The properties of the pion-pion interaction are discussed by the use of dispersion relation techniques. New dispersion relations for the pion-nucleon scattering are obtained, by applying the reduction formula to the initial pion and final nucleon operators. In each of these dispersion relations the absorptive part is divided into two parts, one of which is written as an integral of the pion-pion scattering-amplitude multiplied by a certain matrix element similar to the pion-nucleon scattering-amplitude, while the other is written in terms of the phase shifts for the pion-nucleon scattering. Therefore, if the pion-pion scattering-amplitude is expressed in terms of unknown parameters, such as scattering lengths and effective ranges, their values can be determined by the use of these dispersion relations. Thus one can obtain some knowledge about the sign and the strength of the pion-pion interaction. In this way we are led to the conclusion that, in the isotopic spin state $I=0$ of the pion-pion system, the pion-pion interaction is attractive and its strength is such that the scattering length for the pion-pion scattering is of the same order of magnitude as that of the pion Compton wave length, and that, in the isotopic spin state $I=1$ of the pion-pion system, the interaction is also attractive and is of the strength at least comparable to that of the pion-nucleon interaction in the $(3, 3)$ state.

§ 1. Introduction and summary

From the existence of the strong pion-nucleon interaction, the existence of strong pion-pion interactions is theoretically expected. Such an interaction, if it exists, will play an important role in the various pion phenomena such as the pion-nucleon scattering, the electromagnetic structure of the nucleon, the multiple production of pions from a nucleon, and so on. Unfortunately we have neither experimental nor theoretical methods to attack this interaction, hence little has been known about this until now. Since a nucleon has pion cloud around it, the pion-nucleon scattering must include the effect due to the pion-pion interaction. If this effect can be separated, the analysis of the pion-nucleon scattering will give some information concerning the pion-pion interaction. Indeed, this separation is possible, if one uses new dispersion relations for the pion-nucleon scattering, which will be explained in this section.

As is well known, the amplitude for the pion-nucleon scattering can be written in the form

\[ T(p_2, k_2 ; p_1, k_1) = \bar{u}(p_2) \left[ -\delta_{\bar{I}_2 I_1} A^{(+)} - \frac{1}{2} \left[ \tau_{\bar{I}_2}, \tau_{I_1} \right] A^{(-)} \right] \]
where \( p_1 \) and \( k_1 \) are four-momenta of the initial nucleon and the pion respectively, \( p_2 \) and \( k_2 \) are those of final ones, and \( i_1 \) and \( i_2 \) are charge indices of pion before and after the scattering. \( A^{(\pm)} \) and \( B^{(\pm)} \) are functions of two independent scalars composed of the four-momenta \( p_1, p_2, k_1 \) and \( k_2 \). Usually the independent variables are taken as:

\[
\nu = -(p_1 + p_2)(k_1 + k_2)/4M, \quad \kappa^2 = (k_1 - k_2)^2/4.
\] (1.2)

Then it is proved that \( A^{(\pm)} \) and \( B^{(\pm)} \), as functions of \( \nu \), are analytic in the upper half of the complex \( \nu \)-plane. The ordinary dispersion relations\(^9\) are obtained from this fact.

In order to get alternative dispersion relations, we write the scattering amplitude as

\[
T(p_2, k_2 i_2; p_1, k_1 i_1) = -i(2\rho_0 k_{20}/M)^{1/2} \int \! d^4x \exp[-i(p_2 + k_2)x/2] \\
\times \bar{u}(p_2) \langle k_2 i_2|\theta(x)|\eta(x/2), O_i(-x/2) \rangle \\
- \hat{\sigma}(x_0)[\eta(x/2), \phi_i(-x/2)|p_1),
\] (1.3)

with the abbreviations:

\[
\eta = (\gamma \partial/\partial x + M) \psi, \quad O_i = (\mu^2 - \Box) \phi_i,
\]

where \( \psi \) and \( \phi_i \) are the Heisenberg field operators for nucleon and pion respectively. Using (1.3), and taking the Lorentz frame in which \( p_1 + k_2 = 0 \), we can prove in the usual way that, if \( A^{(\pm)} \) and \( B^{(\pm)} \) are written as functions of the two variables,

\[
\omega = -(p_1 + k_1)(p_2 + k_2)/4M \quad \text{and} \quad \sigma^2 = -(p_1 + k_2)^2/4,
\] (1.4)

they are analytic in the upper-half of the complex \( \omega \)-plane for fixed value of \( \sigma^2 \). The proof is not rigorous, as in the usual treatment, and, moreover, it can be carried out only for a restricted region of \( \sigma^2 \). However, we have found that the first few terms in the perturbation expansion of the scattering amplitude actually satisfy this analyticity property for any fixed \( \sigma^2 \). Therefore we will assume this in the following.

Denoting the total kinetic energy and the scattering angle in the center-of-mass system by \( \omega \) and \( \theta \) respectively, we have

\[
(\omega + M + \mu)^2/2 = M\omega + \sigma^2 = M\nu + \kappa^2 + (M^2 + \mu^2)/2,
\] (1.5)

\[
1 - \cos \theta = \frac{2(M\omega + \sigma^2)(M\omega - \sigma^2)}{[M\omega + \sigma^2 - (M + \mu)^2/2][M\omega + \sigma^2 - (M - \mu)^2/2]} = \frac{4\kappa^2[M\nu + \kappa^2 + (M^2 + \mu^2)/2]}{(M\nu + \kappa^2 - M\mu)(M\nu + \kappa^2 + M\mu)}.
\] (1.6)

The physical region is determined by the conditions \( \omega \geq 0 \) and \( -1 \leq \cos \theta \leq 1 \).
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Fig. 1 Physical and unphysical regions in the $\omega-\sigma^2$ plane. Various lines are determined as follows:

- $AB$: $\cos \theta = 1$
- $AC$: $\cos \theta = -1$
- $DE$: $M_{\omega} + \sigma^2 = (M + \mu)^2/2$, i.e., $\omega = 0$
- $PQ$: $M_{\omega} + \sigma^2 = M^2/2$
- $RS$: $\omega = \omega_r$, where $\omega_r$ is the resonance energy.
- $XY$: $\sigma^2 - M_{\omega} = 2\mu^2$, i.e., $\kappa^2 = -\mu^2$

This is represented in Fig. 1 by the area shaded with horizontal lines.

The absorptive part of the scattering amplitude is obtained from (1.3) by replacing $H(x)$ by $-i/2$ and by dropping the term with $\delta(x_0)$. Then, the absorptive part is the sum of the following two terms:

$$F_+ = -\left(1/2\right)(2\rho_{10}k_{30}/M)^{1/2}\int d^4x \exp[-i(\not{P}+\not{k})x/2]$$

$$\times \bar{u}(\not{p}_2)\langle k_2i|\gamma(x/2)O_{t_0}(-x/2)|p_1\rangle, \quad (1.7)$$

$$F_- = \left(1/2\right)(2\rho_{10}k_{30}/M)^{1/2}\int d^4x \exp[-i(\not{P}+\not{k})x/2]$$

$$\times \bar{u}(\not{p}_2)\langle k_2i|O_{t_0}(-x/2)\gamma(x/2)|p_1\rangle. \quad (1.8)$$

First, we consider $F_+$. Introducing the complete set of eigenstates $|n\rangle$ of the total energy-momentum operator, we may rewrite $F_+$ as

$$F_+ = -\left(1/2\right)(2\pi)^4(2\rho_{10}k_{30}/M)^{1/2}\sum_n \delta(P_n - \not{P} - \not{k})$$

$$\times \bar{u}(\not{p}_2)\langle k_2i|\gamma(0)|n\rangle \langle n|O_{t_0}(0)|p_1\rangle, \quad (1.9)$$
where \( P_n \) is the total energy-momentum of the state \(|n\rangle\). Only the intermediate states with nucleon number one give nonvanishing contributions to \( F_+ \), of which the total mass is \([-\left(p_1+k_1\right)^2]\). Among the states with nucleon number one, only the single-nucleon state has the discrete mass \( M \), and all the others have masses \( M_n \) continuously varying in the region \( M_n \leq M+\mu \). We denote the contribution of the single-nucleon intermediate state by \( F_{+}^{(s)} \) and the rest by \( F_{+}^{(+)} \), writing

\[
F_+ = F_+^{(s)} + F_+^{(+)} ,
\]

then \( F_+^{(s)} \) is different from zero at \(-\left(p_1+k_1\right)^2=M^2\), and \( F_+^{(+)} \) does not vanish, only when \(-\left(p_1+k_1\right)^2\geq(M+\mu)^2\). \( F_+^{(p)} \) is easily calculated, and is given as

\[
F_+^{(p)} = \frac{\pi g^2}{2M} \delta(\omega+\sigma^2/M-M/2)
\]

\[\times \bar{u}(p_s)(i/2)\gamma(k_1+k_2)\left[ \delta_{\tau_3\tau_4} + (1/2)[\tau_{\tau_3\tau_4}] \right] u(p_i),\]

where \( g \) is the renormalized and rationalized pseudoscalar coupling constant. This is different from zero only on the line \( PQ \) in Fig. 1. The region in which \( F_+^{(+)} \) is different from zero is determined by the condition

\[
\omega \geq -\frac{(M+\mu)^2-2\sigma^2}{2M},
\]

because \(-\left(p_1+k_1\right)^2=2(M+\omega+\sigma^2\rangle\). This region is to the right side of the line \( DE \) in Fig. 1. Especially when point \((\omega, \sigma^2)\) lies in the physical region, \( F_+^{(+)} \) can be written as

\[
F_+^{(+)} = \bar{u}(p_s) \left[ -\delta_{\tau_3\tau_4} \text{Im} A^\text{(+)}(\omega, \sigma^2) + (1/2)[\tau_{\tau_3\tau_4}] \text{Im} A^\text{(-)}(\omega, \sigma^2) \right] u(p_i).
\]

Moreover, the imaginary parts can be expanded in the partial waves. Lehmann showed that this partial wave expansion is valid in an area larger than the physical region. This region is shown in Fig. 1 by the area shaded with vertical lines. More strictly speaking, \( F_+^{(+)} \) has the expression obtained by omitting the single-nucleon intermediate state from (1·9). By virtue of this expression one can show, following Lehmann, that (1·12) holds in the above-mentioned region provided that in the right-hand side \( \text{Im} A^\text{(±)} \) and \( \text{Im} B^\text{(±)} \) are being expanded in the partial waves. Since the outside of this region is found to be not important in our dispersion relations, we will use (1·12) for all values of \( \omega \) and \( \sigma^2 \).

Next we write \( F_+ \) in the form similar to (1·9). Since operator \( \gamma \) lowers the nucleon number by unity, the intermediate states must have nucleon number 0, and the total mass of these states is \(-\left(k_1-k_2\right)^2\). Moreover, among the intermediate states consisting of pions, only the intermediate states with even number of pions contributes to \( F_+ \). This follows from the invariance under charge conjugation and a rotation around the 2 axis in the isotopic spin space. Therefore, the
least massive state is a two-pion state, and consequently $F_-$ differs from zero only when $-(k_1-k_2)^2 \geq 4\mu^2$. $F_-$ can be written analogously to (1.1) in the following form:

\[
F_- = \bar{u}(p_2) \left\{ - \delta_{\epsilon_1 \epsilon_2} u^{(+)}(\omega, \sigma^2) - (1/2) \{ \tau_{\epsilon_1} \tau_{\epsilon_2} \} u^{(-)}(\omega, \sigma^2) \right. \\
+ (i/2) \gamma(k_1 + k_2) \{ \delta_{\epsilon_1 \epsilon_2} v^{(+)}(\omega, \sigma^2) + (1/2) \{ \tau_{\epsilon_1} \tau_{\epsilon_2} \} v^{(-)}(\omega, \sigma^2) \} u(p_1). \tag{1.13}
\]

Since $-(k_1-k_2)^2 = 2(\sigma^2-M\omega)$, $u^{(+)}$ and $v^{(\pm)}$ are different from zero only when $\omega \leq (\sigma^2-2\mu^2)/M$. In Fig. 1, this region is to the left side of the line $XY$.

From the above consideration, we have the following dispersion relations$^9$:

\[
\text{Re} A^{(\pm)}(\omega, \sigma^2) = (P/\pi) \int_{(\sigma^2-2\epsilon)^2-2\epsilon^2)/M}^{(M+\epsilon)^2-2\epsilon^2)/M} d\omega' \text{Im} A^{(\pm)}(\omega', \sigma^2)/(\omega'-\omega) \\
+ (1/\pi) \int_{-\infty}^{0} d\omega' u^{(\pm)}(\omega', \sigma^2)/(\omega'-\omega), \tag{1.14}
\]

\[
\text{Re} B^{(\pm)}(\omega, \sigma^2) = -g^2 [1/(2M\omega + 2\sigma^2-M^2) \pm 1/(4\sigma^2-M^2-2\mu^2)] \\
+ (P/\pi) \int_{(\sigma^2-2\epsilon)^2-2\epsilon^2)/M}^{(M+\epsilon)^2-2\epsilon^2)/M} d\omega' \text{Im} B^{(\pm)}(\omega', \sigma^2)/(\omega'-\omega) \\
+ (1/\pi) \int_{-\infty}^{0} d\omega' v^{(\pm)}(\omega', \sigma^2)/(\omega'-\omega). \tag{1.15}
\]

Here, no subtraction has been made. The first term between brackets in the right-hand side of (1.15) comes from $F_+^{(p)}$. The second term does not depend on $\omega$. We have added this term so that (1.15) holds in the lowest order perturbation theory. But it may be doubtful that the coefficient of this term is really the square of a renormalized coupling constant, $g^2$. To avoid this ambiguity, we will carry out one subtraction in the above dispersion relations.

Let us consider the last terms in (1.14) and (1.15). Owing to the denominator of the integrand, $u^{(\pm)}$ and $v^{(\pm)}$ at small values of $|\omega|$ are important in these terms, and this becomes surer if the subtraction is carried out. Now, for small values of $|\omega|$, the most important contribution to $F_-$ comes from the two-pion intermediate states. Indeed, for $2\mu^2-\sigma^2 \leq -M\omega < 8\mu^2-\sigma^2$, only the two-pion intermediate states contribute to $F_-$. (For $-M\omega \geq 8\mu^2-\sigma^2$, contribution of the four-pion intermediate state appears.) Therefore, we approximate $F_-$ by the contribution of two pion intermediate states. Then we have

\[
F_-(p_2, k_2 i_2; p_1, k_1 i_1) \\
= (1/4) (2\pi)^{-2} (2p_0 k_{0\beta}/M)^{1/2} \sum_{i_1'i_2'} dk_1' dk_2' \delta^4(k_2-k_1-k_1'-k_2') \times \langle k_2 i_2 | O_{\alpha}(0) | k_1'i_1', k_2'i_2' \rangle \bar{u}(p_2) \langle k_1'i_1', k_2'i_2' | \gamma(0) | p_1 \rangle. \tag{1.16}
\]
Here, the integral over intermediate states should be understood to be the average of the integral over “in” states and the one over “out” states. The matrix element of $O_i(0)$ in (1.16) is the scattering amplitude for the pion-pion scattering, and is written by the requirement of Lorentz invariance in the following form:

$$
\langle k_2 i_2 | O_i(0) | k_1' i_1', k_2' i_2' \rangle_{in} \rangle
= 32\pi (2k_2 k_{2'} k_1) \frac{-1/2}{(8 k_2 k_{2'} k_1)} \left[ L_0(-\kappa^2, \xi) (1/3) \delta_{i_1 i_2} \delta_{i_{1'} i_{2'}}, \right.

+ L_1(-\kappa^2, \xi) (1/2) (\delta_{i_1 i_1'} \delta_{i_{1'} i_{2'}} - \delta_{i_1 i_2} \delta_{i_{1'} i_2'})

+ L_2(-\kappa^2, \xi) [(1/2) (\delta_{i_1 i_1'} \delta_{i_{1'} i_{2'}} + \delta_{i_1 i_2} \delta_{i_{1'} i_2'})

- (1/3) \delta_{i_1 i_2} \delta_{i_{1'} i_{2'}}]\right],

(1.17)

where

$$
\kappa^2 = (k_1' + k_2')^2/4, \quad \xi = (k_1' + k_2' - 2k_2)(k_1' - k_2')/4.

(1.18)

Here, $L_i$ is a Lorentz invariant scattering amplitude in the state where the total isotopic spin is $I$. This is related to the pion-pion phase shifts $\delta_{I,i}$ as

$$
L_i(\omega^2, k^2 \cos \theta) = (\omega / k) \sum_{I} (2l + 1) \exp(i \delta_{I,i}) \sin \delta_{I,i} P_l(\cos \theta),

(1.19)

where $\omega$ and $k$ are energy and magnitude of momentum of either pion and $\theta$ is the scattering angle in the center-of-mass system. The sum over $l$ should be understood as the sum over even $l$ for even $I$ and the sum over odd $l$ for odd $I$. This follows from the invariance of the matrix element (1.17) under the exchange of $k_1' i_1'$ and $k_2' i_2'$.

It is easily found that $L_2$ does not contribute to $F_-$. This follows from the fact that the system of a nucleon and an antinucleon cannot have a total isotopic spin larger than one.

Next we assume that all the terms with $l > 1$ can be neglected in (1.19). The assumption may be reasonable from the following reason. The range of the pion-pion interaction is considered to be $(2\mu)^{-1}$ at most. Then the condition under which the $d$-wave plays an important role in the scattering is that $k \approx 4\mu$, which is also written as $M \omega \lesssim \sigma^2 - 34\mu^2$. From this, we see that for small values of $|\omega|$, the $d$-wave can be neglected more safely than the contribution of four-pion intermediate states which has already been neglected. For $s$- and $p$-waves, we write

$$
L_0(\omega^2) = (\omega / k) \exp(i \delta_0) \sin \delta_0;

L_1(\omega^2) = \mu^2 (\omega / k^3) \exp(i \delta_1) \sin \delta_1,

(1.20)

(1.21)

where we have written $\delta_0$ and $\delta_1$ in place of $\delta_{0,0}$ and $\delta_{1,1}$ respectively. Then $L_0$ and $L_1$ become

$$
L_0(-\kappa^2, \xi) = L_0(-\kappa^2), \quad L_1(-\kappa^2, \xi) = 3\mu^2 L_1(-\kappa^2) \xi.

(1.22)

At low energies $L_0$ and $L_1$ can be characterized by a few parameters such as
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the scattering lengths and the effective ranges. Therefore, if the matrix element of $\gamma(0)$ in (1.16) has been calculated, and if $F_-$ and, consequently, $u(\pm)$ and $v(\pm)$ can be expressed in terms of these parameters, then we can estimate these parameters numerically by means of our dispersion relations (1.14) and (1.15), since all the terms in (1.14) and (1.15) other than those containing $u(\pm)$ and $v(\pm)$ can be computed by using pion-nucleon data. Numerical calculations will be given in § 3. The results obtained in this way lead us to the following conclusions: the pion-pion interaction is attractive for the both states $I=0$ and $I=1$. For the state $I=0$, the scattering length of the pion-pion scattering is of the same order as that of the pion Compton wave length. For the state $I=1$, strength of the interaction is at least comparable to that of the pion-nucleon interaction in the (3, 3) state.

The matrix element of $\gamma(0)$ in (1.16) can be written formally, as the pion-nucleon scattering amplitude, but it must be noticed that the matrix element lies in the unphysical region. As has been done in the problem of the electromagnetic structure of nucleon,4) the scattering amplitude in the unphysical region can be related to that of the physical region by virtue of the ordinary dispersion relations3) and the expansion in the partial waves. In § 2, we first calculate the matrix element in question in this method. In this calculation, however, as has already been pointed out,4) we have to use the partial-wave expansion in the region where its convergence has not been proved. To avoid such an unjustified procedure, we next make in § 2 a calculation in the static theory. We shall see in § 3 that both calculations yield qualitatively the same results.

§ 2. Calculations of $u(\pm)$ and $v(\pm)$

By the requirement of invariance, the matrix element of $\gamma(0)$ in (1.16) can be written in the form

$$\bar{u}(p_3) \langle k'_3 i'_3, k'_4 i'_4 \text{ out} | \gamma(0) | p_1 \rangle$$

$$= - (1/2) \left( p_{10} k_{10} k_{20} / M \right)^{-1/2} \bar{u}(p_3) \left\{ - \delta_{i'_3 i'_4} A^{(+)}(\nu', \kappa^2) 
- (1/2)[\tau_{\nu'} \tau_{i'_4}] A^{(-)}(\nu', \kappa^2) 
+ (i/2) \gamma(k'_3 - k'_4) \delta_{i'_3 i'_4} B^{(+)}(\nu', \kappa^2) 
+ (1/2) \left[ \tau_{i'_4} \tau_{i'_3} B^{(-)}(\nu' \kappa^2) \right] u(p_4), \right.$$  

(2·1)

where

$$\nu' = - (p_1 + p_2) (k'_3 - k'_4) / 4M$$

and $\kappa^2$ is given by (1.18). If the "out" state in the left-hand side is changed into "in" state, $A^{(\pm)}$ and $B^{(\pm)}$ in the right-hand side are changed into their complex conjugates. This follows from the invariance under charge conjugation. For the same reason, if the "in" state in the left-hand side of (1.17) is changed into the "out" state, $L_i$'s in the right-hand side are changed into their complex conjugates.

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Taking account of these facts, we substitute (2·1) and (1·17) with (1·22) into
(1·16). Transforming the three-dimensional integrals into four-dimensional ones
by adding the delta-function $\delta (k_2^2 + \mu^2)$ to the integrand, and carrying out the inte­
gration over $k'_1 + k'_2$, we have

\[
F_\nu(p_3, k_2 i_2 ; p_1, k_1 i_1) = - \frac{1}{(1/\pi)} \int d^4Q \theta (k_\infty - k_\infty + 2Q_\infty) \theta (k_\infty - k_\infty - 2Q_\infty) \\
\times \delta (Q^2 + \kappa^2 + \mu^2) \delta [-(k_2 - k_1)Q] \\
\times \bar{u}(p_2) \left\{ - \delta_{\nu\kappa} \text{Re} \left( l_{\nu}^\dagger (-\kappa^2) A^{(+)} (\nu', \kappa^2) \right) \\
+ \frac{1}{2} \left[ \tau_{\nu\kappa} \text{Re} \left( l_{\nu}^\dagger (-\kappa^2) B^{(-)} (\nu', \kappa^2) \right) \right] \\
+ \frac{1}{2} \left[ \tau_{-\nu\kappa} \text{Re} \left( l_{\nu}^\dagger (-\kappa^2) B^{(-)} (\nu', \kappa^2) \right) \right] \right\} u(p_1),
\]

(2·2)

where

\[
\kappa^2 = (k_2 - k_1)^2/4, \quad \nu' = - (p_1 + p_2)Q/2M.
\]

It follows from (2·2) that

\[
\nu^{(+)}(\omega, \sigma^2) = 0.
\]

This is shown as follows: comparing (2·2) with (1·13) we see that only the
term with factor $B^{(+)}$ in (2·2) can contribute to $\nu^{(+)}$. This term contains $\gamma$-matrices
only through the factor $i\gamma Q$, and depends on $Q$ only through $i\gamma Q$ and $(p_1 + p_2)Q$.
Taking a particular Lorentz frame in which $k_2 - k_1 = 0$, we carry out the integration
over $Q_\infty$. Then $Q_\infty$ in the integrand becomes zero on account of the delta-function
$\delta [(k_2 - k_1)Q]$, hence $i\gamma Q$ and $(p_1 + p_2)Q$ become $i\gamma Q$ and $(p_1 + p_2)Q$ respectively.
Therefore, on carrying out the integration over directions of $Q$, the term under
consideration contains $\gamma$-matrices only through the factor $i\gamma (p_1 + p_2)$, which becomes
$i\gamma (p_1 + p_2)$ in the general Lorentz frame, because we have $p_1 + p_2 = 0$ in the Lorentz
frame where $k_2 - k_1 = 0$ on account of the relation $(k_2 - k_1) (p_1 + p_2) = 0$. However,
$i\gamma (p_1 + p_2)$ reduces to $-2M$ because of the Dirac equations satisfied by $u(p_1)$ and $\bar{u}(p_2)$.
Thus the term in question does not contain $\gamma$-matrices ultimately, and does
not contribute to $\nu^{(+)}$. This proves (2·3).*

As was mentioned in § 1, (2·2) has been obtained under the following
approximations: contributions of all the intermediate states other than the two-pion
states are neglected in $F_\nu$ and all the partial waves higher than the $p$-wave in the
pion-pion scattering amplitude are neglected. Therefore (2·3) is a consequence of
these approximations. Since (2·3) makes the integral containing $\nu^{(+)}$ in (1·15)

* Though we are considering the unphysical region $(k_2 - k_1)^2 < 0$, the "real" Lorentz frame
in which $k_2 - k_1 = 0$ does not exist, unless $\omega$ and $\sigma^2$ satisfy a certain condition. However, we have
verified for the first few terms in the perturbation expansion that the formal procedure used here
always gives the correct results.
vanish, the numerical difference between Re$B^{(+)}$ and a sum of the Born terms and the integral of Im$B^{(+)}$ in (1·15) will give an idea about the error caused by our approximation.

In order to proceed further, we must calculate $A^{(+)}$ and $B^{(+)}$ in (2·1). Perturbation theory is not adequate for our purpose, because this theory cannot describe the behavior of $s$-wave pions correctly, and consequently it will yield incorrect results particularly for $A^{(+)}$ and $B^{(+)}$. We therefore adopt the following two methods, the one is the use of the dispersion relations, and the other, the static theory.

a) Calculation with the Dispersion Relations

The matrix element (2·1) can be regarded as the amplitude for an unphysical pion-nucleon scattering, in which a pion with energy-momentum $-k'$ is scattered into the one with energy-momentum $k''$. Therefore, it is formally proved that $A^{(±)}$ and $B^{(±)}$ in (2·1) satisfy the dispersion relations for the pion-nucleon scattering given by Chew et al. By means of these dispersion relations $A^{(±)}$ and $B^{(±)}$ can be expressed as the integrals of their imaginary parts in the region where the energy is physical but the scattering angle is unphysical. (When the variables $\nu'$ and $\kappa'$ are transformed into $\nu$ and $\cos \theta$ by the use of (1·5) and (1·6) with $\nu'$ and $\kappa'$ in place of $\nu$ and $\kappa$, this region is in which $\nu \leq 0$ but $|\cos \theta| > 1$.) In spite of the unphysical scattering angle, however, these imaginary parts can be written in terms of the phase shifts in the physical region by means of the partial-wave expansion of the former. In this way we can compute $A^{(±)}$ and $B^{(±)}$ using experimental phase shifts. This method has already been applied to the problem of the electromagnetic structure of nucleon by Chew et al. and Federbush et al.

In the present case one should be more careful about the convergence of the dispersion integrals than in the case of the nucleon structure, because we must calculate all of $A^{(±)}$ and $B^{(±)}$, while the form factors of nucleon involve $A^{(-)}$ and $B^{(-)}$ only. Therefore, we use here the subtracted dispersion relations:

\[
A^{(±)}(\nu', \kappa') = A^{(±)}(\nu_0, \kappa_0) + (1/\pi) \int_{\nu_0}^{\infty} d\nu'' \text{Im} A^{(±)}(\nu'', \kappa') \\
\times \left\{ 1/(\nu''-\nu) \mp 1/(\nu''+\nu) - 1/(\nu''-\nu_0) \mp 1/(\nu''+\nu_0) \right\} ; \quad (2·4)
\]

\[
B^{(±)}(\nu', \kappa') = B^{(±)}(\nu_0, \kappa_0) \\
+ (g^2/2M) \left\{ 1/(\nu_0'-\nu') \mp 1/(\nu_0'+\nu') - 1/(\nu_0'-\nu_0) \mp 1/(\nu_0'+\nu_0) \right\} \\
+ (1/\pi) \int_{\nu_0}^{\infty} d\nu'' \text{Im} B^{(±)}(\nu'', \kappa') \\
\times \left\{ 1/(\nu''-\nu') \mp 1/(\nu''+\nu') - 1/(\nu''-\nu_0) \mp 1/(\nu''+\nu_0) \right\} ; \quad (2·5)
\]

where
In the present case, it is not necessary to add an infinitesimal imaginary part to each of the denominators in these formulas, because, on substituting them into (2·2) and combining terms appropriately, all the denominators become positive definite. In fact, this is true only when \( \kappa^2 + M^2 > 0 \), i.e., \( \omega > (\sigma^2 - 2M^2)/M \). This condition is the one under which the intermediate states including nucleon-antinucleon pair does not contribute to \( F_\pi \). We may consider this condition to hold, since in later calculations we shall cut off the integrals of \( u(\pm) \) and \( v(\pm) \) at \( \omega' = (\sigma^2 - 2M^2)/M \).

It should be noticed that the meaning of \( \text{Re} \) in (2·2) must be reinterpreted as to take real part of the whole expression in order to obtain correct results; otherwise \( u(\pm) \) and \( v(\pm) \) would become imaginary. It is verified that in the lowest order perturbation theory correct results are obtained, indeed, by this treatment. Then, it is found that \( u(\pm) \) and \( v(\pm) \) contain only the real parts of \( \lambda_0 \) and \( \lambda' \).

We expand \( A(\pm) \) and \( B(\pm) \) in (2·4) and (2·5) in the partial waves using the formulas given in reference 1), and neglect all the partial-wave amplitudes other than the (3·3) amplitude, for which, following Federbush et al., we make the narrow-resonance approximation:

\[
\text{Im} f_{33} = \frac{g^2}{6M^2} \left( \frac{(M \nu'' + \kappa^2)^2 - M^2 \mu^2}{2M \nu'' + 2\kappa^2 + M^2 + \mu^2} \right) \times \{ 1 + (\omega_r/M) \} \delta(2M \nu'' + 2\kappa^2 + \mu^2 - 2M \omega_r - \omega_r^2),
\]

where \( \omega_r \) is the resonance energy in the center-of-mass system. We take \( \omega_r = 2\mu \).

We also expand \( A(\pm)(\nu_0, \kappa^2) \) and \( B(\pm)(\nu_0, \kappa^2) \) in (2·4) and (2·5) in the partial waves. Because of \( \kappa^2 \neq 0 \), all partial waves appear in these expansions. However, we neglect all of them other than \( s- \) and \( p- \)waves, and for the latter phase shifts we use experimental values reported by Puppi at the CERN Conference in 1958.

Here the final expressions of \( u(\pm) \) and \( v(\pm) \) are not written, but the numerical results will be given in § 3.

b) Calculation in the static theory

Chew and Low and Miyazawa have developed techniques by means of which various matrix elements are related to the pion-nucleon phase shifts under the assumption that nucleon can be regarded as a fixed and extended source of pion field and that pion-nucleon interaction is described by a particular Hamiltonian. Extending the basic idea of these authors, we now calculate the matrix element of \( \gamma(0) \) appearing in (1·16).

From the definition of \( \gamma(x) \) we have in the nonrelativistic approximation

\[
\tilde{u}(p_s, f) \gamma(x, t) = \chi_f(-i \partial/\partial t + M) \phi(x, t),
\]

where \( f \) is the spin index which has been suppressed up to this time, and \( \chi_f \) is the two-component spin eigenfunction of the final nucleon. From this and
Properties of the Pion-Pion Interaction

\[ \psi(x, t) = \exp(iHt)\psi(x, 0) \exp(-iHt) \]

\((H\) is the total Hamiltonian\), it follows that

\[ \bar{u}(p_s, f) \langle k_1', k_2' \text{ out} | \eta(0) | p_1, i \rangle = (k_{10} + k_{20}) \zeta_f^* \langle k_1', k_2' \text{ out} | \psi(0) | p_1, i \rangle, \]

where \(i\) is the spin index of the initial nucleon, and the charge indices of pions have been suppressed.

We expand the Heisenberg field operators at \(t=0\) in the Fourier series as follows:

\[ \phi(x, 0) = \sum_{p,j} A_{p,j} \zeta_j \exp(i p x), \]

\[ \phi(x, 0) = \sum_{k} \left( \frac{2\omega_k}{2\pi} \right)^{1/2} [a_k \exp(ikx) + a_k^* \exp(-ikx)], \]

\[ \phi(x, 0) = -i \sum_{k} \left( \frac{\omega_k}{2} \right)^{1/2} [a_k \exp(ikx) - a_k^* \exp(-ikx)]. \]

We call the eigenstates of the occupation numbers \(A_{p,j}^* A_{p,j}\) and \(a_k^* a_k\) as the bare-particle states and write them as \(\ldots \text{bare}\). For example,

\[ |p, j \text{ bare} \rangle = A_{p,j}^* |0\rangle, \quad |p, k \text{ bare} \rangle = A_{p,k}^* a_k^* |0\rangle, \]

\[ |k_1, k_2 \text{ bare} \rangle = a_k^* a_k^* |0\rangle \quad \text{and so on,} \]

where \(|0\rangle\) is the vacuum state for the bare particles.

By the substitution of the Fourier expansion of \(\phi\) the matrix element in the right-hand side of (2.6) is rewritten as

\[ \langle k_1', k_2' \text{ out} | \psi(0) | p_1, i \rangle = \langle k_1', k_2' \text{ out} | A_{p',p} | p_1, i \rangle, \]

where \(p = p_1 - k_1' - k_2'\). We approximate the right-hand side of this by

\[ (1/2) \sum_{k_1'' k_2''} \langle k_1' k_2' \text{ out} | k_3'' \text{ bare} \rangle \langle k_1'' k_2'' \text{ bare} | A_{p',p} | p_1, i \rangle. \]

Then, because of the commutativity between \(A_{p'}\) and \(a_k^*\) the second matrix element becomes \(\langle p_f; k_1'', k_2'' \text{ bare} | p_1 i \rangle\), which is more simply written as \(\langle f; k_1'', k_2'' \text{ bare} | i \rangle\) in the static theory. Thus we have

\[ \bar{u}(p_s, f) \langle k_1', k_2' \text{ out} | \eta(0) | p_1, i \rangle = (k_{10} + k_{20}) \langle f; k_1', k_2' \text{ bare} | i \rangle \]

\[ + (k_{10} + k_{20}) (1/2) \sum_{k_1'' k_2''} \langle k_1', k_2' \text{ out} | k_1'', k_2'' \text{ bare} \rangle \]

\[ - \delta_{k_1', k_1''} \delta_{k_2', k_2''} - \delta_{k_1', k_2''} \delta_{k_2', k_1''} \langle f; k_1'', k_2'' \text{ bare} | i \rangle. \]

The appearance of the last term is due to pion-pion interaction. In the static theory, however, this term can be dropped for the following reason: It will be found later that the matrix element \(\langle f; k_1', k_2' \text{ bare} | i \rangle\) is proportional to \((k_{10} + k_{20})^{-1},\)
so that the factor \((k_{10}'+k_{20}')\) in the right-hand side of (2.7) disappears in the first term but survives in the second term. After the substitution of (2.7) into (1.16), this factor becomes \(k_{20}-k_{10}\) on account of the delta function. But in the static theory the energy conservation yields \(k_{20}-k_{10}=0\). Therefore, we may write

\[
\bar{u}(p_2, f)\langle k_1', k_2' \text{ out} | \gamma(0) | p_1, i \rangle = (k_{10}'+k_{20}')\langle f; k_1', k_2' \text{ bare} | i \rangle. \tag{2.8}
\]

For the matrix element in the right-hand side we further make the following approximation:

\[
\langle f; k_1', k_2' \text{ bare} | i \rangle = \langle f; k_2' \text{ bare} | a_{s,v} | i \rangle \\
\approx \sum_{j} \langle f; k_2' \text{ bare} | j \rangle \langle j | a_{s,v} | i \rangle \\
+ \sum_{j,k} \langle f; k_2' \text{ bare} | j ; k \text{ in} \rangle \langle j ; k \text{ in} | a_{s,v} | i \rangle, \tag{2.9}
\]

where \(| j \rangle\) is the single physical-nucleon state, and \(| j ; k \text{ in} \rangle\) is the scattering state of a nucleon with spin \(j\) and a pion with momentum \(k\). The sum in the last term should be understood as half the sum over "in" states and "out" states, though we have not written this explicitly. (2.9) is so-called the one-meson approximation.

We assume that only \(s\) - and \(p\)-wave pions interact with the nucleon. To avoid too lengthy an explanation, however, we leave the \(s\)-wave pions out of consideration, until their effects are added to the final results.

We begin with the consideration of the first matrix element in the last line of (2.9). We denote the total Hamiltonian by \(H\), and the free Hamiltonian at \(t=0\) by \(H_0\). Then \(| j ; k \text{ in} \rangle\) and \(| f; k_2' \text{ bare} \rangle\) are eigenstates of \(H\) and \(H_0\) respectively. From this fact it follows that

\[
\langle f; k_2' \text{ bare} | j ; k \text{ in} \rangle \\
= \langle f; k_2' \text{ bare} | j ; k \text{ bare} \rangle + (k_0-k_{20}'+i\epsilon)^{-1} \\
\times \langle f; k_2' \text{ bare} | [H'+H'(k_0-H+i\epsilon)^{-1}H'] | j ; k \text{ bare} \rangle. \tag{2.10}
\]

where \(H'\) is the interaction Hamiltonian. On the other hand the corresponding \(S\)-matrix element has the form:

\[
\langle f; k_2' \text{ out} | j ; k \text{ in} \rangle \\
= \langle f; k_2' \text{ out} | j ; k \text{ bare} \rangle - 2\pi i \delta(k_0-k_{20}') \\
\times \langle f; k_2' \text{ bare} | [H'+H'(k_0-H+i\epsilon)^{-1}H'] | j ; k \text{ bare} \rangle. \tag{2.11}
\]

In terms of the phase shifts this is also written as

\[
\langle f; k_2' \text{ out} | j ; k \text{ in} \rangle = \delta_{r,j} \delta_{v,k} + 2\pi i \delta(k_0-k_{20}') \\
\times 2\pi (k_0k_{20}')^{-1/2} \sum_{a} h_{a0}(k_0) P_{a0}(k_{20}', k) \chi_j. \tag{2.12}
\]
with the abbreviation
\[ h_{\alpha\beta}(k_0) = (k_0^2 - p^2)^{-3/2} \exp[i\eta_{\alpha\beta}(k_0)] \sin \eta_{\alpha\beta}(k_0), \]
\[ P_{11}(k', k) = (1/3) \tau_{\alpha\beta} \tau_k (\sigma \cdot k') (\sigma \cdot k), \]
\[ P_{12}(k', k) = (1/3) \tau_{\alpha\beta} \tau_k [3(k' \cdot k) - (\sigma \cdot k') (\sigma \cdot k)], \]
\[ P_{31}(k', k) = (\partial_{\alpha\beta} - (1/3) \tau_{\alpha\beta} \tau_k) (\sigma \cdot k') (\sigma \cdot k), \]
\[ P_{33}(k', k) = (\partial_{\alpha\beta} - (1/3) \tau_{\alpha\beta} \tau_k) [3(k' \cdot k) - (\sigma \cdot k') (\sigma \cdot k)]. \]

It is seen from (2·10) that the matrix element in the last term depends on \( k' \) only through the factor \( k_0' k_0 \), so long as \( |k'| \) are much larger than the range of the pion-nucleon interaction. Therefore, if we compare (2·10) with (2·11), and (2·11) with (2·12), we have, to a good approximation,
\[ \langle f; k' \text{bare} | j; k \text{in} \rangle = \delta_{\beta\delta} \delta_{\alpha\eta} + (k_0' - k_0 - i\varepsilon)^{-1} \times 2\pi (k_0) P_{33}(k', k) \phi_j. \] (2·13)

In the same way it is found that, if \( |j; k \text{in} \rangle \) is replaced with \( |j; k \text{out} \rangle \) in the left-hand side of (2·13), \( h_{\alpha\beta} \) is replaced with \( h_{\alpha\beta}^* \) in the right-hand side.

Next we turn to the last matrix element in (2·9). This is written by means of the well known technique\(^6\) as
\[ \langle j; k \text{out} | a_{k'\alpha} | i \rangle = -(k_0 + k_0')^{-1} \langle j; k \text{out} | V_{k'} | i \rangle \] (2·14)
with
\[ V_{k'} = -[H', a_k]. \]

Here we assume that \( H' \) is the same as in the Chew-Low theory\(^6\). Then we have \( V_{k'}^* = -V_k \), so that the matrix element in the right-hand side of (2·14) becomes essentially the scattering amplitude. Thus we have
\[ \langle j; k \text{out} | a_{k'\alpha} | i \rangle = -(k_0 + k_0')^{-1} 2\pi (k_0 k_0')^{-1/2} \phi_j^* \sum_{\alpha\beta} h_{\alpha\beta}(k_0) P_{33}(k, k') \phi_i. \] (2·15)

It follows from the invariance under time reversal that, if \( \langle j; k \text{out} | \) is replaced with \( \langle j; k \text{in} \rangle \) in the left-hand side, \( h_{\alpha\beta} \) is replaced with \( h_{\alpha\beta}^* \) in the right-hand side.

The first term in the right-hand side of (2·9) can be written as
\[ \sum_j \langle f \text{bare} | a_{k'\alpha} | j \rangle \langle j | a_{k\beta} | i \rangle. \]

Here we approximately replace the bare-nucleon state \( \langle f \text{bare} \rangle \) with the corresponding physical-nucleon state \( \langle f \rangle \). Then this term is easily calculated by the well-known method\(^6\), and we obtain the result
\[ \sum_j \langle f' \mid k'_{\text{bare}} \rangle \langle j \mid a_{ki} \mid i \rangle = -2\pi (k'_{20} k'_{20})^{-3/2} (f^2/\mu^2) \chi^*_j (\sigma \cdot k'_s) (\sigma \cdot k'_t) \tau_{k'_s} \tau_{k'_t} \xi, \] (2.16)

where \( f \) is the renormalized and nonrationalized pseudovector coupling constant.

Thus we have expressed all terms in the right-hand side of (2·9) in terms of the quantities known experimentally. If the Chew-Low equation \(^6\) for \( h_{\pi \pi} \) is used, the result obtained by substituting (2·13), (2·15) and (2·16) into (2·9) is found to be proportional to \((k'_{10} + k'_{20})^{-1}\) as was mentioned previously. Furthermore, this result is found to be symmetric in \( k'_t \) and \( k'_s \), though they have not been treated symmetrically during the calculation. We substitute this result into (2·8), and neglect all \( h_{\pi \pi} \)'s other than \( h_{20} \). Further, we add the terms due to the \( s \)-wave pions. Then we have finally

\[ \bar{u}(p, f) \langle k'_s i'_t, k'_s i'_s \text{ out} \mid \tau(0) \mid p_1, i \rangle = -2\pi (k'_{10} k'_{20})^{-3/2} \chi^*_j \left[ \delta_{i't} \right. \left. - \frac{1}{3} (a_1 + 2a_3) \right. \\
+ (k'_s \cdot k'_t) \left( B_+ (k'_{10}, k'_{20}) + 4H_+ (k'_{10}, k'_{20}) \right) \\
+ \frac{i}{2} (\sigma \cdot (k'_s \times k'_t)) \left( B_- (k'_{10}, k'_{20}) - 2H_- (k'_{10}, k'_{20}) \right) \\
+ \frac{(1/2) [\tau_{i't}, \tau_{i's}]}{(1/6) (a_1 - a_3) (k'_{10} - k'_{20})} \\
+ \left. \frac{(k'_s \cdot k'_t)}{(B_- (k'_{10}, k'_{20}) - 2H_- (k'_{10}, k'_{20}))} \right] \chi, \] (2.17)
where

\[ u^{(+)}(\omega, \sigma^2) = -4\pi \text{Re} l_0(-\kappa^2)[(\kappa^2 + \mu^2)/\kappa^2]^{1/2} \left\{ (1/3) (a_1 + 2a_2) + 2\pi \frac{f_2}{\mu^2} \sqrt{-\kappa^2 - \mu^2} \right\} \]

\[ \cdot \int_{\omega_{\text{max}}}^\omega d\omega_k \omega_k |h_{\text{33}}(\omega_k)|^2 \left\{ \frac{\omega_k^2 - 2\kappa^2 - \mu^2}{\omega_k^2 - \kappa^2 - \mu^2} \tan^{-1} \frac{1}{\omega_k^2 - \kappa^2 - \mu^2} \right\}, \]

\[ \cdot \int_{\omega_{\text{max}}}^\omega d\omega_k \omega_k |h_{\text{33}}(\omega_k)|^2 \left\{ \frac{\omega_k^2 - 2\kappa^2 - \mu^2}{\omega_k^2 - \kappa^2 - \mu^2} \tan^{-1} \frac{1}{\omega_k^2 - \kappa^2 - \mu^2} \right\}, \]

\[ u^{(-)}(\omega, \sigma^2) = -(2M)^{-1} (M\omega + 3\sigma^2 - M^2 - \mu^2) v^{(-)}(\omega, \sigma^2), \]

\[ v^{(+)}(\omega, \sigma^2) = 0, \]

where

\[ \kappa^2 = (M\omega - \sigma^2)/2. \]

We have carried out the integration over \( \omega_k \) making the narrow resonance approximation for \( |h_{\text{33}}(\omega_k)|^2 \) as in the case a). It should be noticed that in the present case one must cut off the integrals containing \( u^{(\pm)} \) and \( v^{(\pm)} \) in (1·14) and (1·15) at the lower limit \((\sigma^2 - 2\omega_{\text{max}}^2)/M\) in accordance with the cut-off in the static theory. We take \( \omega_{\text{max}} = 6\mu \). But the results are quite insensitive to the cut-off energy, since we use the subtracted dispersion relations.

Numerical values of the parameters are taken to be

\[ a_1 \mu = 0.173, \quad a_2 \mu = -0.110, \quad f_2^2 = 0.08. \]

The numerical results will be given in § 3.

§ 3. Numerical results and discussions

In this section we attempt to estimate the scattering amplitudes \( l_0 \) and \( l_1 \) for the pion-pion scattering, using our dispersion relations (1·14) and (1·15). We first introduce the following abbreviations:

\[ \text{Re}[A^{(+)}(\omega, \sigma^2), B^{(+)}(\omega, \sigma^2), B^{(-)}(\omega, \sigma^2)] = R_1, R_2, R_3, R_4(\omega, \sigma^2); \]

\[ (P/\pi) \int_{(M+\mu^2-2\sigma^2)/2M}^{\infty} d\omega'/(\omega'-\omega) \text{Im}[A^{(+)}(\omega, \sigma^2), B^{(+)}(\omega, \sigma^2), B^{(-)}(\omega', \sigma^2)] \]

\[ = J_1, J_2, J_3, J_4(\omega, \sigma^2); \]

\[ (1/\pi) \int_{(\sigma^2-2\mu^2)/M}^{-\infty} d\omega'/(\omega'-\omega) [u^{(+)}(\omega', \sigma^2), u^{(-)}(\omega', \sigma^2), v^{(+)}(\omega', \sigma^2), v^{(-)}(\omega', \sigma^2)] \]

\[ = Q_1, Q_2, Q_3, Q_4(\omega, \sigma^2); \]
\[ S_i'(\omega, \sigma^2) = Q_i'(\omega, \sigma^2). \quad (i = 1, \ldots, 4) \]
pion-pion-scattering amplitude. Therefore, the nonvanishing value of \( S' \) must be
due to the effects neglected by this assumption. It is natural to think that, in \( S' \) also
there is a contribution of these neglected effects whose magnitude is about the same
as that of \( S' \) since \( S' \) is of the same dimension as that of \( S' \). When we use (3·7),
this contribution should be regarded as an error in \( S' \), since \( Q' \) has been calculated
under the same assumption as above. This error is estimated to be \( \pm 5 \), i.e. about
30%. (The value of \( |S'| \) can be as large as \( \sim 5 \) on account of error in \( R' \).)
Errors of this kind will appear also in \( S' \) and \( S' \). Though we have no way to
estimate them, we will assume these to be 30% as in \( S' \). Furthermore, there is
another error in each \( S' \). This is mainly due to the inaccuracy of \( R' \), and is
estimated to be \( \pm 3 \). Thus the total error is estimated to be \( \pm 6 \) for \( S' \) and \( S' \),
and \( \pm 8 \) for \( S' \).*

Next we turn to the right-hand side of (3·7). \( Q' \) is given by the expression
obtained from (3·3) by replacing the denominator \( \omega' - \omega \) by its square. \( u^{(\pm)} \) and
\( v^{(\pm)} \) in the integrand have been calculated by two methods; the one is explained
in \( \S \) 2 a) and the other, in \( \S \) 2 b). The former will be referred to as the
relativistic case and the latter, as the static case.

We found that \( Q' \) cannot be calculated with sufficient reliability, since this is
a difference of two terms, both of which contain slowly converging integrals. Here
we will calculate \( Q' \) and \( Q' \) only.

First we consider \( Q' \). To perform the integration over \( \omega' \), it is necessary to
assume the functional form of \( \text{Re} \ l_0 (-\omega^2) \) appropriately. In terms of the phase
shift for the pion-pion scattering, \( \text{Re} \ l_0 \) is written as

\[
\text{Re} l_0 (\omega_c^2) = (\omega_c^2 / 2k_c) \sin 2\delta_0, \tag{3·8}
\]

where \( \omega_c \) and \( k_c \) are respectively energy and momentum of either pion in the
center-of-mass system, and \( \delta_0 \) is the phase shift for the state with \( l=0 \) and \( I=0 \).
Since the range of the pion-pion interaction is thought to be at most \( (2\mu)^{-1} \),
\( \text{Re} \ l_0 \) will not change very rapidly for \( k_c \leq 2\mu \). On the other hand, we have
actually found that the region of \( \omega' \) corresponding to \( k_c > 2\mu \) gives only negligible
contribution to \( Q' \).

* Such large errors are possible, but not so probable.
Taking these facts into consideration, we first set
\[(\omega_c/2k_c) \sin 2\theta_o = \lambda_o\] (3.9)
where \(\lambda_o\) is a real constant. Then we have these results
\[Q'_1 = -5.8 \lambda o \mu^{-2}\] for the relativistic case;
\[= -15.2 \lambda o \mu^{-2}\] for the static case.
Equating these values to the value of \(S'_1\) found above, we have
\[\lambda_o = 1.7 \pm 1.0\] for the relativistic case;
\[= 0.7 \pm 0.4\] for the static case. (3.10)

From this we can say that \(\lambda_o\) is definitely positive. This means that the pion-pion interaction is attractive in the state \(I=0\). The reason for this is as follows: If the pion-pion interaction is repulsive, \(\delta_o\) is negative and its magnitude is smaller than \(\pi/2\), and hence \(\text{Re} \, \lambda_o\) is negative for small values of \(k_c\). Value of \(k_c\) at which \(\text{Re} \, \lambda_o\) changes its sign is at least about the reciprocal of the range of pion-pion interaction, which is \(2\mu\) in the present case. As was mentioned above the region \(k_c > 2\mu\) gives no appreciable contribution to \(Q'_1\). Thus, with the repulsive interaction, we would necessarily have a positive \(Q'_1\), which definitely contradicts the negative \(S'_1\).

To get a more adequate measure of the strength of the interaction than \(\lambda_o\), we next make the scattering-length approximation; namely, we set
\[(k_c/\omega_c) \cot \theta_o = 1/\alpha_o,\] (3.11)
where \(\alpha_o\) is a real constant and is regarded as the scattering length, in units of \(\mu^{-1}\), of the pion-pion scattering in the states \(I=0\). The functional form of \(\text{Re} \, \lambda_o\) is determined by (3.8) and (3.11), and \(Q'_1\) can be computed for any given value of \(\alpha_o\). The results are shown in Table 2. All the values of \(Q'_1\) in this table are smaller in magnitude than \(S'_1\) in Table 1, but, within the error \(S'_1\) mentioned previously, any of them is in agreement with \(S'_1\). The best agreement is obtained for \(\alpha_o = 1 \sim 2\). If \(\alpha_o > 1\), it is possible to increase the magnitude of \(Q'_1\) to some extent by adding the effective-range term to the right-hand side of (3.11),

| \(\alpha_o\) | \(Q'_1\) in units of \(\mu^{-2}\) |
|---|---|---|
|  | relativistic case | static case |
| 0.5 | -4.2 | -6.4 |
| 1 | -5.6 | -9.0 |
| 2 | -5.4 | -8.8 |
| 3 | -3.8 | -7.4 |
and thus to improve the agreement. On the contrary, if \( \alpha_s < 1 \), the addition of the effective-range term will decrease \( \text{Re} \, l_o \), and the agreement will become worse.* Therefore, if \( \alpha_s \) is much smaller than unity, \( Q'_I \) can never agree with \( S'_I \). Thus we are led to the conclusion that, for the pion-pion scattering in the state \( I=0 \), the scattering length is of the same order as the Compton wave length of pion.

Next we consider \( Q'_I \). We can compute this if \( \text{Re} \, l_i \) is given. This is written as

\[
\text{Re} \, l_i (\omega^*_i) = (\mu^2 \omega^*/2k^*_e) \sin 2\delta_i,
\]

where \( \delta_i \) is phase shift for the pion-pion scattering in the state with \( I=1 \) and \( l=1 \). Analogously to the previous case, we first set

\[
(\mu^2 \omega^*/2k^*_e) \sin 2\delta_i = \lambda_1,
\]

where \( \lambda_1 \) is a real constant. Then we have

\[
Q'_I = -60.4 \, \lambda_1 \mu^{-3} \quad \text{for the relativistic case;}
\]

\[
= -79.8 \, \lambda_1 \mu^{-3} \quad \text{for the static case.}
\]

From this and the value of \( S'_I \) given in Table 1, taking account of the error \( \pm 8 \) in \( S'_I \), we get

\[
\lambda_1 = 0.27 \pm 0.13 \quad \text{for the relativistic case;}
\]

\[
= 0.20 \pm 0.10 \quad \text{for the static case.}
\]

Thus \( \lambda_1 \) is seen to be definitely positive. From the same reason as the previous case, this means that the pion-pion interaction is attractive in the state \( I=1 \). Therefore, it is possible that the resonance scattering occurs in this state as in the \((3, 3)\) state in the pion-nucleon scattering.

For the purpose of comparison, we next make the effective-range approximation by setting

\[
(\mu^2 \omega^*/k^*_e) \tan \delta_i = \alpha_i (1 - \omega_e/\omega_i)^{-1},
\]

where \( \alpha_i \) is a real constant, and \( \omega_e \) is the resonance energy. Then we can compute \( Q'_I \) for any given values of \( \alpha_i \) and \( \omega_e \). The results are shown in Table 3. Any value of \( Q'_I \) in this table is in agreement with \( S'_I \) in Table 1 within the error. It is seen by the extrapolation that the lower limit of \( \alpha_i \) giving the agreement is about 0.1 for \( \omega_e = 3\mu \), and is about 0.2 for \( \omega_e = \infty \). Good agreement is, however, obtained for \( \alpha_i = 0.5 \sim 1.0 \). These latter values of \( \alpha_i \) are much larger than the corresponding values of the \((3, 3)\) state in the pion-nucleon scattering, because in the latter case we know that the value of the quantity corresponding to \( \alpha_i \) is

* Since the effective range is in general positive for the attractive interaction, the addition of the effective-range term to the right-hand side of (3 ·11) increases \( \cot \delta_0 \) and decreases \( \delta_0 \). If \( \alpha_s < 1 \), \( \delta_0 \) determined by (3 ·11) never exceeds \( \pi/4 \), therefore \( \text{Re} \, l_0 \) given by (3 ·8) decreases with \( \delta_0 \).
Table 3. Numerical values of $Q'_i$ in the effective-range approximation

<table>
<thead>
<tr>
<th>$\alpha_i$</th>
<th>$Q'_i$ in units of $\mu^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>relativistic case</td>
</tr>
<tr>
<td></td>
<td>$\omega_r=3\mu$</td>
</tr>
<tr>
<td>0.25</td>
<td>-10.0</td>
</tr>
<tr>
<td>0.50</td>
<td>-12.0</td>
</tr>
<tr>
<td>0.75</td>
<td>-13.6</td>
</tr>
<tr>
<td>1.00</td>
<td>-14.4</td>
</tr>
</tbody>
</table>

about 0.1. In the present case it is possible that the smaller values of $\alpha_i$ are sufficient for good agreement between $Q'_i$ and $S'_i$, if more appropriate values of $\omega_r$ are chosen. Though we can say nothing quantitatively, we can conclude that the pion-pion interaction in the state $I=0$ is the strength at least comparable to that of the pion-nucleon interaction in the $(3,3)$ state.

Finally we add one remark. As was mentioned previously, we set $w=0$ and $\cos \theta=1$ in computing both sides of the dispersion relations. Then the path of integration over $\omega'$ is a horizontal straight line passing through the point A in Fig. 1, and consequently it is seen that the partial-wave expansion of $\text{Im} A^{(\pm)}$ and $\text{Im} B^{(\pm)}$ made in computing $J'_i$ can be justified by the Lehmann's theorem only below the resonance energy. We see also that, if we shift this path slightly above, the energy region, in which the partial wave expansion is allowed, is far more increased. Indeed, if we set $w=63$ Mev and $\cos \theta=1$, this energy region covers almost all energies which contribute appreciably to $J'_i$. However, we have not made calculations with $w=63$ Mev, because, as was mentioned previously, there is a difficulty in computing $J'_i$ accurately, unless $w$ is zero. Instead of doing this, we have attempted to justify our calculations with $w=0$ in the following way: in contrast with $J'_i$, $J_i$ can be computed with sufficient accuracy even if $w \neq 0$. We therefore compute both sides of (3.6), the dispersion relations without subtraction, for $w=0$ and $w=63$ Mev keeping $\cos \theta=1$. If both of the two choices of $w$ yield the same result, we may consider that the calculations using (3.7) at $w=0$ have been justified. For that purpose it is convenient to take $i=1$ in (3.6), since $S_1$ does not contain the ambiguous Born term, and $Q_1$ can be computed with much more reliability than $Q_2$. In the static case $Q_1$ turns out to be independent of $w$ if $\cos \theta$ is kept to unity. In the relativistic case, on assuming (3.9), we have $Q_1=23.4 \mu^{-1}$ for $w=0$, and $Q_1=27.0 \mu^{-1}$ for $w=63$ Mev. On the other hand we have $S_1=26.6 \mu^{-1}$ for $w=0$ and $S_1=25.4 \mu^{-1}$ for $w=63$ Mev. From these results it can be said our treatment has been justified.

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References

3) Dispersion relations of this kind have been discovered also by Mandelstam. S. Mandelstam, Phys. Rev. 112 (1958), 1344.