DISCUSSION

Y. C. Ho[9]

Professor Markus and Dr. Lee should be congratulated for a very fine paper. The discusser feels that it is of definite value to modern day control engineers. This statement may appear somewhat paradoxical when one takes a cursory look at the paper. An engineer might easily argue that the questions of existence of optimal control in most practical problems are trivial. He is probably right. However, the discusser feels that the paper brings out certain aspects of the modern theory of control which are of importance to practicing engineers. It is those aspects of the problem the discusser wishes to stress further.

Professor Markus and Dr. Lee studied the conditions under which solutions to a general optimal control problem exist. First of all, such conditions are of great mathematical interest. However, they often, in addition, convey physical significance. If such conditions fail to be satisfied (i.e., no solution exists to the optimal control problem) then it is usually implied that there is something degenerate about the physical system to be controlled, or the way the control problem is formulated. Thus failure to meet such conditions is often a tip off to the engineer that a closer and more basic look at the problem is required. Furthermore, if such conditions nearly fail to be satisfied (a more probable occurrence in practice), then it is intuitively reasonable to expect that this implies that the problem is nearly degenerate and the effort required for optimal control might be prohibitive. In fact, it is not too far fetched to expect that one can derive some figures of merit which measure the amount such conditions nearly fail to be satisfied and which in turn reflect the difficulty or the cost of control. Such figures of merit can be assigned to any system for a given optimal control problem. Systems can actually be compared on this basis. The engineering significance of such results is obvious. Recently, Drs. Kalman, Narendra, and the discusser have carried out such a study for the general linear system [13] and derived results as previously mentioned. In the present paper Professor Markus and Dr. Lee have made an excellent beginning in the study of the same problem for much more general (nonlinear) systems. Of course, with more general systems, initial results are less useful directly. However, they do represent important steps toward more practical utilization. This is one of the reasons that their work is of engineering importance and should be further pursued by theorists and engineers alike.

Secondly, it should be emphasized that controllability (i.e., the existence of solution to an optimal control problem) is an *intrinsic* property of a physical system. In many ways, it is very much like the well-known property of stability of dynamic systems. They are completely specified by the parameters of the system and are independent of the initial conditions, inputs, and other variables of the system. For this reason, the study of this property of controllability of a system should deserve just as much attention in the future as stability has received in the past. It is, in the discusser's belief, a more difficult but at the same time more rewarding area since it represents virtually an unexplored area for investigation. Some fine examples of such investigation are theorems 5 and 6 of the paper where the class of second-order nonlinear systems are treated.

Additional Reference

13 R. E. Kalman, Y. C. Ho, and K. S. Narendra, "Controllability of Linear Dynamic Systems," Contributions to Differential Equations, vol. I, A University of Maryland and RIAS Publication (to appear).

R. E. Kalman[10]

This is a very interesting paper which will undoubtedly give rise to much further research in the area of optimal control.

Since the paper was written further results were obtained by the discusser concerning the controllability of linear systems [14, 15]. Using these results, one may extend and simplify the treatment of the authors contained in the section, "Domain of Controllability." This is the purpose of this discussion.

In order to avoid restating elementary notations and definitions, we shall adhere to the terminology of [14, 15] but otherwise develop the material in a self-contained way.

We consider the vector differential equation

$$dx/dt = f(t, x, u) \quad \text{where} \quad f(t, 0, 0) \equiv 0, \tag{2}$$

under the hypotheses which the authors state after equation (1) of the paper. If f is linear in x and u, (2) becomes

$$dx/dt = F(t)x + G(t)u, \tag{3}$$

where F and G are matrices.

The system (3) is said to be *completely controllable at time* t_0 if for any initial state $x_0 = x(t_0)$ there exists a measurable control function $u(t)$, defined on a finite interval $t_0 \leqq t \leqq t_1$, but not restricted in magnitude, such that $u(t)$ takes x_0 into the origin at time $t = t_1$.

If we denote the solutions of (2) or (3) explicitly by $\phi(t;\ x_0, t_0, u)$ (with $\phi(t_0;\ x_0, t_0, u) \equiv x_0$), then the preceding definition is expressed symbolically by writing

$$\phi(t_1;\ x_0, t_0, u_{t_0, x_0}) = 0.$$

(The notation u_{t_0, x_0} emphasizes the fact that the function $u(t)$ will depend, in general, both on t_0 and on x_0.)

For the linear system (3), complete controllability is characterized conveniently by the following fundamental:

Controllability Lemma. *A system* (3) *is completely controllable at time* t_0 *if and only if there is a* $t_1(t_0) > t_0$ *such that the matrix*

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)G(t)G'(t)\Phi'(t, t_0)dt \tag{4}$$

is positive definite, in which case the function u_{t_0, x_0} *may be chosen to be of the form*

$$u(t) = -G'(t)\Phi'(t_0, t)W^{-1}(t_0, t_1)x_0. \tag{5}$$

(In (4–5), $\Phi(t, t_0)$ is the *transition matrix* corresponding to $F(t)$, which is uniquely defined by $d\Phi(t, t_0)/dt = F(t)\Phi(t, t_0)$ and $\Phi(t_0, t_0) = I$ for all t_0.)

The proof may be found in [14, 15].

The advantage of this lemma is that it avoids the somewhat complicated construction of the function $u(t)$ which the authors use to prove Theorem 3 in the paper. Note also that as far as the lemma is concerned, there is no need to assume that f does not depend explicitly on t.

We say that the system (2) is *locally completely controllable at the origin at time* t_0 if there exist neighborhoods $M(t_0)$ and $N(t_0)$ of the origin in $\{x\} = R^n$ and Ω, respectively, such that every initial state x_0 in $M(t_0)$ can be taken to the origin during the interval $t_0 \leqq t \leqq t_1$ by means of a measurable control function $u(t)$ (defined on this interval) whose values are in $N(t_0)$.

Then we have the

Local Controllability Theorem. *Let* $F(t) = [\partial f^i(t, 0, 0)/\partial x^j]$ *and* $G(t) = [\partial f^i(t, 0, 0)/\partial u^j]$ *be continuous functions in* t. *Thus* (3) *is the variational equation associated with* (2). *Then* (2) *is locally completely controllable at the origin at time* t_0 *if* (3) *is completely controllable at time* t_0.

Proof. Let $t_1(t_0)$ be a number such that (4) is positive definite. The existence theorem for solutions of differential equations together with continuity arguments shows (just as in the case discussed by the authors) that the solution of (2)

$$\phi(t;\ x_0, t_0, u)$$

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exists in the interval $t_0 \leqq t \leqq t_1$, provided that $x_0 \in M(t_0)$ and $u(t) \in N(t_0)$ for all $t \in [t_0, t_1]$, where $M(t_0)$ and $N(t_0)$ are sufficiently small neighborhoods of the origin in R^n and Ω, respectively. We let u_ξ be the function

$$u(t, \xi) = G'(t)\Phi'(t_0, t)\xi. \tag{6}$$

An elementary calculation, identical with that given in the paper, shows that the jacobian $[\partial\phi^i(t_1;\ x_0, t_0, u_\xi)/\partial\xi^j]$ is equal to $\Phi(t_1, t_0)W(t_0, t_1)$, which is nonsingular by the hypothesis of the theorem.

Hence the implicit function theorem can be applied to the equation

$$\phi(t_1;\ x_0, t_0, u_\xi) = 0$$

and we can conclude that this equation has a unique solution

$$\xi = \psi(x_0)$$

($\psi(0) = 0$ and ψ continuous at 0) in some neighborhood $M'(t_0) \subset M(t_0)$ of the origin in R^n. Of course, $M'(t_0)$ is to be taken so small that the graph of $u(t, \psi(x_0))$ remain in $N(t_0)$; this can always be done since $\|u(t, \xi)\| \leqq \alpha(t_0)\|\xi\|$ by the assumed continuity of F and G in t. Q.E.D.

A completely trivial modification of the implicit function theorem applies also to the equation

$$\phi(t_1;\ x_0, t_0, u_\xi) - x_1 = 0$$

so that (restricting the neighborhood $M'(t_0)$ further, if necessary) we have the following:

Corollary. *Under the hypothesis of the theorem, there exists a neighborhood of $M''(t_0)$ of the origin in R^n such that if x_0, x_1 are in $M''(t_0)$, we can find a control u which takes $x_0 = x(t_0)$ to $x_1 = x(t_1)$.*

In paper [15], various explicit criteria are given which relate complete controllability to Hypothesis (2) in the authors' Theorem 3.

Complete controllability of the variational equation (3) is not a necessary condition, however. For instance, the first-order system

$$dx/dt = x + 2u - u^2$$

is surely locally controllable at the origin but here $G(t) \equiv 0$.

It would be desirable to know the weakest conditions which assure local complete controllability.

Finally, it is worth noting that the preceding considerations, unlike those in the paper, need not be restricted to considering a neighborhood of the origin, which is an equilibrium point of (2). This is because we allow f in (2) to be explicitly a function of time.

Let $u_0(t)$ be any function of t whose values lie in the *interior* of Ω. We can then consider the variational equation (3) along an arbitrary motion $\phi(t;\ x_0, t_0, u_0)$. That is, we let

$$F(t) = [\partial f^i(t, \phi(t;\ x_0, t_0, u_0), u_0(t))/\partial x^j]$$

and $G(t)$ is defined similarly. It is then easy to see that any state lying in a suitable small neighborhood of x_0 at time t_0 can be transferred by an allowable control to any other state lying in a suitable small neighborhood of $x_1 = \phi(t_1;\ x_0, t_0, u_0)$ at time t_1, provided the matrix $W(t_0, t_1)$ associated with the variational equation is positive definite. Thus *complete controllability of the variational equation always implies local complete controllability of the original system.*

Additional References

14 R. E. Kalman, "Contributions to the Theory of Optimal Control," Bol. Soc. Mat. Mex., 1960, pp. 102–119.

15 R. E. Kalman, Y. C. Ho, and K. S. Narendra, "Controllability of Linear Dynamical Systems," Contributions to Differential Equations, Macmillan, 1961, to appear.

J. P. LaSalle[11]

There is a proof that the temperature in hell is uniform. The proof is by contradiction and follows from the very reasonable assumption that there is at least one engineer in hell. If the temperature were not uniform, the engineer could build a refrigerating system. But then it wouldn't be hell. Similar reasoning based on the equally plausible assumption that there is a mathematician in hell leads to the conclusion that hell is nonlinear. A linear hell would be too comfortable for a mathematician. Perhaps this assures such mathematicians as Markus and Lee who can cope with nonlinearities a place in heaven. Let me add that this conclusion causes their friends to doubt the validity of my argument.

This quite elegant paper illustrates the difficulties encountered in developing a nonlinear theory. It can no longer be assumed that a bounded set of controls gives over a finite time a bounded set of responses. Even though the restraint set for the control is compact a sequence of responses can converge to a response that is not admissible. The existence of optimal control cannot be taken for granted and the theorems of this paper are a necessary first step in the development of a mathematical theory. We cannot hope to obtain structural information about optimal control unless we can identify mathematically some well-behaved classes of nonlinear systems. Without additional knowledge about the form of optimal control we cannot expect to devise practical computational methods. Global theorems are rare in nonlinear theory and it is particularly interesting to see a relation established between global stability and global controllability.

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