Again, this is something the authors are perfectly aware of; it is all there, at least implicitly, in the paper.

3 This is a very minor comment. Fig. 2 seems to indicate that dimension $s$ is some nonzero constant, while the symbolic equation lists it as zero. Perhaps the writer is simply guilty of a misreading.

After a severe but fair discussion of an earlier paper by Professors Denavit and Hartenberg, Professor de Jonge concluded with the following generous remark which deserves repetition. “... it is only by investigating all avenues of approach that we can arrive at the simplest and most effective methods for the study of mechanisms and their kinematics. For having so richly contributed to this aim, the authors deserve the highest praise.”

Authors’ Closure

We wish to thank the several discussers for their interest, kind remarks, and informative contributions.

As pointed out by Professor Beggs, the statement following Eq. (34) is erroneous. Application of the orthogonality conditions to the $(3 \times 3)$ submatrix of Eq. (35) does not yield a unique solution, and the $\pm$ signs in Eq. (38) are a consequence of this uncertainty. Thus, if only the three terms $B_{m}$, $B_{n}$, and $B_{a}$ were included, the iteration process would not always converge to an actual solution of the problem. The diagonal elements $B_{m}$, $B_{n}$, and $B_{a}$ have been included in the iteration process to eliminate this difficulty (see statement following Eq. (38)). It is worth noting, however, that the inclusion of only two of the diagonal elements would have been sufficient for convergence to the actual solution. Since all matrices $T_{i}$ have determinants equal to $+1$, the determinant of $B_{i}$ (Eq. 38) is also $+1$. Thus if two of the signs in the diagonal are prescribed positive, the third is also necessarily positive. In other words, the condition $B_{m} = B_{a} = 1$ implies $B_{n} = 1$. The last rows of the matrices $M$ and $V$ are therefore superfluous.

In reply to Professor Zimmerman’s comments, the displacement relations of a number of spatial linkages conveniently may be established by methods of analytic geometry or vector algebra. In some cases, a direct relation between input and output variables may even be established by a simple trigonometric equation. When applicable, these methods are often faster and less cumbersome than the matrix approach. Unfortunately, these methods apply to special cases characterized by “happy” geometric configurations such as intersecting, parallel, or perpendicular axes. In some problems, as in vehicle suspensions, such favorable geometric alignments do not exist and analyses of errors stemming from tolerances or deformation of parts also require general methods typified by the matrix or quaternion algebra procedures.

Finally, in addition to any practical reasons, methods based on matrix or quaternion algebra, which clearly account for relative rotations and translations between links, give a better insight into the whole problem of kinematic analysis than the direct methods in which a mechanism with unique geometry is considered as an individual.

With regard to Fig. 2 and eq. (3), $s_{i}$ in general would be a non-zero constant as shown in the figure. Eq. (3), however, gives numerical data for a particular case used as example at the end of the paper; see Fig. 5 for which $s_{i} = 0$.

Professor Tesar’s questions about other possible techniques cannot be answered on the basis of any personal experiences. Different methods should certainly be investigated, for they may well lead to further insights and possible simpler procedures.

Application of Dual-Number Quaternion Algebra to the Analysis of Spatial Mechanisms

J. S. Beggs. Fashions come and go in engineering as well as in ladies millinery, until they come full circle. Planetary gears are back in Ford transmissions, and quaternions and dyadics are back in mechanics. Even though “the idea of using dual-number quaternions suggested itself,” the writer is sure the authors deserve some of the credit for this truly ingenious concept.

Equation (40) of the paper is the loop-closure equation, written in terms of dual-number quaternions, which was introduced by Denavit and Hartenberg, written in terms of $(4 \times 4)$ coordinate transformation matrices. (To the writer’s way of thinking, this equation is one of the most important contributions to the analysis of mechanisms.)

If $X_{1}Y_{1}Z_{1}$ and $X_{2}Y_{2}Z_{2}$ are coordinate systems fixed in adjacent linkages of a mechanism, and $(T_{2})$ is the $(4 \times 4)$ transformation matrix which transforms the coordinates of a point in system 1 to system 2, then $(T_{2})^{-1}$ is the matrix of the screw (relative to system 1) that will bring a third system from coincidence with system 1 to coincidence with system 2. In fact, the physical elements of the screw, $\theta$ and $s$, used in the screw operator $Q$, can be calculated readily from the elements of $(T_{2})^{-1}$. This is no surprise, since both the quaternion and the matrix are describing the same physical situation.

In his dissertation, Denavit uses $(2 \times 2)$ matrices of dual numbers in the loop-closure equation. The elements of these matrices are complex, with dual components. Both the $(2 \times 2)$ matrix and quaternion methods are very elegant in concept but involve a great deal of hard work, as witness the Appendix. (By the way, many of the space-industry writers are saving a great deal of writing time and journal space by writing “$S$” and “$C$” for sine and cosine.) The writer’s students worked through the detailed analysis of the four-link mechanism of Fig. 2 using first the $(4 \times 4)$ matrices and then the $(2 \times 2)$ matrices. It will be interesting to see their reaction to dual-number quaternions.

The instantaneous screw, or instant screw for short, has zero lead in the case of plane motion and pierces the plane of motion normally at the instant center of the two bodies. Just as the instant center traces out centroids in the two bodies in plane motion, the instant screw, or instant axis for short, traces out surfaces called axodes in the two bodies in spatial motion. For example, in hypoid gears the axodes are hyperboloids of revolution about the shaft centerlines. In general the bodies roll on each other about the instant axis and slide along it.

In spatial motion, the analog of the Arohold-Kennedy theorem may be stated as follows: The instant axis of three bodies taken at a time intersect a common normal. Perhaps the writer is simply guilty of a misreading.

“... it is only by investigating all avenues of approach that we can arrive at the simplest and most effective methods for the study of mechanisms and their kinematics. For having so richly contributed to this aim, the authors deserve the highest praise.”
DISCUSSION

From what we have observed, it is obvious that dual-number quaternions are certainly well adapted to handling problems involving force, torque, and velocity ratios.

The definition of velocity ratio used here is the preferred one, but it is the reciprocal of the usual definition. It is unfortunate that we cannot agree on a standard nomenclature in mechanisms. For example, we have revolute versus turning pair and cylindrical pair versus turn-slides.

Professors Denavit and Hartenberg and the present authors are to be commended for making available their computer programs. The designer can insert his own constants for the job at hand. Tapes generated by the computer can then be put on a cathode-ray curve plotter, such as the Stromberg Carlson 4020, and whole families of curves, representing data like that in Table 1 of the paper, plotted in a matter of seconds.

Discretizations are extremely useful in teaching graduate courses because of the material they contain and because they demonstrate the ever-increasing standards. University Microfilms is performing a very fine service in making them readily available.

It is gratifying to note that more and more firms, like American Machine and Foundry, are granting their men the time and support needed in the monumental task of writing dissertations in modern engineering.

O. BOTTEMA. This is, I believe, an excellent paper and although I did my best I find it difficult to find fault with it. As proof of careful reading of this paper, the writer notes that he would have written the left-hand side of equation (28) as $Q(KQ)$; that equation (40) could have been given in the more elegant form $Q_1Q_2Q_3Q_4 = 1$ or the equivalent.

The section on force and torque analysis seems very important (and new as far as the writer knows). For most readers it will be difficult to follow because the authors express themselves rather concisely. Some remarks might have been added concerning the duality of the velocity distribution of a rigid body on the one hand and of a general system of forces acting on such a body on the other hand. Both may be expressed by a dual vector, but the rotation corresponds to the force and the linear velocity to the moment—a somewhat paradoxical situation (and for this reason, dual velocity and dual force, the terms used by the authors, may give rise to confusion in some minds).

Finally, the following three questions are addressed to the authors:

1. Do the authors agree that the theory of spatial four-bar will never attain the elegance of the planar counterpart, in view of the fact that $(3 \times 3 - 1)$ has 4 as a factor whereas $(3 \times 6 - 6)$ does not? Hence, a spatial four-bar mechanism with one degree of freedom and four identical joints does not exist (Bennett's mechanism being a pathological exception). In principle, a satisfactory theory could be obtained, starting with four turn-slides and accepting two degrees of freedom.

2. Does the tool developed by the authors open up the possibility of studying the path of a point on the floating link, thereby providing information about spatial coupler curves?

3. In the paper, a mechanism with three turn-slides and one turning pair is considered. This means that on each of the four links, two axes are specified; and if it is understood correctly, this circumstance permits introduction of the dual-number apparatus (making use of the minimum distance and the angle between the two axes). What type of analysis could be developed, however, for the case where one of the joints is a spherical joint (and the others, for example, a turn-slide and two turning pairs)?

J. DENAVIT and R. S. HARTENBERG. The displacement equations of spatial mechanisms may be formulated by representing the relative position of two adjacent links in terms of an operator. The product of all operators taken around each closed chain of the mechanism then yields a set of equations from which all kinematic relations of the mechanism, hypothetically at least, may be extracted. A number of matrix operators have been used for this purpose: $(4 \times 4)$ matrices with real elements, $(3 \times 3)$ matrices with dual elements, and $(2 \times 2)$ matrices with complex-dual elements. The authors have shown in the present paper that dual quaternions also may be used. Although the algebraic properties of these operators are somewhat similar, and one would expect their use to be equivalent, practice shows that some are better suited than others to investigate particular situations. Thus, the $(2 \times 2)$ dual matrices and the dual quaternions seem better suited than others to investigate particular situations.

The formulation of displacement equations, however, is only half the problem of spatial kinematic analysis. The other half is the solution of these equations to obtain the output variable explicitly in terms of the input variable. This involves elimination of the intermediate variables. In the example chosen by the authors, this elimination takes place nicely because of the natural separation of rotational and translational variables, and the same occurs in a number of other significant applications. In general, however, this elimination and the solution of the resulting equations may be difficult. The displacement equations for a simple closed chain of seven revolutes are of the form

$$f_i (\cos \theta_i \sin \phi_i) = 0 \quad i = 1, \ldots, 6$$

$$k = 2, \ldots, 7$$

$	heta_i$ being the input variable. Making use of the tangents of the half-angles, $t_i = \tan \frac{\theta_i}{2}$, the displacement relations may be written as six quadratic equations for the six unknowns $t_i$. The elimination of one unknown between two quadratic equations leaves five equations of eighth degree. Eliminating a second unknown will produce four equations of degree 128. After elimination of the fifth unknown, the single equation to be solved is of very high degree.

To be useful, a linkage may have to transmit force as well as motion. Limiting the discussion to statical forces, a force-transmission criterion, such as the purely kinematic transmission angle of the four-bar linkage, would be of value in the selection of spatial linkages. The authors have defined such a factor of merit for their example, but the establishment of a more general criterion applying also to other linkages would seem significant.

The velocity analysis of linkages may be based on the derivatives of equation (1) with respect to the input variable $\theta_i$, which gives a set of equations of the form

$$\sum_{k=1}^{n} \frac{\partial f_i}{\partial \theta_k} \frac{d \theta_k}{d \theta_i} = - \frac{\partial f_i}{\partial \theta_i}$$

Acceleration, error, and static-force analyses yield equations with similar left members. In all cases, solution depends on having a determinant $|\frac{\partial f_i}{\partial \theta_i}|$ that is different from zero. If the determinant is zero, the linkage is in dead-center position and is incapable of force transmission. For small values of the determinant, the velocities, accelerations, static forces, and errors will be large, and the linkage cannot be expected to give good performance. It is interesting to note that in a number of simple cases (such as the planar and spherical four-bar linkages), the determinant is proportional to the sine of the already available transmission angle. It seems reasonable, therefore, that, properly normalized, this determinant might furnish an adequate and general "force-transmission factor" in terms of which the merits of linkages—planar and spatial—could be compared.

G. N. SANDOR. The authors are to be complimented on having

1 Professor, Department of Mathematics, Technological University, Delft, Holland.
2 Associate Professor, Department of Mechanical Engineering and Astronautical Sciences, Northwestern University, Evanston, Ill. Assoc. Mem. ASME.
3 Professor, Department of Mechanical Engineering and Astronautical Sciences, Northwestern University, Evanston, Ill. Mem. ASME.
4 Yale University, New Haven, Conn. Mem. ASME.

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made a very substantial contribution in spatial kinematic analysis. It is hoped that the authors' work will be extended to deal with problems of kinematic synthesis. For both analysis and synthesis, it would be useful if, say, FORTRAN subroutines were worked out to allow dealing with dual quantities by a single symbolic in computer programming. Since dual vectors have six components and dual quaternions have eight components, such "dual mode" of computation would afford considerable economy and reduce the risk of error in programming and would facilitate the application of the authors' methods to new problems in analysis and synthesis. Furthermore, since spherical and planar mechanisms are included in the authors' spatial theory as special cases, programs written in such "dual mode" would be universally applicable in all three areas.

Authors' Closure

The authors wish to thank the discussers for their illuminating comments and generous remarks. Professor Beggs' remark on the concept of the instant screw axis in spatial motion—that it is analogous to the concept of the instant center of a body in plane motion (extension of Aroinhold-Kennedy theorem)—is interesting. Incidentally, the authors believe that, using dual-number quaternion algebra and the concept of the instant screw axis, one may determine analytically the velocity of any point on the floating link of a spatial four-link mechanism.

Professor Beggs' comments on terminology in mechanisms are well taken. The authors share Professor Beggs' hope that kinemacists will continue to work out a set of standard terminology to facilitate communication for the benefit of all, in line with the recent efforts of Professors Artobolevskii, Denavit, Goodman, and others.

The authors would like to take this opportunity to acknowledge their debt to Professor Bottema for suggesting the form \( KQK \) for the left-hand side of equation (29); this form was adopted in the paper as published in the \textit{Journal}. As to equation (40), the authors agree with Professor Bottema that it might well have been written in the more elegant form \( \frac{Q_0Q_0Q_0Q_0}{Q} = 1 \). The expression given in the paper, however, has some computational advantages.

On the somewhat paradoxical situation—rotation corresponding to force and linear velocity to moment—it may be clarified if we consider two points, \( O \) and \( P \), on a moving body whose angular velocity is \( \omega \). We may transfer the linear velocity at \( O \) to \( P \) as follows:

\[
V_p = V_o + \omega \times \vec{OP} = V_o + \vec{PO} \times \omega
\]

If force \( F \) and moment \( M_p \) are applied at point \( P \), the moment about \( O \) is given by

\[
M_o = M_p + \vec{OP} \times F
\]

Comparing the two equations, we note that moment corresponds to linear velocity and force to angular velocity. In the paper, angular and linear velocities are represented in dual vector form; so are force and moment. Hence, the terms dual velocity and dual force. More detailed treatment of the terms can be found in references [5] and [20] of the paper.

And now to Professor Bottema's three specific questions:

1. The statement that the theory of spatial four-link mechanism will not attain the elegance of its planar counterpart is very true. The authors also agree that, in principle, a satisfactory theory could be developed for a spatial four-link mechanism with four identical turn-slides and two inputs (rotation and sliding). In fact, when its sliding input velocity is zero, such a mechanism is the spatial four-link under study in this paper. The most symmetrical expressions derived for four-link spatial mechanism seem to be those of F. M. Dimenbank, reference [10] of the paper.

2. The authors believe that the tools developed in the paper may be used for the study of the path of a point on the floating link; such a point may be specified as a point on the line which is perpendicular to \( \dot{\omega} \) and subtends a constant dual angle with either one of the two floating axes, \( \dot{\omega}_3 \) or \( \dot{\omega}_3 \).

3. In principle, the analysis developed in this paper can be applied to the case where one of the joints is a spherical joint and the others are a turn-slide and two turning pairs; for a spherical joint can be replaced by three turning pairs while a turning pair is but a special case of the turn-slide.

Over the past ten years, the matrix methods developed by Professors J. Denavit and R. S. Hartenberg have become a fundamental contribution to the analysis of spatial mechanisms in general. The investigation presented in this paper, as acknowledged by the authors in the opening paragraph, owes much to their work.

A spatial four-link mechanism with one turning pair and three turn-slides is selected by the authors for study in depth, because of its generality. The authors believe that as the screw operator derived from dual-number quaternions seems to be well adapted for the description of screw motion, and, since the motion of a rigid body in space in general can be considered as a series of screw motions about instant axes, this tool may possess good possibilities for the description of the motion of a link in space, for example, the floating link of a spatial four-link mechanism.

The point raised by Professors Denavit and Hartenberg—that, in the case of a closed spatial chain of seven turning pairs, dual-number quaternion may not lead to analytical solutions—seems well taken. The authors also agree with their comment that, although algebraic properties of many operators are somewhat similar, practice shows some are better suited than others to investigate particular situations. It is the authors' hope that quaternion methods may find a suitable companion place to the matrix methods already in existence. The authors are grateful to Professor George N. Sutor for his valuable suggestion that FORTRAN subroutines should be worked out to allow dealing with dual quantities by a single symbol in computer programming. It is intended to carry this out and, hopefully, in the not-too-distant future.


\textbf{E. J. Gunter, Jr.}\footnote{Numbers in brackets indicate References at end of Discussion.} The concept of indifferent equilibrium was formulated by Rankine [1] in 1869 in the first recorded article on rotor dynamics and has persisted to this date. Rankine examined the equilibrium of a frictionless, uniform shaft and concluded that the motion is stable below the first critical speed, unstable or in "indifferent" equilibrium at the critical speed, and unstable above this speed. The problem of "indifferent equilibrium" was not resolved until 1910 when H. H. Jeffcott [2] analyzed the whirling of a single-mass unbalanced rotor with viscous damping (see Fig. 2 of the paper). His analysis revealed that synchronous precession is the only possible steady-state motion and the rotor deflection becomes unbounded only if the friction coefficient approaches zero. Jeffcott's analysis shows that viscous damping increases the critical speed. Examination of equation (14) with both \( P_1 \) and \( P_2 \) equal to zero indicates that turbulence or damping proportional to the velocity squared will not change the critical speed.

In this case, the rotor amplitude factor is given by

\[
\frac{\delta}{e} = \left( \frac{\mu_1^2}{\omega} + \left[ 1 - \frac{\omega^2}{\omega_1^2} \right]^2 \right)^{1/2}
\]