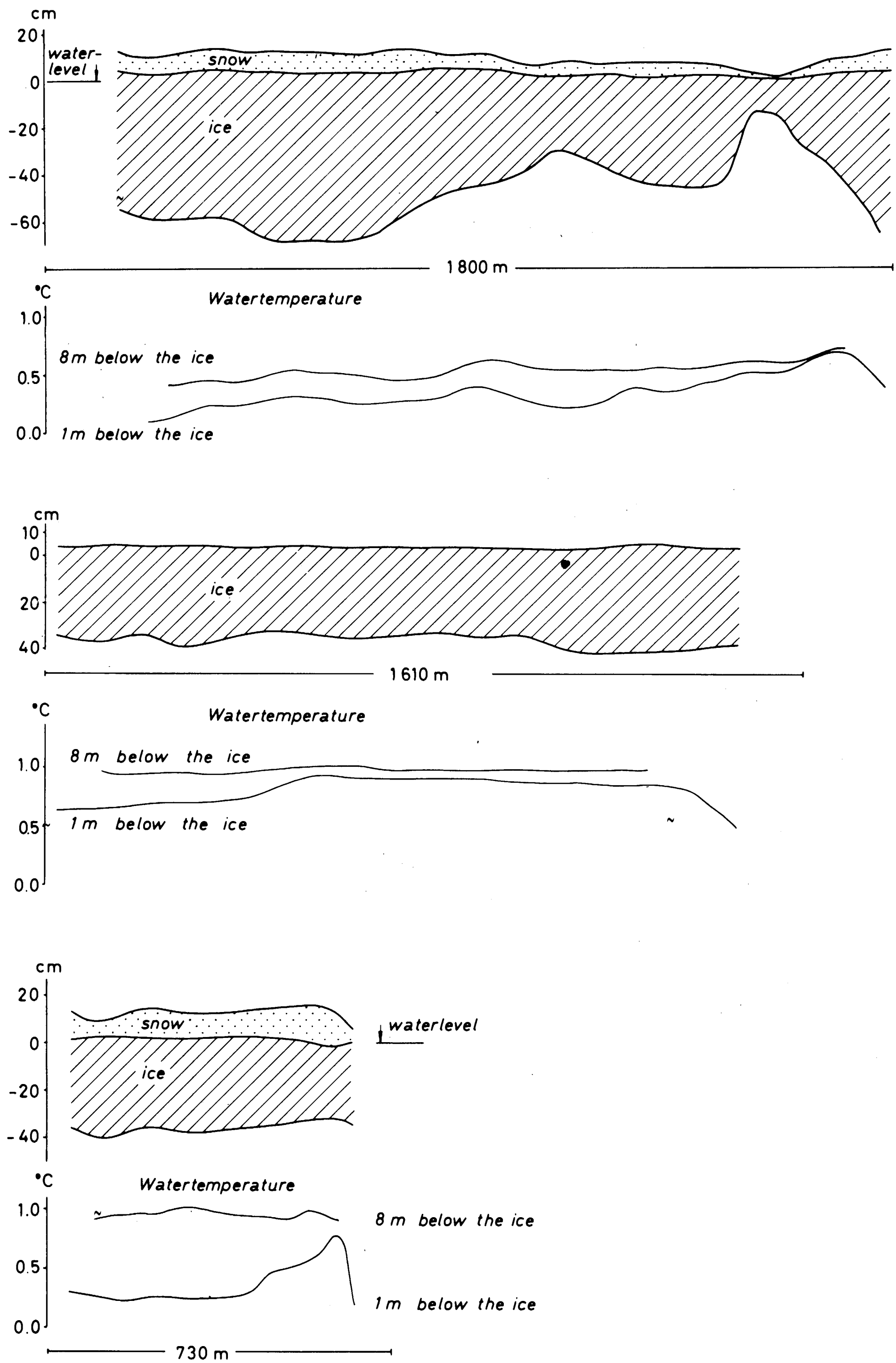
where T_i and T_0 are the temperatures of the incoming and outgoing rivers respectively. We have assumed that specific heat and density can be considered as constants in the temperature interval considered.

A pure filling box can perhaps be said to be dynamically passive, it is driven by some process, which can be treated separately, such as a buoyant plume, a buoyant jet, a selective sink or some other kind of source/sink. For several types of sources and sinks there exist models that can produce an adequate $q(z)$. In those cases there seem to be good reasons to work with the conventional filling box (retaining the z-dependence). For systems believed to behave like a filling box but where knowledge about the sink or source function is lacking there seem to be good reasons to work with a filling box in the temperature space in order to learn more about the source or sink functions.

Conditions in the Lake Proper

Ice covered and flushed inland lakes have become common only in recent time after increased regulation of the water discharge in river systems, necessary for the increased use of hydroelectrical power. After regulation of the river Begna in Southern Norway people living around the lake Sperillen, which is a lake through which the Begna flows, complained that the ice thickness had diminished. Friis (1969) undertook extensive measurements of ice thickness in the lake and he found ice thinner than average in a band along the right bank of the lake (as seen in the flow direction of the river). A map of lake Sperillen is given in Fig.A1. Friis also found that the vertical temperature stratification in the water beneath the ice was abnormal in this band. The isotherms were spread vertically and the temperature just below the ice was higher in the band than in the rest of the lake. Ice thicknesses and temperature distributions in the water in three sections across the lake are shown in Fig. 3. Friis concluded from his temperature measurements that the longitudinal flow in the lake took place in a »flow channel«, a few

Fig. 3. Ice thickness and temperature distribution in the water in three sections across the Sperillen. (After Friis (1966)).
a) Just outside the inlet mixing region (March 16, 1966)
b) Halfway between inlet and outlet (March 14, 1967)
c) Near the outlet (February 16, 1967).


cm
20
water-
level
0
-20
-40
-60
snow
ice
1800 m
°C
Watertemperature
1.0
8m below the ice
0.5
0.0
1m below the ice
cm
10
0
20
40
ice
1610 m
°C
Watertemperature
8 m below the ice
1.0
0.5
1 m below the ice
0.0
cm
20
0
-20
-40
snow
waterlevel
ice
°C
Watertemperature
1.0
8 m below the ice
0.5
1 m below the ice
0.0
730 m

hundred metres wide, along the right bank. Of course Friis explained the deflection of the current to the right by the action of the Coriolis force. Tesaker (1973) placed two thermistor strings in a section perpendicular to the »flow channel« not far from the inlet mixing region (indicated by T on the map in Fig.A1) From the time mean of these temperature recordings he calculated baroclinic geostrophic velocities. The maximal baroclinic velocity was found to occur at a depth of 3 m and it was slightly below 2 cm/s.

Closer examination of Friis' measurements of ice thickness shows that the weakening of the ice is most pronounced near the inlet mixing zone where the ice thickness at the right bank can be less than 30% of average ice thickness. A few kilometers from the inlet mixing zone and further downstream the ice thickness at the right bank is not significantly less than average. At less than about 5 kilometers from the outlet, the weakening of the ice near the right bank begins to be more pronounced again and it can be down to 80% of normal ice thickness.

The picture of the current system in the lake given by Friis (1969) and further underlined by Tesaker (1973) of a concentrated »flow channel« along the right bank of the lake is not supported by the ice thickness data referred above. These rather indicate that high current velocities occur only near the inlet mixing zone and to some degree also near the outlet. An ultrasonic current meter has been developed by T. Gytre at Chr. Michelsens Institute, Bergen, Norway. This instrument can, carefully calibrated, measure currents down to a few millimeters per second. It has been used in the Sperillen and other lakes, a data report is given by Stigebrandt (1977). Currents measured along two cross sections in the Sperillen, see Figs. A4, A6 and the map in Fig. A1, show only a slight tendency for a current concentration towards the right bank. The south-going current essentially fills the whole width of the lake. A counter-current, towards the mixing zone, is in both cases found to exist below the southgoing water at the western side. A weak counter current also seems to exist near the surface at the eastern side.

Geostrophic baroclinic transports have been calculated from the temperature measurements obtained by Friis (1969) (10 sections across the lake, a total of 100 vertical profiles), were measured 4 times during the winter 1968-69). The calculated transports were, however, as a rule not very close to the expected and deviations of several hundred percent were common. These bad results are believed to be caused by internal waves, disturbing the temperature field in the synoptic sections. The temperature recordings, of a duration of nearly two months, obtained by Tesaker (1973) show a rather strong activity of internal waves. All periods from 2 hours (sampling frequency was 1 hour) are found in these. Most of the wave energy certainly comes from the outer part of the inlet mixing zone (where the flow is subcritical). Internal waves generated under such circumstances have been reported by Hamblin (1977).

The modified picture of the flow field in the lake proper that has emanated from the discussion above is the following: After mixing with lake water in the inlet mixing zone the mixture leaves the mixing zone in a concentrated current along the right bank. The current is possibly near critical and in geostrophic balance and thus it has a width comparable to the internal (Rossby) radius of deformation. At this stage the current behaves like an inertial current. On its further way towards the south, the current is acted upon by friction, both at the water-ice interface and at lower levels. This leads to a transport of water to the left (east) and the width of the current increases and the mean velocity decreases. We can look at part of this transport to the left as occurring in an Ekman layer just below the ice, pumping away colder water from the ice-water interface at the right bank. This horizontal transport is driving an upwelling of warmer water along the right bank. In Appendix E we have treated a boundary current as a stream tube in order to calculate the incerase in width downstream. We find that the boundary current, initially supposed to be about 250 m wide, about 4 km downstream has expanded to a width of 1 km which approximately is the width of the lake. Thus a concentrated »flow channel« should only exist in a region near the inlet mixing zone. This explains why the ice-thickness at the western bank is reduced only down to a few kilometers downstream of the inlet mixing zone. The slight reduction of ice-thickness found upstream of the outlet zone can perhaps be explained by the smaller width of the lake in the southern end, leading to higher velocities and thereby also to a more active Ekman layer which promotes the upwelling of warmer water along the western bank.

Appendix A – General Description of the Sperillen

The southern end of the north-south oriented lake Sperillen is situated about 70 km NNW of Oslo. It has a surface area of about 38 km^2 and the greatest length is 25 km. The mean depth is 38.5 m and the greatest measured depth is 129 m. The Begna river enters the lake in the northern end and leaves it in the southern. Normal winter discharge of the river is about $60 m^3/s$. In Fig. A1 is a map of the Sperillen with some of the measured sections shown. The width of the outlet channel, the Kongsströmmen, is about 25 m and its depth is about 3 m.

The Sperillen is normally ice covered from the middle of December to the end of April and for most of this time the inflowing river water has a temperature near 0^oC (slightly increasing during early spring). When the ice cover first develops the lake is stratified but the vertical temperature stratification in the upper 25 m of the lake can be quite different from year to year dependent on the weather conditions in the period between the autumnal vertical overturning and the development of an ice cover. During the winter the tempeature in the upper 25 m changes because of inflow and outflow and from late winter (approximately from the end of March) the water layers nearest the ice are heated by solar radiation through the ice.

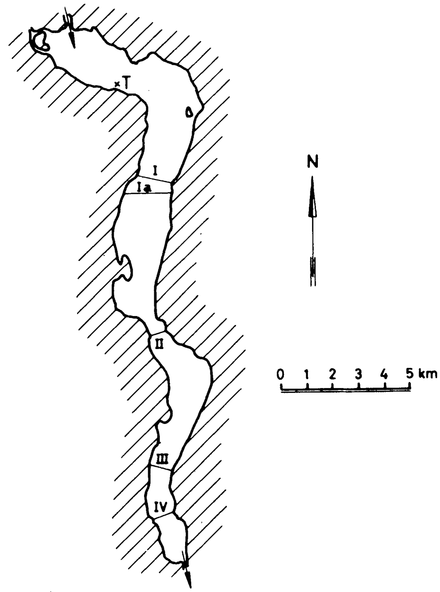


Fig. A1. Map over the Sperillen with measured sections indicated.

Even during extremely cold periods there exist two ice-free areas in the lake, one in the inlet mixing region and one at the outlet. These are further discussed in Appendix D. The rest of the lake has relatively uniform ice thickness except the transition area between the ice-free area in the inlet mixing region and the lake proper. Besides there exists a band along the western (right) bank with weaker ice near the inlet and outlet areas as described in the section on conditions in the lake proper.

Appendix B – Entrance Mixing in the Sperillen 10/1-16/2 1969

Temperature measurements from the Sperillen show that the temperature changes during the winter for depths less than 25 m but remains very stable below that. During the winter 1968-1969 the temperature in Sperillen was measured four times (in December, January, February and March). 100 vertical profiles were measured each time and a mean profile was calculated for each occasion, see Friis (1969). From the knowledge of the horizontal area of the lake for all pertinent depths it is possible to calculate $\partial v(T,t)/\partial t$ see page 227. The distribution of the sink flow, out of the lake, with respect to depth is calculated as already described.

Knowing the outflow with respect to depth and the vertical temperature stratification we can immediately calculate the outflow with respect to temperature, i.e. $q_0(T)$. A rough check on the estimation of selective withdrawal can for this case be obtained because the temperature T_0 was measured a small distance downstream of the outlet. Eq. (17) is used for this purpose. With the help of Eq. (14) it is now possible to estimate $q_i(T)$ which gives the net effect of mixing in the inlet region. Heat exchange with the atmosphere and bottom sediments has to be small if the estimate shall be reliable. The data show that such heat losses were small during the actual period, about ten per cent of the heat lost by the lake through the outflowing water. This period should therefore be well suited for a study of the entrance mixing.

We divide the volume of the lake water into parts according to Eq. (13) and with $\Delta T \equiv .2$°C. In Table 1 the distances between the bounding surfaces (isothermal) $h(T,t)$, the depth to the isothermal surface $T + \Delta T/2$ (denoted by $H/(T,t))$ and the calculated partial volume $v(T,t)$ are given for the two occasions 10/1 and 16/2. The mean values for $h(T,t)$ and $H(T,t)$ are also given. The mean temperature and density profiles has been drawn in Fig. A2. From the values given in Table I we are now able to calculate $\partial v(T,t)/\partial t$ for the period and the results are given in Table 2.

The calculated $q_i(T)$ and $q_0(T)$ have been plotted in Fig.A3. We find that the flow out of the inlet mixing region has increased by almost 60% compared to the

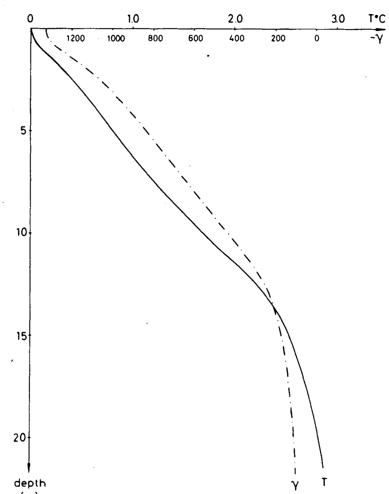


Fig. A2. Mean temperature T, and mean density γ, in the Sperillen during the period 10/1-16/1 1969. γ is defined by $\gamma = (\rho - 1)\, 10^7$.

Table 1–Distribution of temperature and volume in the Sperillen Jan-Feb 1969

		10/1-69			16/2-69		Mean	
T (°C)	$h(T)$ (m)	$H(T)$ (m)	$v(T)\times10^{-6}$ (m³)	$h(T)$ (m)	$H(T)$ (m)	$v(T)\times10^{-6}$ (m³)	$h(T)$ (m)	$H(T)$ (m)
0.0	0.6	0.3	22.5	0.75	0.37	28.4	0.68	0.34
0.2	0.6	0.9	21.0	0.75	1.12	26.0	0.68	1.01
0.4	1.0	1.7	34.9	1.3	2.15	45.4	1.15	1.93
0.6	1.0	2.7	34.2	1.4	3.5	46.8	1.2	3.10
0.8	0.8	3.6	26.7	1.6	5.0	52.6	1.2	4.30
1.0	1.0	4.5	33.1	1.7	6.65	54.0	1.35	5.58
1.2	0.9	5.45	29.6	1.7	8.35	53.0	1.3	6.90
1.4	1.0	6.4	32.3	1.3	9.85	39.8	1.15	8.13
1.6	0.9	7.35	28.6	1.0	11.0	30.2	0.95	9.18
1.8	1.0	8.3	31.2	0.7	11.85	20.9	0.85	10.08
2.0	1.2	9.4	37.5	0.4	12.4	11.9	0.80	10.90
2.2	1.3	10.65	39.3	0.6	12.9	17.6	0.95	11.78
2.4	1.6	12.1	47.7	1.0	13.7	29.0	1.3	12.90
2.6	2.4	14.1	69.6	1.8	15.1	51.5	2.1	14.60
2.8	3.5	17.05	97.5	2.8	17.4	78.0	3.15	17.23
3.0	6.2	22.4	161.2	6.2	21.9	161.2	6.2	22.15

Table 2 – Fluxes of volume and temperature in Sperillen 10/1-16/2 1969

T (°C)	$\partial v/\partial t$ (m³/s grad)	q_0 (m³/s grad)	q_i (m³/s grad)	q_0T (m³/s)	q_iT (m³/s)	$(\partial v/\partial t)T$ (m³/s)
0.0	1.9	-6.6	8.5	-0.7	0.9	0.2
0.2	1.6	-6.2	7.8	-1.8	2.3	0.5
0.4	3.4	-9.6	13.0	-4.8	6.5	1.7
0.6	4.1	-8.9	13.0	-6.2	9.1	2.9
0.8	8.3	-7.7	16.0	-6.9	14.4	7.5
1.0	6.7	-7.2	13.9	-7.9	15.3	7.4
1.2	7.5	-5.5	13.0	-7.2	16.9	9.8
1.4	2.4	-3.8	6.2	-5.8	9.3	3.6
1.6	0.5	-2.2	2.7	-3.8	4.6	0.9
1.8	-3.3	-1.4	-1.9	-2.6	-3.6	-6.3
2.0	-8.2	-0.7	-7.5	-1.5	-15.8	-17.2
2.2	-7.0	-0.2	-7.2	-0.6	-16.6	-16.1
2.4	-6.0		-6.0		-15.0	-15.0
2.6	-5.8		-5.8		-15.7	-15.7
2.8	-6.3		-6.3		-18.3	-18.3
3.0	0.0		0.0		0	0

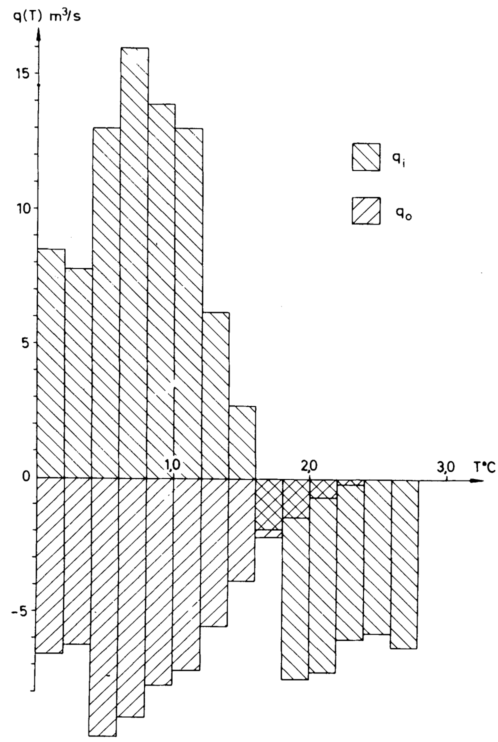


Fig. A3. $q_i(T)$ and $q_0(T)$ for Sperillen 10/1-16/2 1969. (Calculated).

river flow (60 m³/s). From the plot it is also evident that the entrance mixing affects much deeper (warmer) water layers than the process of selective withdrawal at the outlet does. If we had more such computations of inlet mixing from other points in the N, F_d-space (N = buoyancy frequency which gives the stratification in the lake and F_d is the densimetric Froude number for the entering river) we could fit a curve to $q_i(T)$ (or $q(z)$) which should be governed by the two parameters N and F_d. This is a task for future research.
In order to control the calculation of $q_0(T)$ we calculate

$$T_0 = \frac{\int_0^4 q_0(T)\,T\,dT}{\int_0^4 q_0(T)\,dT}$$

Inserting the appropriate values from Table 2 we find that T_0 = 0.83°C. This result is very good because the measured value of T_0 is 0.8C.

Appendix C – Selective Withdrawal in Sperillen

We shall now use the theory for selective withdrawal from a lake, to calculate the thickness of the layer flowing towards the outlet for the situation which was measured on the 20th of February 1975. Measurements of temperature and current were conducted at section II, see map in Fig. A1. These and other measurements are given in detail in Stigebrandt (1977). The outflow from Sperillen was in this case about 70 m^3/s and the inflow was about 60 m^3/s. The stratification was nearly linear with $N = 7x10^{-3}$ s^{-1}. In Fig. A4 are shown velocities parallel to the lake axis. The arrows at the surface show positions for the measurements. The water in the upper 15 m moves towards the outlet and below there are weak movements towards the inlet. There are also weak return currents at the left bank, seen in the primary current direction. To give an impression of the quality of the current measurements we can mention that the transport obtained from the measured velocities over the section is 71 m^3/s (it should be about 67 m^3/s). In Fig. A5 the vertical current profiles for the central verticals are shown. From this figure we see that the thickness of the layer flowing towards the outlet is about 15 m. This thickness must not be exactly the same as that given by the process of selective withdrawal at the outlet because the mixing process at the inlet will also have some influence at section II. The partial lake surface area between section II and the outlet is essentially smaller than the rest

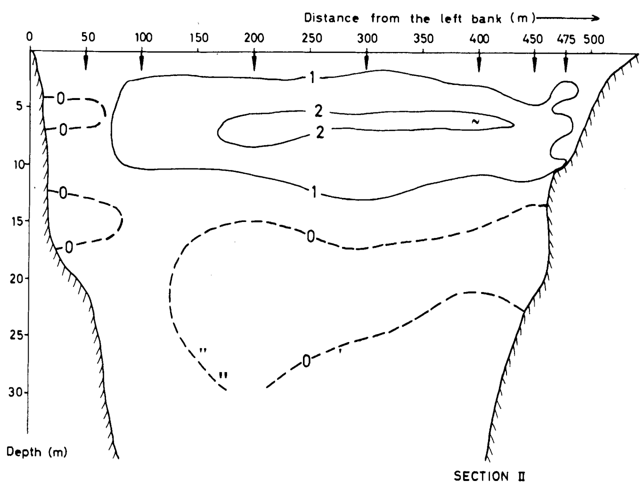


Fig. A4. Velocities in cm/s parallel to the lake axis in the Sperillen, section II on 20/2 1974. The current is directed towards south except in the area bounded by the broken line, where the current is towards north. The location of the measured section II is found in Fig. A1.

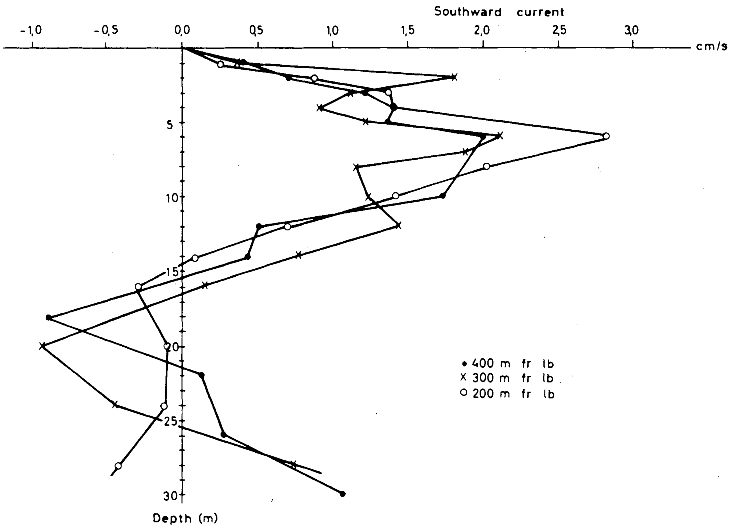


Fig. A5. Vertical velocity profiles at section II 20/2 1975 200, 300 and 400 m from the left bank (Sperillen).

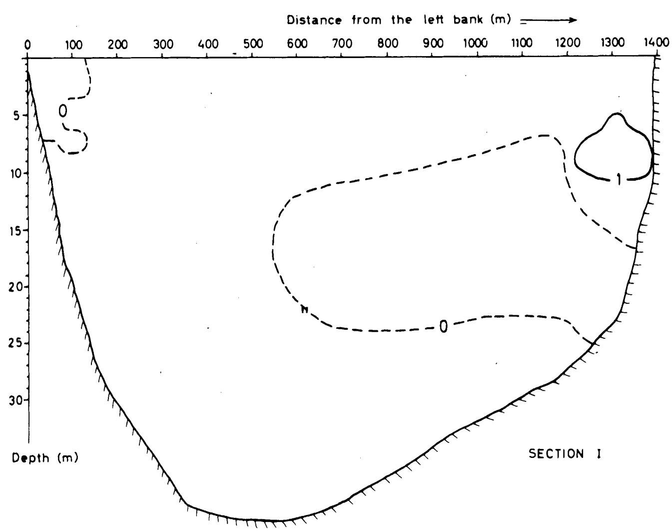


Fig. A6. Velocities in cm/s parallel to the lake axis in the Sperillen, section I on 19/2 1974. The current is directed towards south except in the area bounded by the broken line, where the current is towards north. The location of the measured section I is found in Fig.A1.

of the lake surface area and therefore the currents in section II should be dominated by the process of selective withdrawal at the outlet. Furthermore friction against the ice will give a tendency for larger layer thickness than the calculated one. The thickness of the withdrawn layer. δ , calculated as described in the section on the temperature budget, using the parameters given in Table 3 below, is found to be 13 m.

Table 3 Sperillen parameters 20/2 1975

$Q_0 = 70\ m^3/s$	$N = 0.007\ s^{-1}$	$B_0 = 25\ m$
$D_0 = 3\ m$	$k_1 = 0.035\ m^{-1}$	$k_2 = 0.045\ m^{-1}$

The method outlined thus seems to work rather well for this case. We have used the same method in Appendix B in order to calculate the flux $q_0(T)$. For simplicity we have used a linear velocity profile over the flowing layer.

Appendix D – Ice Free Areas in the Lake

There exist two areas in the Sperillen, and very often also in other flushed lakes, that always are ice-free. One is situated in the inlet mixing zone and the other just at the outlet. The mechanisms that create these ice-free areas seem to be quite different. One can imagine at least two circumstances under which part of a lake surface can remain ice-free even during extremely cold periods. The first and most obvious occur when warmer lake water is transported to the surface by strong mixing. This occurs in the inlet region where the entering river creates mixing. The warmer lake water from lower levels is pumped to the surface and ice can not form before the surface water has been cooled down to 0°C. With constant stratification in the lake and constant water discharge (and constant densimetric Froude number) from the river the ice-free area in the inlet mixing zone expands during warmer periods and contracts during colder. Since the ice thickness increases more or less gradually towards the lake proper, this region is the most treacherous in the whole lake.

The second case when part of the lake can remain ice-free even during extremely cold periods seems to occur in areas where the water has large velocities. When ice first forms on the lake, ice-crystals attach to each other and to the banks of the lake. The ice will rapidly be land-fast and rigid. If there exists a region with high velocities the ice crystals formed there do not attach to the rigid, land-fast ice but are transported away by the current. In rivers, it is known, that the ice crystals formed is such high-speed regions attach to other land- (or bottom-) suspended ice crystals downstream where velocities are lower. Rivers can remain ice-free in regions where velocities are higher than about 0.6 m/s. In the lake relatively high velocities can be found near the outlet where the water accelerates towards the outlet river. Ice crystals formed in the region nearest the river can apparently not attach to the land-fast lake ice but are transported out of the lake. This ice-free area is generated even in turbulence-free flow, in contrast to the ice-free area in the inlet mixing region. The size of this ice-free area is probably determined when the lake ice first develops. Measurements in lake Sperillen showed that the water velocity right at the ice-edge was about 0.06 m/s this is one order of magnitude smaller than the velocity found under similar conditions in rivers. When the ice thickness grows in the lake it is growing at approximately the same rate right to the ice edge at the outlet and the ice thickness is really discontinuous at the outlet, as if it were sawed. The ice at the outlet end is not dangerous, it is possible to walk right to the ice edge. When the ice is thick enough the ice edge generates a turbulent wake. Heat is transported to the surface in the ice-free area and ice crystals will then no longer form there. Carstens (personal communication) warns that the good ice-conditions at the outlet of the Sperillen is not a general feature for all lakes. He has several examples of weak ice near the outlet. Use ice-prods when walking on the ice near the outlet of an unknown lake!

Appendix E – A Simple Streamtube Model of a Boundary Current

We shall derive the equations for a streamtube model of a boundary current, cf. Smith (1975), starting with the following set of equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(E1)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\tau_{xz}}{\rho_0}\right) \qquad \text{(E2)}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial z}\left(\frac{\tau_{yz}}{\rho_0}\right) \qquad \text{(E3)}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} - g\frac{\rho}{\rho_0} \qquad \text{(E4)}$$

where Eq. (E1) is the continuity equation and Eqs. (E2-4) are the momentum equations in the x, y, and z-directions respectively. We have used the Boussinesq and the hydrostatic approximations. The coordinate system is oriented with x towards the south, y towards the east and z vertically upwards. We use a two-layer system with the flowing upper layer of density ρ_1 and the lower resting layer of density ρ_2. The free surface has the coordinate $z = \xi(x, y)$ and the interface is described by $z = \eta\ (x, y)$. The coast, a vertical wall (see Fig. A7) is situated at $y = a$ and the current is following that straight coast. The width of the current is B and the horizontal mean velocity in the x direction is u. Friction acts at the interface, no side friction, but entrainment is supposed to be insignificant.

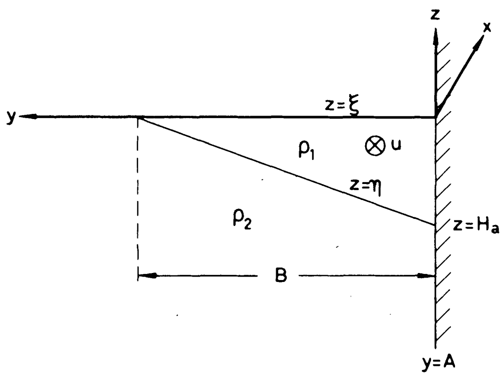


Fig. A7. Definitions for the streamtube model of a boundary current.

After integrating vertically and laterally we obtain the following equations

$$\frac{d}{dx} uB\bar{H} \equiv 0 \tag{E5}$$

$$\frac{d}{dx} u^2 B\bar{H} \equiv -g' B\bar{H} \frac{d\bar{H}}{dx} + B\left[\frac{\tau_\xi}{\rho_0} - \frac{\tau_\eta}{\rho_0} \right] \tag{E6}$$

$$fuB\bar{H} = \frac{1}{2} g' H_a^2 \tag{E7}$$

Here we have supposed that the lateral mean velocity is small compared to the longitudinal. $g' \equiv g\,(\rho_2 - \rho_1/\rho_0)$ is the reduced gravity and we have supposed a triangular cross section of the lighter water and $H = ½H_a$ is the mean depth of the flowing layer.

With constant transport $Q = uBH$, no entrainment and no ß-effect (effect of the variation of the Coriolis parameter with latitude) we find that

$$\frac{dH_a}{dx} = \frac{d\bar{H}}{dx} = 0$$

With

$$\frac{\tau_\eta}{\rho_0} = ku^2 \equiv k \frac{Q^2}{B^2 \bar{H}^2}$$

we can write Eq. (E6).

$$\frac{d}{dx}\ \frac{2Q^2}{BH_a} = -4k\frac{Q^2}{BH_a^2}$$

or

$$\frac{1}{B}\ \frac{dB}{dx} = \frac{2k}{H_a} \qquad \text{(E8)}$$

the solution to this equation is

$$B = B_0\ \exp\frac{2kx}{H_a} \qquad \text{(E9)}$$

Thus if we know the initial width, B_0, the depth of the current at the border, H_a, and the friction coefficient k we can determine the downstream width B. With the same friction coefficient, k, we find from Eq. (E9) that the downstream lateral spreading of a deep current is much less than the spreading of a shallow one. This is probably one of the reasons why steady currents in the ocean also are deep. We have ignored wind stress in this simple variant but we can see, from Eq. (E6), that a wind stress acting in the current direction helps the current keep narrow and an opposing windstress increases the spreading.

The current leaving the mixing zone in an ice covered lake was supposed to have a width approximately equal to the internal radius of deformation. For Sperillen this is about 250 m and the flowing layer is approximately 15 m deep. The stresses against the ice and against the lower resting water are both acting in the same direction so we can replace the stress coefficient in Eq. (E9) by a sum $k_{ice} + k_\eta$ where k_{ice} is the coefficient modelling the stress against the ice and k_η models the internal stress. It is believed that $k_{ice} \approx 5k_\eta \approx 5 \times 0.5 \times 10^{-3}$ are acceptable values of the friction coefficients so that $k_{ice} + k_\eta \approx 3 \times 10^{-3}$. From Eq. (E9) we then get

$$B = 250\ \exp(0.0004\,x)$$

This equation gives $B = 1000$ m for $X = 3500$ m. Thus after approximately 4 km the current has a width which is comparable to the width of the Sperillen (1 km).

Acknowledgement

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