Inconsistency among the Properties of Renormalizability, Analyticity, and Regularity at Zero Charge

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The photon propagator is analyzed by means of general properties of the present theory of electrons and photons. It can be shown, by consideration of the charge renormalization group, that the photon propagator is independent of physical charge in the high energy limit. If, in addition to renormalizability, regularity at vanishing physical charge is assumed, then it follows that the bare charge vanishes. On the other hand, the commonly assumed analyticity properties require that bare charge exceeds physical charge. Thus at least one of the general properties assumed is inadmissible. This relationship of expressions satisfying the charge renormalization group equations to analyticity properties and definite sets of Feynman diagrams (which correspond to an expansion about zero charge) is illustrated by a simple example.

§ 1. Introduction

Gell-Mann and Low\(^1\) were the first to demonstrate clearly the importance of renormalizability as a tool in going beyond the perturbation approach to quantum field theory.\(^2\) Bogoliubov and Shirkov\(^3\) have applied this tool to a variety of propagators while Blank and Shirkov\(^4\) have treated vertex functions in this way. The study of analyticity properties of the theory represents another attempt to transcend perturbation solutions. It is natural to try to combine results from these two general lines of investigation: the charge renormalization group properties and analyticity properties. We shall confine our attention to the photon propagator, which appears to be the simplest object one can study with such methods.

Section 2 contains a derivation of the result we shall need from the charge renormalization group theory of the photon propagator, namely, that the high energy limit is independent of the physical charge when only photons and electrons are considered. The derivation is quite simple because we are here concerned only with this general result, not with the group differential equations or detailed solutions thereof.

In § 3, we consider two common sets of assumptions: (1) The usual analyticity properties are valid. As a consequence, bare charge exceeds physical charge. (2) The theory is renormalizable (so that § 2 is relevant), and regular at vanishing charge in the high energy limit. From these assumptions and the result of § 2, it follows easily that the bare charge vanishes.

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Inconsistency among the Properties of Renormalizability

Clearly, (1) and (2) are incompatible. The results thus obtained on the basis of very general properties of the theory is reminiscent of the heuristic argument of Dyson, who suggested that the quantum electrodynamic vacuum is unstable when \( e^2 < 0 \), which means that the theory is singular at \( e^2 = 0 \).

Section 4 contains an illustration, drawn from existing explicit calculations, of the relation between expressions satisfying the charge renormalization group equation, analyticity properties, and Feynman diagrams (which correspond to an expansion about zero charge). The illustration behaves as expected from § 3. It should be noted that the incompatibility of assumptions (1) and (2), established generally in § 3 and verified for a specific case in § 4, becomes numerically evident in § 4, only at energies far beyond those for which quantum electrodynamics has received experimental verification. Also, our methods suffice to establish the incompatibility only for the theory of electrons and photons (this point is discussed at the end of § 2).

§ 2. The charge renormalization group

Let us consider quantum electrodynamics. This theory is invariant under the simultaneous transformations

\[
e_1^2 \rightarrow e_2^2 = Z_3 e_1^2, \tag{2·1}
\]

\[
D_1(e_1^2) \rightarrow D_2(e_2^2) = Z_3 D_1(e_1^2), \tag{2·2}
\]

which relate two different renormalizations of the electric charge \( e^2 \) and photon propagator \( D \). The photon propagator is, in explicitly gauge invariant form,***

\[
D^{\mu \nu}(k, m^2, e^2) = \frac{1}{ik^2} \left( g^{\mu \nu} - \frac{k^{\mu} k^{\nu}}{k^2} \right) d(k^2, m^2, e^2). \tag{2·3}
\]

We shall concentrate our attention on the unknown \( d \), which contains all the dependence of \( D \) on charge and the electron mass \( m \). From (2·1), (2·2), and (2·3) it follows that \( e^2 d \) is invariant under renormalization:

\[
e_1^2 d_1(k^2, m^2, e_1^2) = e_2^2 d_2(k^2, m^2, e_2^2). \tag{2·4}
\]

To each photon 4-momentum \( k^2 = \lambda^2 \) there corresponds a definite physical value of \( e^2 d \). If we call this value \( e_1^2 \), then \( d_1(\lambda^2, m^2, e_1^2) = 1 \). In particular, the physical

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** For another case where the solution is not regular at \( e^2 = 0 \), see the discussion of the Klein paradox on p. 124 of W. Thirring, Principles of Quantum Electrodynamics, Academic Press, London and New York (1958).

*** The corresponding subscripts, 1 and 2 in this case, will be explicitly indicated only when necessary: e.g. in (2·1) and (2·2) but not (2·3).

† We use \( k^2 = \hbar^2 - \hbar^2 \), \( \hbar = c = 1 \).

**** We are following here the very clear exposition of Bogoliubov and Shirkov.
charge \( e_p^2 \) corresponds to \( \lambda_p^2 = 0 \), and the bare charge \( e_b^2 \) corresponds to \( \lambda_b^2 = \infty \). Since the definition of \( d \) depends on \( \lambda \) (\( e \) is otherwise undetermined), it is convenient to include the dependence explicitly in (2.4), which allows us to drop the subscript on \( d \):

\[
e_i^2 d \left( k^2/\lambda_i^2, \ m^2/\lambda_i^2, \ e_i^2 \right) = e_b^2 d \left( k^2/\lambda_b^2, \ m^2/\lambda_b^2, \ e_b^2 \right) \tag{2.5}
\]

with

\[
d(1, \ m^2/\lambda^2, \ e^2) = 1. \tag{2.6}
\]

Equation (2.5) expresses the invariance of the theory under simultaneous changes of charge and momentum scale.

Now let us introduce ultraviolet cutoffs \( \Lambda \) so that \( \lim_{k^2 \to \infty} e^2 d \) is finite*. Then \( d \) has a limit \( d \left( k^2/\Lambda^2, \ 0, \ e^2, \ \Lambda^2 \right) \), as \( m^2/\Lambda^2 \to 0 \). Thus, for \( k^2, \ k^2 \gg m^2 \) Equation (2.5)

\[
\]

becomes asymptotically

\[
e_i^2 d \left( k^2/\lambda_i^2, \ 0, \ e_i^2, \ \Lambda^2 \right) = e_b^2 d \left( k^2/\lambda_b^2, \ 0, \ e_b^2, \ \Lambda^2 \right). \tag{2.7}
\]

The cutoffs \( \Lambda \) have been introduced only to justify setting \( m^2/\Lambda^2 = 0 \) in the case where \( \lim_{k^2 \to \infty} e^2 d \) diverges. Perhaps this justification is unnecessary because, as remarked by Gell-Mann and Low², the smaller of \( k^2 \) and \( \Lambda^2 \) provides an infrared cutoff on each integral. In any case the use of \( \Lambda \) implies states of negative norm which are inconsistent with the analyticity properties we wish to use in § 3, so from now on we consider the limit of (2.7) as \( \Lambda \to \infty \) (and we drop \( \Lambda \) from the explicit notation). Clearly \( e \) and \( \lambda \) are not independent; in fact the limit of (2.7) as \( \Lambda \to \infty \) implies that

\[
e^2 d \left( k^2/\Lambda^2, \ 0, \ e^2 \right) = F \left\{ k^2/\Lambda^2, \ \phi(e^2) \right\}. \tag{2.8}
\]

Now as \( k^2 \) varies (\( k^2 \gg \Lambda^2 \gg m^2 \)) we see that the physical charge plays the role of a scale factor (for any finite \( \Lambda^2 \), the \( e^2 \) defined by

\[
e_p^2 d \left( \Lambda^2, \ m^2, \ e_p^2 \right) = e^2
\]

is a function of the physical charge \( e_p^2 \)). The limit \( F \to e_b^2 \) as \( k^2 \to \infty \) is independent of this scale factor, so the bare charge \( e_b^2 \) (which may be finite or infinite) is independent of physical charge \( e_p^2 \).

Similar arguments could be applied to other quantities \( X \) which are renormalization-invariant (i.e., not explicitly dependent on renormalization constants), at least as far as (2.4). If \( X \) is dimensionless one can proceed to (2.5). The step which cannot be extended to general \( X \) seems to be the dropping of finite momenta (e.g., \( m^2 \) in (2.7)) as \( \Lambda^2 \) becomes large, which allows us to assert that the charge appears only as a scale factor of the momentum which becomes infinite. In the case of two masses \( m_1^2 \) and \( m_2^2 \), for example, \( X \) might depend on \( m_1^2/m_2^2 \) in a charge-

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* Gell-Mann and Low², among others, have shown how to do this in a gauge invariant way.
Inconsistency among the Properties of Renormalizability

dependent but $\mathcal{L}$-independent manner. Even in the simple case of the vertex (multiplied by the square of the electron wave function renormalization) connecting two physical electrons (with 4-momenta $m^2$) and a virtual photon ($k^2$), an infrared cutoff must be used and cannot be dropped at large $k^2$. Likewise the proof of (2·7) for the photon propagator seems to fail when two or more different charged particle masses are introduced. So the conclusions ((2·7), § 3, § 4) reached in this paper for the theory of electrons and photons may break down as soon as more elements are included in the theory.

§ 3. Proof of the contradiction

First, we shall see what happens if $e^2d$ is assumed to be regular (uniformly convergent) as $e_1^2 \to 0$. In that case $e^2d$ and $e_s^2$ are identically zero. As $e_p^2$ is increased, $e_s^2$ remains identically zero because it is independent of $e_p^2$ (from § 2). So we have

\[ e_p^2 > e_s^2 = 0. \]  
(3·1)

But if the analyticity properties assumed for the spectral representation\(^{11,10}\) of $d$ are correct,

\[ e_s^2 \geq e_p^2. \]  
(3·2)

One or more of the assumptions must therefore be wrong. Although we have not directly proved anything general about the behavior at finite momenta, one would certainly expect the discrepancy between (3·1) ($e^2d=0$ at $k^2=\infty$) and (3·2) ($e^2d \geq e_p^2$ at $k^2=\infty$) to persist smoothly into a range of finite momenta, which is the behavior exhibited by the examples discussed in § 4.

§ 4. Explicit calculations of the photon propagator

One can be more specific than we have been in § 2, and derive differential or integral equations expressing invariance under the charge renormalization group. These equations can be used to modify any perturbation theory result into an expression invariant under the renormalization group.\(^{\text{1,8}}\) The aim of this section is pedagogical: we attempt to provide insight into the general conclusions of the preceding sections by reviewing the relation of some particular expressions for the photon propagator, which satisfy the renormalization group equations, to diagrams and the usual analyticity properties. From § 3 we expect that these two aspects are complementary; a non-trivial expression satisfying the charge renormalization group equations can have the usual analyticity properties\(^{\text{9,11,12}}\) or a power series expansion about zero coupling constant\(^{\text{9,12}}\) (diagram description), but not both.

It is well to begin by indicating what sets of diagrams can be treated by the charge renormalization group.* Such sets must conspire to multiply each of the re-

* This subject has already been touched upon by Gell-Mann and Low\(^{\text{1}}\) in their footnote (16).
normalizable quantities (charge, propagator, vertex, wave function) by a consistent amount (renormalization) in each place where that quantity occurs. This condition is satisfied by any set which includes all the possible reducible* and improper* corrections to itself.** Thus to each collection of irreducible, proper parts (employed consistently) there corresponds a renormalizable set of diagrams, obtained by adding all possible reducible and improper parts. In addition, if the set is to exhibit the full symmetry properties of the theory (e.g., gauge invariance), a suitably symmetric choice of irreducible, proper parts should be made.

By way of illustration, consider the quantity \( d \) defined by Eq. (2.3). The lowest order correction to the photon propagator is the bubble diagram (Fig. 1). The corresponding value for \( d \) at high energies (with correct analyticity properties) is approximately

\[
d = 1 + \frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2}
\]

(4.1)

where \( e^2 = 1/137 \). The solution of the group equation, regular at \( e^2 = 0 \), which reduces to (4.1) when expanded in powers of \( e^2 \) has been computed by a method which makes no reference to diagrams\(^1,3\). The result is

\[
d = \frac{1}{1 - \frac{e^2}{3\pi} \ln \left( \frac{4m^2 - k^2}{4m^2} \right)}
\]

(4.2)

In accordance with (3.1), Eq. (4.2) vanishes as \( k^2 \rightarrow \infty \). From the geometrical series expansion of (4.2) in powers of \( e^2 \) valid when \( 1 > \frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2} \) we verify that (4.2) includes all the improper parts (Fig. 2) corresponding to Fig. 1.\(^1,8\)

As is well known, (4.2) contains a singularity, at \( \frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2} = 1 \), not in accordance with the general analyticity properties of \( d \)\(^1,10\). In order to remove this

* A skeleton is a diagram with all self-energy and vertex parts omitted\(^7\). A graph which is its own skeleton is called irreducible; all other graphs are called reducible. A proper part is one which cannot be divided into two parts joined by a single line; all other parts are improper.

** Note that the bare photon propagator is itself invariant under the charge renormalization group. But in this trivial case \( e^2 d = e^u \) is independent of \( k^2 \), so (2.8) is not valid and the conclusions of § 2 and § 3 (which follow from (2.8)) are not fulfilled.
difficulty Redmond\textsuperscript{*} has used (4·2) to determine
\[ 2\pi i I(z) = \lim_{\epsilon \to 0} [d(z+i\epsilon) - d(z-i\epsilon)] \] (4·3)
in the spectral representation\textsuperscript{1,10}.
\[ d = 1 + k^2 \int_0^{\infty} \frac{I(z)}{k^2 - z + i\epsilon} \frac{dz}{z}. \] (4·4)
The result,
\[ e^2 d = \frac{3\pi}{e^2} - \ln \frac{4m^2 - k^2}{4m^2} \frac{3\pi}{1 - \exp \left\{ \frac{3\pi}{e^2} - \ln \frac{4m^2 - k^2}{4m^2} \right\}}, \] (4·5)
has the right analyticity properties as a function of $k^2$, exhibits an essential singularity at $e^2=0$, and is finite and independent of $e^3$ in the high energy limit $(e^2 d(\infty)=3\pi)$. Bogoliubov, Logunov, and Shirkov\textsuperscript{5}) have shown that (4·5) can be modified to agree with the charge renormalization group equations, without changing the other properties of (4·5) (in particular $e^2 d(\infty)=3\pi$ remains independent of $e^2$ in accordance with §2, and greater than $e^2=1/137$ in agreement with §3). In view of the singularity at $e^2=0$, (4·5) cannot represent an expansion in series of diagrams, although it bears some relationship to Fig. 2. The difference between quantum electrodynamics and meson physics is that (4·5) allows an asymptotic expansion in powers of $e^2=1/137$, whereas the corresponding expression in meson physics\textsuperscript{11,12} probably has no useful expansion in powers of $g^2 \approx 15$.
Equation (4·5) is very well approximated by the lowest order expression, (4·1), for all practical purposes at laboratory energies. The asymptotic expansion of (4·5) in powers of $e^3$ at moderate energies represents a weak coupling, which builds up to a strong coupling expansion in inverse powers of $e^3$ at high energies.

To complete the trilogy, consider (4·1), which is not a solution of the charge renormalization group equations. It is easily seen that $d$ in (4·1) is regular at $e^2=0$ and satisfies the analyticity condition (4·3), (4·4).

\section{5. Conclusion}

In §3 we found on general grounds that, if only electrons and photons are considered, the photon propagator cannot have all three of the following properties: renormalizability, analyticity, and regularity at $e^2=0$. In §4, we quoted particular expressions for the propagator, related to the restricted set of diagrams in Fig. 2, which exhibit the behavior predicted in §3.

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\textsuperscript{*} Actually Redmond considered the meson propagator, but his discussion can be adopted to the photon propagator without any difficulty.
References

1) M. Gell-Mann and F. E. Low, Phys. Rev. 95 (1954), 1300.
2) For an earlier attempt see S. F. Edwards, Phys. Rev. 90 (1953), 284.
3) N. N. Bogoliubov and D. V. Shirkov, Nuovo Cim. 3 (1956), 845.
4) V. Z. Blank and D. V. Shirkov, Nuclear Phys. 2 (1956), 356.
5) N. N. Bogoliubov, A. A. Logunov, and D. V. Shirkov, Dubna preprint.
7) F. J. Dyson, Phys. Rev. 75 (1949), 1736.
8) A. Salam, Phys. Rev. 84 (1951), 426.