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R. L. Mallett.[4] The author derives a necessary and sufficient condition on the damping matrix of a linear dynamic system which insures that the system's second order differential equations of motion can be uncoupled. Two examples are given to show that criteria previously stated by Rayleigh [5][5] and by Caughey [6] are sufficient but not necessary. Unfortunately, the author's criterion is not new and the two examples do not accomplish their stated purpose. The examples will be discussed first.

In the author's notation, **A**, **B**, and **C**, respectively, denote the symmetric $n \times n$ mass, damping, and stiffness matrices. Caughey's criterion, which includes Rayleigh's criterion as a special case, assumes, without loss of generality [7], that **A** is the identity matrix and it requires that

$$\mathbf{B} = \sum_{r=0}^{n-1} \beta_r (\mathbf{C}^{1/s})^r \qquad (1)$$

where the β's are arbitrary constants, s is any positive integer, and $\mathbf{C}^{1/s}$ designates any symmetric s-root of **C**. In the author's first example it is claimed that $\mathbf{B} = \mathbf{C}^{-1}$ permits the desired uncoupling but does not satisfy (1). However the Cayley-Hamilton theorem guarantees that one can express the inverse of any nonsingular square matrix **M** of order n as a polynomial in **M** of degree at most $n - 1$. Thus such as expression for $\mathbf{C}^{-1}$ satisfies (1) with $s = 1$, and Caughey's criterion is satisfied.

In his second example, the author gives explicit 3×3 matrices which permit uncoupling of the equations of motion. But once again it is easy to show that Caughey's criterion with $s = 1$ is satisfied. If **A** is not taken as the identity matrix, then Caughey's criterion with $s = 1$ becomes

$$\mathbf{B} = \beta_0 \mathbf{A} + \sum_{r=1}^{n-1} \beta_r \mathbf{C}(\mathbf{A}^{-1}\mathbf{C})^{r-1} \qquad (2)$$

For the matrices of the author's example, simple calculation shows that

$$\mathbf{B} = 3\mathbf{A} + \frac{3}{2}\mathbf{C} - \frac{1}{2}\mathbf{C}\mathbf{A}^{-1}\mathbf{C}$$

which is clearly an example of equation (2).

It is easy to construct examples of any order which demonstrate the nonnecessity of Caughey's criterion. If **A** and **C** both equal the identity matrix, for example, then (1) requires that **B** be diagonal, but the equations of motion can clearly be uncoupled for any symmetric **B** matrix, since any similarity transformation which diagonalizes **B** also diagonalizes the identity matrix.

Finally, the author's necessary and sufficient condition for the existence of uncoupled equations of motion (his equation (19))

$$\mathbf{B}\mathbf{A}^{-1}\mathbf{C} = \mathbf{C}\mathbf{A}^{-1}\mathbf{B} \qquad (3)$$

was given by Caughey and O'Kelly in 1965 [8, equation (18)]. They made the important additional step of showing, in effect, that (2) is a solution of (3) and that it becomes the most general real symmetric solution in the case where $\mathbf{A}^{-1}\mathbf{C}$ has no multiple eigenvalues. In the case of multiple eigenvalues, it can be shown that (3), which is linear and homogeneous in **B**, has more than n linearly independent solutions whereas (2) gives fewer than n linearly independent solutions. To be precise, for each distinct natural frequency of multiplicity m of the undamped system, there corresponds $m(m + 1)/2$ linearly independent damping matrices satisfying (3). The proof of this result and the development of a complete solution of (3) will be submitted for publication as a Brief Note by the discusser.

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Author's Closure

The author would like to thank Mr. Mallett for his interest in the paper, and for his discussion. The comments mentioned, however, seem to be motivated by an objective which is different from that of the paper. The main objective of the paper was to provide a simple means of finding out whether certain given **A**, **B**, and **C** matrices admit decoupling of the equations of motion or not. If they do the solution is greatly simplified and easily expressed in terms of the modal shapes of the undamped system. To this extent it is not feasible, if at all possible, to apply Caughey's criterion in a general systematic way.

Such a process does require the previous knowledge of the normalized modal shapes of the system. But if the modal shapes of the system are already known nothing much is gained by knowing whether the original equations of motion can be decoupled or not.

The fact that $\mathbf{B} = \mathbf{C}^{-1}$ satisfies Caughey's criterion along the lines suggested in the discussion cannot be accomplished without realizing beforehand that the damping matrix in a certain system is indeed the inverse of its stiffness matrix. Mr. Mallett does not say how this can be done. Also the relationship which he gave for the matrices of the second example of the paper can not be reached, neither by simple calculation nor otherwise, without making use of the modal shapes that are only known after the system is solved.

As for the example which is proposed to demonstrate the nonnecessity of Caughey's criterion it really belongs to a very special category unsuitable for a general analysis. When both **A** and **C** are identity matrices the undamped system has one nautral frequency of multiplicity n.

In short the paper sets to investigate the matrices **A**, **B**, and **C** as a first step toward the analysis of a general damped system. The discussion relies upon solving the system first and then correlating between the natural frequencies and the possible forms of damping matrices that satisfy the uncoupling criterion.

Having said that it remains to thank Mr. Mallett again for pointing out his reference [8] which was not known to the author beforehand.

Elastic Interface Waves Involving Separation[1]

L. B. Freund.[2] In this paper the authors consider the possibility that an unbonded interface between two elastic solids, across which tensile tractions cannot be transmitted, can sustain steady interface

[1] By Maria Comninou and J. Dundurs, and published in the June, 1977, issue of ASME JOURNAL OF APPLIED MECHANICS, Vol. 44, pp. 222–226.
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waves which involve local separation of the two materials. On the basis of an elegant mathematical analysis of the problem, they conclude that such plane strain interface waves are indeed possible provided that the solids are pressed together by remotely applied tractions. As noted by the authors, the interface waves involving separation exhibit features which are similar to those encountered in dynamic fracture. Thus it would seem worthwhile to consider the solution of this particular problem in light of certain general results which have emerged from dynamic crack propagation analysis. Such an examination reveals that the interface waves exhibit a feature which seems to be physically unrealistic and, consequently, suggests that the question of existence of these interface waves warrants further consideration. To keep the discussion as simple as possible, consideration is restricted primarily to the symmetric separation of an unbonded interface between *identical* elastic materials.

The leading and trailing edges of the localized separation regions of the interface waves are essentially the same as moving crack tips, and the stress and velocity fields very near the moving separation point (i.e., the leading edge of the separation zone) would therefore have the universal spatial dependence established for arbitrarily moving crack tips in [1].[3] In particular, the particle velocity of points on the interface within the separation zone near a right-going separation point and the normal stress transmitted across the interface in front of the separation point are, from [1],

$$\dot{u}_2(x_1, \pm 0, t) = \mp \frac{c^3}{c_T{}^2} \frac{\zeta_1 K_I}{\mu R \sqrt{-2\pi x_1}}, \quad \sigma_{22}(x_1, 0, t) = \frac{K_I}{\sqrt{2\pi x_1}} \qquad (1)$$

for $x_1 < 0$ and $x_1 > 0$, respectively, where the authors' notation is followed, the moving point in question is at $x_1 = 0$, and K_I is the so-called dynamic stress-intensity factor of fracture mechanics. It appears from equations (52) and (54) of the paper that the solution obtained exhibits the general property represented by (1). For speed c in the range $c_R < c < c_T$, the Rayleigh wave function R is positive. The fact that the material surfaces move away from each other behind the separation point implies a negative stress-intensity factor, that is, the interface traction ahead of the separation point is compressive and thus admissible. On the other hand, for speed c in the range $0 < c < c_R$, R is negative and separation requires positive traction ahead of the separation point, which is inadmissible. It is thus concluded from this result, along with a similar result for the closure point, that interface waves satisfying the specified conditions cannot exist for speeds c in the range $0 < c < c_R$, but that speeds in the range $c_R < c < c_T$ have not yet been ruled out.

The energy flux through a contour which begins on one traction-free face of the separation zone, surrounds the separation or closure point, and terminates on the opposite face of the separation zone is discussed in detail in [2]. If the local elastic fields are square root singular at the end point, then in general the energy flux F toward the end point has a finite, nonzero limit as the loop is shrunk down onto the end point, viz.,

$$F = \mp (2\zeta_1 c^3/\mu R c_T{}^2) K_I{}^2, \qquad (2)$$

the upper and lower signs corresponding to right-going separation and closure points, respectively. Recall that $R > 0$ for $c_R < c < c_T$. The result (2) leads immediately to the physically unrealistic conclusion that, if a separation zone of any type propagates with speed c in the range $c_R < c < c_T$, then the separation (closure) point necessarily acts as a source (sink) of mechanical energy. For the separation waves considered by the authors the energy created at the separation point is exactly absorbed at the closure point for each separation zone so that overall energy is constant, but the physical mechanism for this energy creation-absorption process at a point is not at all clear and the result might be regarded as evidence against the existence of such interface waves.

The transient processes of separation and closure of a cohesionless interface have been considered in [3, p. 187] and in [4, p. 334], respectively. In each case, the condition that a *cohesionless* interface could be neither a source nor sink of mechanical energy was imposed, with the result that a separation or closure point could travel only with the Rayleigh wave speed, $c = c_R$. In these cases, if the separation or closure point had been allowed to be a source or sink of mechanical energy, then any propagation speed in the interval $c_R \leq c < c_T$ would also have been admissible. This is a common situation in dynamic crack propagation analysis. For a specific geometrical configuration and applied loading, mathematical solutions can be found for a wide variety of crack motions. Because of the energy sink (or source) characteristics of a propagating crack tip, however, the mathematical solution is considered to represent a physically realizable process *only* when the amount of energy created or absorbed is consistent with an additional physical principle, a so-called fracture criterion. In other words, *mechanical* energy is not conserved for a solution with a source or sink and, unless another form of energy (e.g., surface energy in brittle fracture) can be identified so that *total* energy is conserved, the solution does not represent a physically realizable process. It is also of interest to note that if Yoffe's classical steady-state crack propagation solution [5] is evaluated for a *negative* remote tension and for a crack speed c in the range $c_R < c < c_T$, then the solution represents an interface wave of precisely the type considered in the present paper, except that there is a single separation zone rather than an infinite periodic array of separation zones. When evaluated in this range, this solution also has the physically unrealistic feature that the separation and closure points act as a source and sink, respectively, of mechanical energy.

Finally, the points raised here concerning the existence of separation waves at the interface of identical solids seem to apply as well to the case of dissimilar frictionless solids, for which the dominant singular parts of the stress and velocity fields near a separation or closure point have the form (1) in each material. The total energy flux into a separation or closure point in this case seems to vanish when $c = c_A$ or when $K_I = 0$. Aside from the trivial case of vanishing applied load, it would seem that $K_I = 0$ when $c = c_R$ or $c = \bar{c}_R$. This last condition is difficult to establish with certainty from a steady-state analysis, however, and final resolution of the question must await the analysis of the appropriate transient wave propagation problems involving a smooth, cohesionless interface between dissimilar materials.

Acknowledgment

I am grateful to Professor Dundurs for making a preprint of the paper on interface waves available to me.

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Authors' Closure

The discussion provided by Professor Freund points not only to a quandary in our paper, but also to a general dilemma in elastodynamics: What do we do with moving singularities that emit or absorb energy? When should such singularities be admitted, and when do they become illegitimate in a problem that appears reasonably well posed?

Suppose that the displacement components in a coordinate system attached to the moving singularity are

[3] Numbers in brackets designate References at end of Discussion.

$$u_i = O(r^{1-\lambda}), \quad r \to 0$$

where r is the distance from the singular point in a plane problem. Then

$$\partial_i u_j, \sigma_{ij}, \dot{u}_i = O(r^{-\lambda}), r \to 0$$

If the singularity is to be integrable, we must insist that $\lambda < 1$. Taking for S a circle of radius ϵ, so that $dS = \epsilon d\theta$, it follows from the Freund formula [6, equation (13)][4] that the rate at which the singularity emits energy is

$$E = O(\epsilon^{1-2\lambda}), \quad \epsilon \to 0$$

Therefore, all singularities that are stronger than the inverse square root ($\lambda > 1/2$) emit energy at an infinite rate. The inverse square root singularity ($\lambda = 1/2$) is exceptional, as it emits or absorbs energy at a finite rate. Singularities weaker than the inverse square root ($\lambda < 1/2$) do not emit energy at all.

It so happens that moving cracks, just as their static counterparts, involve inverse square root singularities, and thus E is finite. The point of view, now well entrenched in fracture mechanics, is that only energy absorption can be admitted, and that it must be related *at each individual singularity* to the energy required for fracture. We would by no means argue that this is not a valid point of view in fracture mechanics. The same point of view adopted in other situations, however, leads to certain dilemmas.

Consider, for instance, a flat frictionless punch with sharp corners that is pressed against an elastic half plane and made to move with a constant velocity. The steady-state solution [7, Section 7.13] again involves inverse square root singularities at the corners. If the velocity of the punch is above the Rayleigh speed, the leading corner is an energy sink and the trailing corner a source, but the same amount of energy is absorbed by the sink as is emitted by the source. In this case one can identify the tractions exerted by the punch as doing work on the elastic material at highly intensified rates near the corners, but the external agent moving the punch has to do no work, because the punch stays on the same level at all times. Even if one considered the transient problem of a punch that, say, starts to move from rest, the singularities are bound to be of the same type, because they are already present in the equilibrium problem, and energy sources and sinks will appear at the corners. Rounding the corners just slightly, would remove the singularities and the energy sources and sinks, but is not likely to affect the fields except very near the corners. Should the solution for the flat punch be discarded? We do not know.

The situation in our problem is very similar to that for the punch, as there is apparently a focusing effect leading to singularities at the edges of the separation zones. In fact if the half planes are made of identical materials, the fields are the same as induced by a periodic array of moving flat punches. An overall energy balance is preserved, as the energy emitted by the leading edge of a separation zone is absorbed at its trailing edge. It may be noted in this context that the uniqueness theorem derived by Freund and Clifton [8] in connection with crack motion does not require more than an overall balance of source and sink strengths.

Finally, we do not believe that the peculiarities in our solution are due to the fact that a steady-state motion is considered. When the separation of the interface is forced by incident waves causing a steady-state motion of the gaps, we have found that singularities appear in some cases, but the solutions can be chosen so that the singularities are weaker than inverse square root [9].

In our opinion, most of the points raised in the discussion still remain open. We are not as pessimistic as Professor Freund, however, and think that there is a fair chance that interface waves involving separation will eventually be observed.

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The Problem of Internal and Edge Cracks in an Orthotropic Strip[1]

K. Arin.[2] This paper and also other recent papers by the same authors [1–2][3] lack complete referencing to the previous work done on the subject which might have been inadvertently omitted. Hence, for the sake of completeness, those references [3–5] which deal with solutions in orthotropic media have been added here. It should also be noted that the formulation developed previously in [3–5] and subsequently used in this paper and [1–2] is based on the general formulation of the theory of elasticity of anisotropic bodies by Lekhnitskii [6] and the proper use of the principle of superposition in the linear theory of elasticity [7].

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[2] General Electric Company, 53-332, Schenectady, N. Y.

[3] Numbers in brackets designate References at end of Discussion.

Stress-Induced Radial Pressure Gradients in Liquid-Filled Multiple Concentric Cylinders[1]

C. M. Rodkiewicz.[2] I would like to compliment the authors on their work and encourage them to pursue the subject further.

The presentation of results in Fig. 4 conveys the information well. However, it could be more compact and consequently the influence

[1] By M. Munro and K. Piekarski and published in the June, 1977, issue of ASME JOURNAL OF APPLIED MECHANICS, Vol. 44, pp. 218–221.

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