Use of complete gamma function in accurate evaluation of Einstein integrals

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ABSTRACT

By the use of the binomial expansion theorem, the series expansion relations in terms of the complete gamma function are obtained for Einstein integrals arising in the hydraulic and modern sediment transport mechanics. The approach presented for Einstein integrals is accurate enough over the whole range of parameters. The computational time for calculation of the series with respect to the literature is fast. Furthermore, the comparison of the method with numerical calculations demonstrates the applicability and accuracy of the method.

Key words | bed load, binomial coefficients, complete gamma function, sediment transport, suspended load

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NOTATION

$J_1(z,E)$, $J_2(z,E)$	Einstein integrals
E	relative bed-layer thickness to water depth
z	Rouse number that expresses the ratio of the sediment properties to the hydraulic characteristics of the flow
$F_i(n)$	binomial coefficient
$\Gamma(\sigma)$	Gamma function

INTRODUCTION

In the analysis of hydraulic and modern sediment transport mechanics, the Einstein integrals (EI) play a central role (Einstein 1950; Akiyama & Fukushima 1986; Guo & Hui 1991; Guo & Wood 1995; Julien 1995; Garcia 1999, 2005; Garcia & Parker 1991). The theoretical efficient backgrounds of these integrals have been given by Einstein (1950). Recently, several efficient techniques and discussions have been proposed for the calculation of EI (Einstein 1950; Itakura & Kishi 1980 and references therein; Nakato & Asce 1984; Guo 2002; Guo & Julien 2004, 2006; Abad & Garcia 2006; Roland & Zanke 2006; Srivastava 2006). In spite of all the developments of EI so far, the analytical evaluation of EI as a function of z and E is still one of the main problems in engineering mechanics. We note that most of the published approximations for EI involve many short intervals of z and E in each of which the different expressions are used.

In this work, an attachment is made to obtain general analytical expressions for EI with the arbitrary values of the parameters z and E. The proposed new analytical approach for evaluating EI is conceptually simpler than those existing in the literature at present. The application of formulae for the EI to the calculation of problems arising in hydraulics and modern sediment transport mechanics presents no difficulties when a computer is used. Moreover the formulae can easily be implemented with an algebraic computer language.

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SERIES EXPANSION RELATIONS FOR FIRST AND SECOND KINDS OF EINSTEIN INTEGRALS

The first and second kinds of Einstein integrals are defined as (Einstein 1950; Guo & Julien 2004)

$$J_1(z,E) = \int_E^1 \left(\frac{1-\xi}{\xi}\right)^z d\xi \tag{1}$$

and

$$J_2(z,E) = \int_E^1 \left(\frac{1-\xi}{\xi}\right)^z \ln\xi\, d\xi \tag{2}$$

We can rewrite Equations (1) and (2) in the following unified form:

$$J_k(z,E) = \int_E^1 \left(\frac{1-\xi}{\xi}\right)^z (\ln\xi)^{k-1} d\xi = \begin{cases} J_1(z,E) & \text{for} \quad k=1 \\ J_2(z,E) & \text{for} \quad k=2 \end{cases} \tag{3}$$

For the evaluation of Einstein integrals, Equation(3), we use the following binomial expansion theorem for an arbitrary real z and $|x| > |y|$ (Gradshteyn & Ryzhik 1980; Guseinov & Mamedov 2002, 2005):

$$(x \pm y)^z = \sum_{i=0}^{\infty} (\pm 1)^i F_i(z) x^{z-i} y^i \tag{4}$$

where $F_0(z) = 1$ and

$$F_i(z) = \begin{cases} z!/[i!(z-i)!] & \text{for integer } z \\ \frac{(-1)^i \Gamma(i-z)}{i!\Gamma(-z)} & \text{for noninteger } z \end{cases} \tag{5}$$

For $i < 0$ the binomial coefficient $F_i(z)$ occurring in Equation (4) is zero and the positive integer z terms with negative factorials do not contribute to the summation. The quantities $\Gamma(\sigma)$ in Equation (5) are well-known complete gamma functions defined by (Gradshteyn & Ryzhik 1980)

$$\Gamma(\sigma) = \int_0^{\infty} t^{\sigma-1} e^{-t} dt \tag{6}$$

Now we can move on to the evaluation of Einstein integrals J_1 and J_2 for any integer and noninteger z. For this purpose we use the following relation (Gradshteyn & Ryzhik 1980):

$$I_\alpha^{(k)}(E) = \int_E^1 x^\alpha (\ln(x))^{k-1} dx = (-1)^k \frac{E^{\alpha+1}-1}{(\alpha+1)^k} - \delta_{k2} \frac{E^{\alpha+1}\ln E}{\alpha+1} \tag{7}$$

where δ_{k2} is the Kronecker symbol. Taking into account Equations (4) and (7) in (3) we obtain for Einstein integrals the following combined relations:

$$J_k(z,E) = z! \sum_{i=0}^{z-2} \frac{(-1)^i I_{i-z}^{(k)}(E)}{i!(z-i)!} + (-1)^z \times \left(\frac{z}{1+\delta_{k2}} (\ln E)^{\delta_{k2}+1} - \delta_{k2} E \ln E + (-1)^k (E-1)\right) \quad \text{for integer } z \tag{8}$$

$$J_k(z,E) = \frac{1}{\Gamma(-z)} \lim_{N\to\infty} \sum_{i=0}^{N} \frac{\Gamma(i-z) I_{i-z}^{(k)}(E)}{i!} \quad \text{for noninteger } z \tag{9}$$

In Equation (9) the index N is the upper limit of summation.

We note that, using binomial expansion theorem for the calculation of Einstein integrals in the case of any integer values $z = n$, Guo & Julien (2004) proposed the same general efficient formulae (their Equations (12) and (19)).

NUMERICAL RESULTS AND DISCUSSION

For the rapid and accurate calculation of Einstein integrals in the large intervals of z amd E, we proposed a new algorithm. The efficiency and numerical stability of the code suggest that this approach may be quite useful for performing HEC-RAS and HEC-6 (USArmy Corps of Engineers 1993, 2003) using the Einstein bed load function. From the viewpoint of computational efficiency, accuracy and computation time of ours and various approximations, the Einstein integrals were evaluated on the Maple 7.0 international mathematical software. As seen from Equation (9), the problem of the Einstein

Table 1 | The comparative values of first Einstein integrals $J_k(z,E)$ for $k = 1$

z	*E*	Simpson method (Nakato & Asce, 1984)	Equation (9)	*N*	Equation (11) of Guo & Julien (2004)	Nakato & Asce (1984)
0.2	0.00001	1.069289	1.06888	800	1.06883433	1.07729
0.6	0.00001	2.0637468	1.9569634	800	1.95695964	1.9578539
0.2	0.0001	1.068211	1.068217	800	1.068170	1.075033
0.6	0.0001	1.92707	1.919167	800	1.919167	1.920035
0.2	0.001	1.063966	1.0640303	800	1.063983	1.06468122
0.6	0.001	1.8242991	1.82425113	800	1.8242473	1.844063
0.2	0.01	1.03756583	1.0376356	800	1.037588733	1.03684628
0.6	0.01	1.5863776	1.586420161	800	1.586416336	1.587325827
1.3	0.00001		100.3616538	800	100.36165391	
2	0.001		984.18548944203		984.18548944203	
4	0.05		1969.4004042391		1969.4004042391	
2.7	0.00001		186004145.83717	800	186004145.83717	
1.8	0.0001		1972.9222509	800	1972.9222509	
3.6	0.0001		9655451862.7177	800	9655451862.7177	
1.5	0.001		58.6280246	800	58.628024646	
3.8	0.001		89181596.66431	800	89181596.66431	
1.7	0.01		30.70486633	800	30.70486633	
4.5	0.01		2682275.2522864	800	2682275.2522864	

integrals determination reduces to the calculation of a basic integral. The examples of computer calculations for the Einstein integrals are shown in Tables 1 and 2. As can be seen from these tables, our numerical results are in agreement with the literature (Nakato & Asce 1984; Guo & Julien 2004).

We note that, in most stand-bed natural rivers, E is very small and z is rather large. For example, typical ranges of E

Table 2 | The comparative values of the second Einstein integrals $J_k(z,E)$ for $k = 2$

z	*E*	Equation (9)	*N*	Equation (18) of Guo & Julien (2004)	Nakato & Asce (1984)
0.2	0.00001	−1.4818990844	300	−1.482610025	−1.482288702
0.6	0.00001	−5.56417958	300	−5.568132919	−5.5648601295
0.2	0.0001	−1.4752444814	300	−1.47595542265	−1.498143196
0.6	0.0001	−5.179136752	300	−5.183090089	−5.179803722
0.2	0.001	−1.442901990	300	−1.44361293118	−1.447589225
0.6	0.001	−4.430735254	300	−4.43468859138	−4.435712406
0.2	0.01	−1.2997948097	300	−1.3005057508	−1.30016919
0.6	0.01	−3.102885409	300	−3.10683874748	−3.103541442
1.5	0.00001	−6014.81016162	300	−6014.79003887	
3	0.005	−93425.746652994		−93425.746652994	
2	0.0004	−17000.89580154		−17000.89580154	
2.5	0.00001	−228644169.010411	300	−228644169.314605	
1.7	0.0001	−7002.237381931205	300	−7002.20543860	
3.7	0.001	−303091132.1288627	300	−303091131.9042094	
4.5	0.01	−11604690.24577824	300	−11604690.442888776	

Table 3 | Convergence of expression for $J_2(2.8, 0.001)$ as a function of summation limits N

N	Equation (9) for $k = 2$	Equation (18) of Guo & Julien (2004)
50	−881491.381844265237	−881492.190620148
100	−881491.381844263870	−881491.794307176
150	−881491.381844263828	−881491.658932990
200	−881491.381844263823	−881491.590612962
250	−881491.381844263822	−881491.549415770
300	−881491.381844263822	−881491.521864867
400		−881491.487328836
500		−881 491.466 555 044
600		−881491.452684030

and z for the Sacramento River near Butte City, CA are $E = 0.0001-0.0004$ and $z = 1.8-5.0$. Similarly, the Mississippi River below Keokuk, IA has the typical ranges of $E = 0.0001-0.0004$ and $z = 3.5-5.0$ (Nakato & Asce 1984). Any desired degree of precision is completely obtained in the present paper. As will be clear from our tests the formula (8) yields the desired accuracy of arbitrary values of integral parameters E and z. We note that the values obtained from Equation (9) are sufficiently accurate for the exact evaluation of sediment motion. Tables 1 and 2 show that the convergence properties of Equation (9) are considered to vary widely. As can be seen from Table 1, Equation (11) in Guo & Julien (2004) displays the most rapid convergence to the numerical results for $E < 0.5$ and arbitrary values of z. In Table 1 the indexes L are the upper limits of summations in Equation (11) of Guo & Julien (2004). Finally, the reason for the empty columns in Tables 1 and 2 is that the computational results for the indicated equations are much lower than the accuracies in the present paper. As shown in Table 3 the arbitrary values of E and z in Equation (9) for $k = 2$ also converge at a good rate compared with Equation (18) in Guo & Julien (2004).

The computer time required for the calculation of Einstein integrals are not given in the tables due to the fact that the comparison cannot be made with different computers used in the literature. It is seen from the algorithm presented for Einstein integrals that our CPU times are satisfactory. For instance, for Einstein integrals with sets $z = 4.9$, $E = 0.8$ and $N = 600$ the CPU time taken is about 0.022 ms on a Pentium 4 PC at 800 MHz provided with 128 MB of RAM.

CONCLUSION

To illustrate the utility of the approximations in applications, the numerical examples were presented where modest accuracy is sufficient and the analytical simplicity of the approximation formulae is important. The validity of Equation (9) has been tested by numerical calculations. Equation (9) gives more accurate values of the Einstein integrals than other approximations through numerical analyses. It can be concluded that this newly proposed approximation leads to reasonable use as integral methods of sediment transport calculation. In conclusion, by means of a complete gamma function, we have obtained new simple accurate general expressions of J_1 and J_2 for arbitrary z and E.

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