The Green function method is applied to the problems of electron gas. It is emphasized that there is a perfect parallelism between the Green function methods in zero-temperature problems and in finite-temperature problems of systems in thermal equilibrium. Further it is pointed out that the Green function method is also useful for the calculation of transport quantities.

§ 1. Introduction

Much advances have been made recently in the study of many-body problems in quantum mechanics and quantum statistics. Particularly noteworthy is the computation of the exact correlation energy of an electron gas at high density by Gell-Mann and Brueckner (GB). The high density result of GB has subsequently been derived by other methods by Sawada et al., Hubbard, and others. The field theoretical techniques are quite useful in summing certain subseries of terms in the so-called “linked-cluster expansion”. Recently Galickij and Migdal, and Klein and Prange have applied the Green function method to many-body problems in quantum mechanics. This method is quite suitable for the study of many-body problems. From the one-particle Green function one may compute the energy and damping of quasiparticles and from the two-particle Green function one may obtain the potential energy due to two-body forces and the energy spectrum of excitations which are not describable as a sum of energies of quasi-particles.

Montroll and Ward have developed a generalization of the cluster integral theory of Mayer to deal with the quantum statistics of many-particle systems. They have shown that in the case of an electron gas the classical limit of the contribution of ring integrals to the grand partition function yields the Debye-Hückel theory, while the low temperature limit leads to the GB result.

Matsubara has applied a formalism of quantum field theory to the calculation of the grand partition function in quantum statistics. Matsubara’s method has been developed recently by Bloch and Dominicis, Fradkin, and others. The applicability of the field theoretical techniques to quantum statistics is based on the existence of the theorem corresponding to that of Wick in quantum field theory. There is a close correspondence, which was first pointed out by Matsubara, between the Green function methods in quantum mechanics and in quantum statistics.
The present paper is mainly concerned with an exposition of the methods. We obtain the two-particle Green function of a dense electron gas and derive the result of GB. From the Bethe-Salpeter equation for the two-particle Green function for an electron-hole pair the dispersion relation of plasmons is derived. This method is applied to the problem of the exciton and the plasmon in insulators. The improvement of the dispersion relation of plasmons by including higher order effects will be carried out in II. The calculation of the diamagnetic susceptibility of a dense electron gas, which will be published in III, is based on the two-particle Green function obtained in § 2. In § 3 we show that the grand partition function of an electron gas at high density can be calculated by the use of the two-particle Green function which is obtained in quite the same way as in § 2 and that the result of Montroll and Ward is derived quite simply. In § 4 we introduce Green functions which are dependent on time and temperature and we point out that these Green functions are suitable for the calculation of transport coefficients. We emphasize that there is a close resemblance between the Green function methods in zero temperature problems and in finite-temperature problems of systems in thermal equilibrium and in the problems of transport processes.

§ 2. Time-dependent Green functions

The Hamiltonian for the system of electrons is

\[ H = H_0 + H_1, \]

where

\[ H_0 = \frac{\hbar^2}{2m} \int \psi^*(r) \cdot \Gamma \psi(r) \, dr, \]  
\[ H_1 = \frac{1}{2} \int \int \psi^*(r) \psi^*(r') \rho(r-r') \psi(r') \psi(r) \, dr \, dr', \]

\[ \rho(r-r') = \frac{1}{V} \sum_k \nu(k) \exp(ik(r-r')), \quad \nu(k) = 4\pi e^2/k^2. \]

\( \psi(r), \psi^*(r) \) are the field operators which satisfy the well-known anticommutation relations

\[ [\psi(r), \psi^*(r')] = \delta(r-r'), \]

\[ [\psi(r), \psi(r')] = [\psi^*(r), \psi^*(r')] = 0. \]

The one- and two-particle Green functions are defined as

\[ G(x_1, x_2) = i \langle T \{ \psi(x_1) \psi^*(x_2) \} \rangle, \]

\[ G(x_1, x_3; x_2, x_4) = i \langle T \{ \psi(x_1) \psi(x_2) \psi^*(x_3) \psi^*(x_4) \} \rangle, \]

where \( x \) represents a point in space and time, and \( T \) is the chronological ordering symbol of Wick. \( \psi(x), \psi^*(x) \) are the Heisenberg operators; \( \psi(x) = \exp[i/\hbar \cdot Ht] \)
H. Kanazawa and M. Watabe

\[ \psi(r) \exp[-i/\hbar \cdot Ht]. \] The average is taken over the ground state function of the Hamiltonian \( H \). From (2.1) and (2.6) we obtain

\[ E_0 = \langle H \rangle \]

\[ = \varepsilon_0 + \frac{i}{2} \int \frac{d\gamma}{g} \int \lim_{t' \to t} G(\gamma, \gamma') \psi(\gamma) \psi^*(\gamma') \psi(\gamma) \psi^*(\gamma') \ d\gamma \ d\gamma', \quad (g = e^\gamma) \quad (2.7) \]

where \( \varepsilon_0 \) is the ground state energy without interaction. As is well known, (2.5) and (2.6) may be written in the form

\[ G(x_1, x_2) = i \langle T \{ \psi(x_1) \psi^*(x_2) S \} \rangle_0 / \langle S \rangle_0, \quad (2.8) \]

\[ G(x_1, x_2; x_3, x_4) = i \langle T \{ \psi(x_1) \psi(x_2) \psi^*(x_3) \psi^*(x_4) S \} \rangle_0 / \langle S \rangle_0, \quad (2.9) \]

where \( \psi(x) = \exp[i/\hbar \cdot H_0 t] \psi(r) \exp[-i/\hbar \cdot H_0 t] \) and \( S \) is the \( S \) matrix; the average is taken over the ground state function of the unperturbed Hamiltonian \( H_0 \).

As was shown by Schwinger\(^{13}\) and Gell-Mann and Low\(^{14}\), the equation for the two-particle Green function has the form

\[ G(x_1, x_2; x_3, x_4) = i G(x_1, x_2) G(x_3, x_4) - i G(x_1, x_4) G(x_3, x_2) \]

\[ + i/\hbar \cdot \int \cdots G(x_1, x_2) G(x_3, x_4) \Gamma(x_5, x_6; x_7, x_8) G(x_7, x_8; x_3, x_4) d\gamma_1 d\gamma_2 d\gamma_3 d\gamma_4, \]

\[ (2.10) \]

where \( \Gamma \) is the irreducible interaction part. At high density limit we may obtain the two-particle Green function without solving the integral equation (2.10). Let us define the effective interaction \( \check{V}(x_1 - x_2) \) by\(^5\)

\[ \check{V}(x_1 - x_2) = v(x_1 - x_2) + 1/i \hbar \cdot \int \psi(x_1 - x_2) G_0(x_3, x_4) G_0(x_4, x_3) \check{V}(x_4 - x_2) d\gamma_3 d\gamma_4 \]

\[ (2.11) \]

Fig. 1. Wavy and dotted lines represent the effective and Coulomb interactions respectively

where \( v(x_1 - x_2) = v(r_1 - r_2) \delta(t_1 - t_2) \). The Feynman diagram corresponding to (2.11) is shown in Fig. 1. \( G_0(x, x') \) is the one-particle Green function without interaction and is given by

\[ G_0(x, x') = \frac{1}{2\pi V} \sum_k \int G_0(k, \varepsilon) \exp[i\hbar \cdot (r - r') - i\varepsilon (t - t')] d\varepsilon, \quad (2.12) \]

where
Green Function Method for Electron Gas. I

\[ G_0(k, \varepsilon) = \frac{-1}{\varepsilon - \varepsilon_0(k)/\hbar + i\delta(k)} \quad (\delta \to 0) \quad (2.13) \]

and

\[ \varepsilon_0(k) = \hbar^2 k^2 / 2m, \]

\[ \theta(k) = 1 - 2n_0(k). \quad (2.14) \]

Here \( n_0(k) \) is the occupation number of unperturbed state. We can solve (2.11) at once by introducing the Fourier transform of \( \mathcal{G} (x_1 - x_2) \):

\[ \mathcal{G} (x_1 - x_2) = \frac{1}{2\pi V} \sum_{q, \omega} \mathcal{G} (q, \omega) \exp\{i\mathbf{q} \cdot (r_1 - r_2) - i\omega(t_1 - t_2)\} d\omega. \quad (2.15) \]

We get

\[ \mathcal{G} (q, \omega) = v(q) / \left[ 1 + v(q)Q(q, \omega) \right], \quad (2.16) \]

where

\[ Q(q, \omega) = \frac{\hbar}{2\pi} \sum_{k} \frac{n_0(q + k) - n_0(k)}{\omega + (\varepsilon_0(k) - \varepsilon_0(q + k))/\hbar - i\delta[n_0(k) - n_0(q + k)]}. \quad (2.17) \]

Then the two-particle Green function is given by

\[ G(x_1, x_2; x_3, x_4) = iG_0(x_1, x_3)G_0(x_2, x_4) - iG_0(x_1, x_4)G_0(x_2, x_3) \]

\[ + \frac{\hbar}{2} \int G_0(x_1, x_3)G_0(x_2, x_4) \mathcal{G} (x_3 - x_6) \left\{ iG_0(x_5, x_3)G_0(x_6, x_4) \right\} dx_5 dx_6 \]

\[ + \frac{\hbar}{2} \int G_0(x_1, x_3)G_0(x_2, x_4) v(x_3 - x_6) \left\{ -iG_0(x_5, x_4)G_0(x_6, x_3) \right\} dx_5 dx_6. \quad (2.18) \]

Inserting (2.18) into (2.7), we get the result of GB for the energy of the ground state. The first term of the right-hand side of (2.18) does not contribute to the energy. The second term gives the exchange energy (Fig. 2a). The contribution from the third term is represented by the diagram shown in Fig. 2b. The last term gives the second order exchange energy which is denoted as \( \varepsilon_{ex}^{(2)} \) in GB's paper (Fig. 2c).

The Green function which yields the result of GB (except the last term in (2.18)) may also be obtained by solving the following Bethe-Salpeter equation,
The interaction part of the second term in (2·19) is described by the diagram shown in Fig. 3a. The dispersion relation of plasmons can be derived from (2·19). Let \( t_1, t_3 > t_2, t_4 \). Then, using the completeness of the set of eigenfunctions \( \mathcal{F}_n \), we rewrite (2·6) as

\[
G(x_1, x_3; x_5, x_6) = -i \sum_n \mathcal{Z}_n(x_1, x_3) \mathcal{Z}_n(x_5, x_6),
\]

where

\[
\mathcal{Z}_n(x_1, x_3) = \langle \mathcal{F}_0, T \{ \phi(x_1) \phi^*(x_3) \} \mathcal{F}_n \rangle,
\]

\[
\mathcal{Z}_n(x_5, x_6) = \langle \mathcal{F}_0, T \{ \phi(x_5) \phi^*(x_6) \} \mathcal{F}_n \} \rangle^*.
\]

Here \( T \) orders the operators in the reversed chronological order to \( T \). Following Gell-Mann and Low’s procedure, we obtain the following equation for the bound state (plasmon state) amplitude \( \chi(x_1, x_3) \) for a pair of an electron and a hole,

\[
\chi(x_1, x_3) = -i/\hbar \int G_0(x_1, x_3) G_0(x_5, x_6) v(x_5 - x_6) \chi(x_5, x_6) dx_5 dx_6.
\]

For simultaneous time \( t_1 = t_3 \), the function \( \chi(x_1, x_3) \) has the physical meaning of wave function describing the behavior of the electron-hole pair and may be written as

\[
\chi(r_1, r_3) = \exp \{ i \hbar \cdot (r_1 + r_3) - i \omega t \} \frac{1}{V} \sum_p \exp \{ i p \cdot (r_1 - r_3) \} f_{pk}^p \phi^1(p),
\]

where \( \hbar \omega \) is the excitation energy, and \( \hbar k \) and \( \hbar p \) are the total and relative momentum. Inserting (2·23) into (2·22), we get the dispersion relation of Bohm and Pines

\[
1 = \frac{1}{2\pi \hbar V} v(k) \sum_p G_0 \left( p + \frac{k}{2}, \varepsilon \right) G_0 \left( p - \frac{k}{2}, \varepsilon - \omega \right) d\varepsilon
\]

\[
= \frac{4\pi e^2}{m V} \sum_p \frac{n_0(p)}{(\omega - \hbar k \cdot p/m)^2 - \hbar^2 k^4 / 4m^2}. \tag{2·24}
\]

The improvement of the dispersion relation by including the higher order effects will be discussed in II. It should be noted that the normalization of the bound state amplitude is derived from the sum rule (see Appendix). We can also obtain the dispersion relation of plasmons in solids, which was derived by Kanazawa, and Nozières and Pines, by expanding \( \phi(r) \), \( \phi^*(r) \) in terms of Bloch functions.
Next, we apply the above-mentioned method to the problem of exciton and plasmon in insulators. Adding the interaction part which is described by the diagram shown in Fig. 3b to (2·19), we get the equations for the singlet and triplet states of an electron-hole pair

\[ \chi(x_1, x_2) = -i/\hbar \int \int G_0(x_1, x_3) G_0(x_4, x_2) v(x_3 - x_4) \chi(x_4, x_1) dx_3 dx_4 + i/\hbar \int \int G_0(x_1, x_3) G_0(x_4, x_2) v(x_3 - x_4) \chi(x_3, x_4) dx_3 dx_4, \]  

(2·25a)

(for singlet state)

\[ \chi(x_1, x_2) = i/\hbar \int \int G_0(x_1, x_3) G_0(x_4, x_2) v(x_3 - x_4) \chi(x_2, x_4) dx_3 dx_4. \]  

(2·25b)

(for triplet state)

As the unperturbed Hamiltonian we adopt that of the Hartree-Fock approximation and denote the eigenfunctions (Bloch functions) and eigenvalues by \( \varphi_{n\sigma}(r) \) and \( \varepsilon_n(p) \) respectively, where \( n, p \) and \( \sigma \) denote the band index, the reduced wave vector and the spin. The Hamiltonian is

\[ H = H_0 + H_1, \]  

(2·26)

\[ H_0 = \sum_{n\rho} \varepsilon_n(p) a_{n\rho}^{\dagger} a_{n\rho}, \]  

(2·27a)

\[ H_1 = \frac{1}{2} \sum_{\langle n_1 n_2 n_3 n_4 \rangle} \langle n_1 p_1 + k | p_2 - k \rangle \langle n_2 p_1 | p_2 \rangle \langle n_3 p_2 | p_1 \rangle \langle n_4 p_2 | p_1 \rangle \varphi_{n_1 p_1 + k \sigma_1} \varphi_{n_2 p_2 - k \sigma_2} \varphi_{n_3 p_2} \varphi_{n_4 p_2} a_{n_1 p_1 \sigma_1} a_{n_2 p_2 \sigma_2} a_{n_3 p_2 \sigma_3} a_{n_4 p_2 \sigma_4}, \]  

(2·27b)

where

\[ \langle n_1 n_2 n_3 n_4 \rangle \langle p_1 p_2 | p_3 p_4 \rangle = \int \varphi_{n_1 p_1}^*(r_1) \varphi_{n_2 p_2}^*(r_2) \frac{e^2}{| r_1 - r_2 |} \varphi_{n_3 p_3}(r_3) \varphi_{n_4 p_4}(r_4) dr_1 dr_2, \]  

(2·28)

and \( N_{np} \) is the occupation number of unperturbed state, and \( N_{np} = 1 \) and \( N_{np} = 0 \) correspond to the valence band and conduction band respectively.

We expand \( \chi(x, x') \) for simultaneous times in terms of \( \varphi_{np} \):

\[ \chi_{np}(r, r') = \sum_{n_{12} p_{12}} \varphi_{n_{12} p_{12}}(r) \varphi_{n_{12} p_{12}}^*(r') \exp(-i\omega t) f_{n_{12} n_{12}}^{\omega_{np}}(p). \]  

(2·29)

Considering the spin states, we write \( f \) as \( \mathbf{f} \) where \( \mathbf{j} \) and \( \mathbf{j} \) correspond to the singlet and triplet respectively. (2·25a) and (2·25b) lead to

\[ \mathbf{f}^{\omega_{np}}_{n_{12} n_{12}}(p) = \left[ \sum_{n_{12} p_{12}} \langle n_1 | p + k/2 \rangle \langle n_2 | p' - k/2 \rangle \langle n_1' | p - k/2 \rangle \langle n_2' | p' + k/2 \rangle \right] \]

\[ -2\delta_{n_1 n_1'} \sum_{n_2 n_2'} \langle n_1 | p + k/2 \rangle \langle n_2 | p' - k/2 \rangle \langle n_1' | p' + k/2 \rangle \langle n_2' | p - k/2 \rangle \]
where
\[ \delta_M = \begin{cases} 1 & \text{for } M=1, \\ 0 & \text{for } M=3. \end{cases} \] (2.31)

Rewriting (2.30) by using the abbreviations \( c \) and \( v \) for the indices corresponding to the conduction and valence band, respectively, we have

\[
M_f^{cc}(p) = \frac{1}{\hbar \omega - \varepsilon_c(p+k/2) + \varepsilon_v(p-k/2)} \times \left[ \sum \langle \begin{array}{c} c \\ p+k/2 \\

\end{array} | \begin{array}{c} \bar{c} \\ p'-k/2 \\

\end{array} \rangle V \langle \begin{array}{c} \bar{v} \\ p-k/2 \\

\end{array} | \begin{array}{c} c' \\ p'+k/2 \\

\end{array} \rangle \right] \cdot f_{f_{cc}}(p),
\] (2.32a)

\[
M_f^{vc}(p) = \frac{1}{\hbar \omega - \varepsilon_c(p-k/2) + \varepsilon_v(p+k/2)} \times \left[ \sum \langle \begin{array}{c} v \\ p+k/2 \\

\end{array} | \begin{array}{c} \bar{v} \\ p'-k/2 \\

\end{array} \rangle V \langle \begin{array}{c} \bar{c} \\ p-k/2 \\

\end{array} | \begin{array}{c} c' \\ p'+k/2 \\

\end{array} \rangle \rangle \cdot f_{f_{cc}}(p'),
\] (2.32b)

These eigenvalue equations are the same as those derived by Horie.17)

§ 3. Temperature-dependent Green functions

The one- and two-particle Green functions are defined as

\[
G(x_1, x_2) = \langle T \{ \phi(x_1) \bar{\psi}(x_2) \} \rangle,
\] (3.1)

\[
G(x_1, x_2 ; x_3, x_4) = \langle T \{ \phi(x_1) \phi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4) \} \rangle,
\] (3.2)

where

\[
\phi(x) = \exp(\lambda t) \phi(r) \exp(-\lambda t),
\] (3.3a)

\[
\bar{\psi}(x) = \exp(\lambda t) \phi^*(r) \exp(-\lambda t)
\] (3.3b)
Green Function Method for Electron Gas. I

\[ \mathcal{H} = H - \mu N; \quad (3.4) \]

\( \mu \) and \( N \) are the chemical potential and the total number operator respectively. In this section \( t \) is the variable corresponding to inverse temperature (multiplied by \( k \)) and \( T \) is the corresponding ordering symbol. The average is the grand canonical one:

\[ \langle \cdots \rangle = \text{Tr} \{ \exp(-\beta \mathcal{H}) \cdots \} / \Xi, \quad (3.5) \]

where \( \beta = (kT)^{-1} \), \( k \) and \( T \) being the Boltzmann constant and temperature respectively. \( \Xi \) is the grand partition function:

\[ \Xi = \text{Tr} \exp(-\beta \mathcal{H}) = \exp(-\beta \Omega), \quad (3.6) \]

where \( \Omega \) is the grand potential. From (3.2) and (3.6) we get

\[ \Omega = \Omega_0 - \frac{1}{2} \int \frac{d\mathcal{y}}{g} \int_{\mathcal{H}_{\mathcal{G}_1}} \lim_{\mathcal{H} \to \mathcal{H}_1} \mathcal{G}(r\beta_1, r'\beta_1; \mathcal{r}\beta_2, r'\beta_2) v(r-r') dr dr', \quad (3.7) \]

where \( \Omega_0 \) is the grand potential without interaction, i.e. the grand potential of a perfect gas: \( \Xi_0 = \text{Tr} \exp(-\beta \mathcal{H}_0) = \exp(-\beta \Omega_0) \). (3.1) and (3.2) can be written as

\[ \mathcal{G}(x_1, x_2) = \langle T \{ \psi(x_1) \bar{\psi}(x_2) U(\beta) \} \rangle_0 / \langle U(\beta) \rangle_0, \quad (3.8) \]

\[ \mathcal{G}(x_1, x_2; x_3, x_4) = \langle T \{ \psi(x_1) \psi(x_3) \bar{\psi}(x_2) \bar{\psi}(x_4) \bar{\psi}(x_3) U(\beta) \} \rangle_0 / \langle U(\beta) \rangle_0, \quad (3.9) \]

where

\[ \psi(x) = \exp(\mathcal{H}_0 t) \psi(r) \exp(-\mathcal{H}_0 t), \quad \bar{\psi}(x) = \exp(\mathcal{H}_0 t) \bar{\psi}(r) \exp(-\mathcal{H}_0 t), \quad (3.10) \]

\[ U(\beta) = T \left\{ \exp \left( - \int_0^\beta H_1(t) dt \right) \right\}, \quad (3.11) \]

\[ H_1(t) = \exp(\mathcal{H}_0 t) H_1 \exp(-\mathcal{H}_0 t), \quad (3.12) \]

and

\[ \langle \cdots \rangle_0 = \text{Tr} \{ \exp(-\beta \mathcal{H}_0) \cdots \} / \Xi_0. \quad (3.13) \]

Comparing these formulae with those in § 1, we see that there is a close correspondence between the time-dependent and the temperature-dependent Green functions. Therefore we may use Feynman diagrams in the same way as in § 1. The integral equation for the two-particle Green function has the form

\[ \mathcal{G}(x_1, x_2; x_3, x_4) = -\mathcal{G}(x_1, x_3)\mathcal{G}(x_2, x_4) + \mathcal{G}(x_1, x_4)\mathcal{G}(x_2, x_3) \]

\[ -\int \cdots \int \mathcal{G}(x_1, x_2)\mathcal{G}(x_2, x_3) \mathcal{F}(x_5, x_6; x_7, x_8) \mathcal{G}(x_7, x_8; x_3, x_4) dx_3 dx_4 dx_5 dx_6 dx_7 dx_8. \quad (3.14) \]
Now we calculate the grand potential in the approximation corresponding to GB’s approximation or Mayer’s ring approximation in classical statistics. Our approximation corresponds to the following simplification in (3.14)

\[
\bar{G}(x_1, x_2) \rightarrow \bar{G}_0(x_1, x_2) = \left\langle T \{ \phi(x_1) \bar{\phi}(x_2) \} \right\rangle_0
\]

\[
= \left\{ \frac{1}{V} \sum_k (1-f_k) \exp \{ ik \cdot (\mathbf{r}_1 - \mathbf{r}_2) - (\varepsilon_0(k) - \mu) (t_1 - t_2) \} \right\} (t_1 > t_2),
\]

\[
= \left\{ -\frac{1}{V} \sum_k f_k \exp \{ ik \cdot (\mathbf{r}_1 - \mathbf{r}_2) - (\varepsilon_0(k) - \mu) (t_1 - t_2) \} \right\} (t_1 < t_2),
\]

\[
(f_k = 1/[\exp(\beta(\varepsilon_0(k) - \mu)) + 1])
\]

\[
\bar{G}(x_7, x_8; x_3, x_4) \rightarrow \bar{G}_0(x_7, x_8) \bar{G}_0(x_3, x_4),
\]

\[
\bar{\Gamma}(x_0, x_6; x_1, x_5) \rightarrow \bar{\psi}(x_0 - x_6) \delta(x_0 - x_1) \delta(x_0 - x_5),
\]

where \(\delta(x_0 - x_1) \delta(x_0 - x_5)\) is the effective interaction determined by the following equation,

\[
\bar{\psi}(x_0 - x_6) = \mathcal{V}(x_0 - x_6) - \int \int \int \mathcal{V}(x_0 - x) \bar{\psi}(x_0 - x_6) dx dx dx',
\]

\[
(3.18)
\]

where

\[
\mathcal{V}(x_0 - x_6) = \mathcal{V}(r_0 - r_6) \delta(t_0 - t_6).
\]

Making the approximations given above and adding the term corresponding to the second order exchange effect, we get the two-particle Green function as follows:

\[
\bar{G}(x_1, x_3; x_2, x_4) = -\bar{G}_0(x_1, x_3) \bar{G}_0(x_2, x_4) + \bar{G}_0(x_1, x_4) \bar{G}_0(x_2, x_3)
\]

\[
+ \int \int \int \int \bar{G}_0(x_1, x_5) \bar{G}_0(x_2, x_6) \bar{\psi}(x_0 - x_6) \bar{G}_0(x_5, x_3) \bar{G}_0(x_6, x_4) dx_5 dx_6 dx_3 dx_2
\]

\[
- \int \int \int \int \bar{G}_0(x_1, x_5) \bar{G}_0(x_2, x_6) \mathcal{V}(x_5 - x_6) \bar{G}_0(x_0, x_3) \bar{G}_0(x_6, x_4) dx_5 dx_6 dx_3 dx_2.
\]

\[
(3.20)
\]

In the limit of classical statistics the second and fourth terms expressing the exchange effects are unnecessary.

Now we calculate the effective interaction \(\bar{\psi}\). Taking account of the periodicity of the one-particle Green function that \(\bar{G}(r, \tau) = -\bar{G}(r, \tau + \beta)\), where \(r = r_1 - r_2\) and \(\tau = t_1 - t_2\), we may change the definition of \(\mathcal{V}(x_1 - x_2)\) as follows,

\[
\mathcal{V}(x_1 - x_2) = \mathcal{V}(r_1 - r_2) \sum_n \delta(t_1 - t_2 + n\beta).
\]

\[
(3.21)
\]

Then, by making the Fourier transform the integral equation for \(\bar{\psi}\) becomes

\[
\bar{\psi}(q, \omega) = \mathcal{V}(q, \omega) - \mathcal{V}(q, \omega) \bar{Q}(q, \omega) \bar{\psi}(q, \omega),
\]

\[
(3.22)
\]
where
\[ \bar{\mathcal{G}}(q, \omega_n) = \frac{1}{\beta} \int_{-\beta}^{\beta} d\tau \int d\mathbf{r} \bar{\mathcal{G}}(\mathbf{r}, \tau) \exp(-i\mathbf{q} \cdot \mathbf{r} + i\omega_n \tau), \] (3.23)
\[ (\omega_n = n\pi/\beta; n = \text{integer}) \]
\[ v(q, \omega_n) = v(q) \left\{ 1 + (-1)^n \right\}/2, \] (3.24)
\[ \bar{Q}(q, \omega_n) = \frac{1}{V\beta} \sum_n \sum_{\omega'} \bar{G}_0(k, \omega_n) \bar{G}_0(k+q, \omega_n+\omega'), \] (3.25)
\[ \bar{G}_0(k, \omega_n) = \frac{1}{\beta} \int_{-\beta}^{\beta} d\tau \int d\mathbf{r} \bar{G}_0(\mathbf{r}, \tau) \exp(-i\mathbf{q} \cdot \mathbf{r} + i\omega_n \tau) \]
\[ = \begin{cases} \frac{1}{\varepsilon_0(k) - \mu - i\omega_n} & \text{for } n: \text{odd}, \\ 0 & \text{for } n: \text{even}. \end{cases} \] (3.26)

From (3.22) we get
\[ \bar{\mathcal{G}}(q, \omega_n) = v(q, \omega_n) / \{ 1 + v(q, \omega_n)\bar{Q}(q, \omega_n) \}. \] (3.27)

Using the above expression for \( G_0(k, \omega_n) \), we have
\[ \bar{Q}(q, \omega_n) = \frac{1}{V} \sum_k \frac{f_{k+q} - f_k}{\varepsilon_0(k + q) - \varepsilon_0(k) - i\omega_n}. \] (3.28)

The following two limiting cases are interesting.

i) Low temperature limit (GB limit): in this case \( f_{k \to n_0(k)} \) and the corresponding \( \bar{Q}(q, \omega_n) \) was calculated by GB in the limit of \( q \ll k_0 \):
\[ \bar{Q}^{\text{GB}}(q, \omega_n) = \frac{m_0 k_0}{\pi^2 \hbar^2} R(u), \] (3.29)
\[ R(u) = 1 - u \tan^{-1} u^{-1}, \] (3.30)
\[ u = m_0 \omega_n / \hbar^2 k_0^2. \] (3.31)

ii) Classical limit (Debye-Hückel limit): in this case using \( f_{k \to \exp[\beta(\mu - \varepsilon_0(k))]}, \) we easily get
\[ \bar{Q}^{\text{DH}}(q, \omega_n) = -\beta n \delta_{n,\omega} + O(q^2), \] (3.32)
\[ (n: \text{electron density}). \]

From (3.27) we see that, as in the case of the ground state, the interaction is effectively screened by the "polarization" of the medium. This screening is statistical, while in the case of the ground state dynamical. This screening was first pointed out by Matsubara. In particular \(-4\pi e^2 \bar{Q}^{\text{DH}}(0, 0)\)^{-1/2} = [4\pi e^2 n\beta]^{-1/2} = \lambda_D \) is called Debye's shielding length.
Using (3·7) and (3·20), we finally get the following expression for the grand potential:

$$
\Omega = \Omega_0 + \Omega^{(1)}_{\sigma} + \Omega^{(2)}_{\sigma} + \Omega_{\eta},
$$

(3·33)

where

$$
\Omega_0 = -\frac{1}{\beta} \sum_k \ln \left[ 1 + \exp \left\{ -\beta (\varepsilon_k - \mu) \right\} \right],
$$

(3·34a)

$$
\Omega^{(1)}_{\sigma} = -\frac{1}{2V} \sum_{k<k'} \nu(k-k') f_k f_{k'},
$$

(3·34b)

$$
\Omega^{(2)}_{\sigma} = -\frac{1}{2V^2} \sum_{k<k'q} \nu(q) \nu(k_1-k_2-q)
\times \frac{(1-f_{k_1}) (1-f_{k_2}) f_{k_1+q} f_{k_2+q}}{\varepsilon_0(k_1) + \varepsilon_0(k_2) - \varepsilon_0(k_1-q) - \varepsilon_0(k_2+q)}
$$

(3·34c)

$$
\Omega_{\eta} = -\frac{1}{2\beta} \sum_{q} \sum_{\omega} \left\{ \ln \left[ 1 + \nu(q) \bar{Q}(q, \omega_\eta) \right] - \nu(q) \bar{Q}(q, \omega_\eta) \right\}.
$$

(3·34d)

Taking the suitable limiting procedure and using (3·29) and (3·32) for $\bar{Q}(q, \omega_\eta)$, we get from (3·33) GB's result for the correlation energy in the low temperature limit and DH's equation of state in the limit of classical statistics.

The higher order effects in the above two limits have different characters from each other. In the low temperature limit the exchange correction to the polarization effect is the next higher order contribution, while in the classical limit it is needed to include the higher order effects of two-body scattering, that is, the "ladder type diagrams" make the contribution to the next higher order. We will discuss this problem later in this series.

§ 4. Time- and temperature-dependent Green functions

We define one- and two-particle Green functions which depend on time and temperature as

$$
\hat{G}(x_1, x_2) = i \, \text{Tr} \exp(-\beta \mathcal{H}) \hat{T} \{ \phi(x_1) \bar{\phi}(x_2) \} / \text{Tr} \exp(-\beta \mathcal{H}),
$$

(4·1)

$$
\hat{G}(x_1, x_2 ; x_3, x_4) = i \, \text{Tr} \exp(-\beta \mathcal{H}) \hat{T} \{ \phi(x_1) \phi(x_2) \bar{\phi}(x_3) \bar{\phi}(x_4) \} / \text{Tr} \exp(-\beta \mathcal{H}).
$$

(4·2)

Here

$$
\bar{\phi}(x_1) \equiv \bar{\phi}(r_1, \lambda_1) = \exp(i/\hbar \cdot \mathcal{H} \lambda_1) \psi^*(r_1) \exp(-i/\hbar \cdot \mathcal{H} \lambda_1),
$$

(4·3)

where $\lambda_1 = x_1 - i\hbar \beta \lambda_1$. $\hat{T}$ is defined in the analogous way as Wick's chronological operator:

$$
\hat{T} \{ \phi(x_1) \bar{\phi}(x_2) \} = +\phi(x_1) \bar{\phi}(x_2) \quad \text{if} \quad \lambda_1 > \lambda_2
$$

$$
-\phi(x_2) \bar{\phi}(x_1) \quad \text{if} \quad \lambda_1 \leq \lambda_2
$$

(4·4)
Green Function Method for Electron Gas. I

where \( \lambda_1 > \lambda_2 \) means that \( t_1 > t_2 \) and \( \beta_1 > \beta_2 \), or \( t_1 > t_2 \) and \( \beta_1 > \beta_2 \).

We consider operators \( U(\lambda, \lambda_0) \) and \( U^+(\lambda, \lambda_0) \) which satisfy the following equations,*

\[
\begin{align*}
\ih \left( \partial U(\lambda, \lambda_0) / \partial \lambda \right) &= \mathcal{K}_1(\lambda) U(\lambda, \lambda_0), \quad (4\cdot5a) \\
-\ih \left( \partial U^+(\lambda, \lambda_0) / \partial \lambda \right) &= U^+(\lambda, \lambda_0) \mathcal{K}_1(\lambda), \quad (4\cdot5b)
\end{align*}
\]

with

\[
U(\lambda_0, \lambda_0) = U^+(\lambda_0, \lambda_0) = 1. \quad (4\cdot6)
\]

Here

\[
\mathcal{K}_1(\lambda) = \exp(i/\hbar \cdot \mathcal{K}_0 \lambda) \cdot \mathcal{K}_1, \exp(-i/\hbar \cdot \mathcal{K}_0 \lambda). \quad (4\cdot7)
\]

From (4\cdot5a), (4\cdot5b) and (4\cdot6) we get

\[
U^+(\lambda, \lambda_0) U(\lambda, \lambda_0) = 1. \quad (4\cdot8)
\]

The solution of (4\cdot5a) is given by

\[
U(\lambda, \lambda_0) = \sum_{n=0}^{\infty} \left( -\frac{i}{\hbar} \right)^n \int_{\lambda_0}^{\lambda} d\lambda_1 \int_{\lambda_0}^{\lambda_{n-1}} d\lambda_2 \cdots \int_{\lambda_0}^{\lambda_n} d\lambda_n \mathcal{K}_1(\lambda_1) \mathcal{K}_1(\lambda_2) \cdots \mathcal{K}_1(\lambda_n), \quad (4\cdot9)
\]

which may be rewritten as follows,

\[
U(\lambda, \lambda_0) = \sum_{n=0}^{\infty} \left( -\frac{i}{\hbar} \right)^n \frac{1}{n!} \int_{\lambda_0}^{\lambda} d\lambda_1 \cdots d\lambda_n \left[ \mathcal{K}_1(\lambda_1) \cdots \mathcal{K}_1(\lambda_n) \right], \quad (4\cdot10)
\]

where the integration over \( \lambda \) is to be performed along a path shown in Fig. 4. Using the relation

\[
\exp(i/\hbar \cdot \mathcal{K}_0 \lambda) \mathcal{K}_1(\lambda') \exp(-i/\hbar \cdot \mathcal{K}_0 \lambda) = \mathcal{K}_1(\lambda + \lambda'),
\]

we find

\[
\exp(i/\hbar \cdot \mathcal{K}_0 \lambda) U(\lambda', \lambda_0) \times \exp(-i/\hbar \cdot \mathcal{K}_0 \lambda) = U(\lambda + \lambda', \lambda + \lambda_0). \quad (4\cdot11)
\]

It then follows from the equations

\[
\begin{align*}
\exp(-i/\hbar \cdot \mathcal{K}_0 \lambda) &= \exp(-i/\hbar \cdot \mathcal{K}_0 \lambda) U(\lambda, 0), \quad (4\cdot12a) \\
\exp(i/\hbar \cdot \mathcal{K}_0 \lambda) &= U^+(\lambda, 0) \exp(i/\hbar \cdot \mathcal{K}_0 \lambda), \quad (4\cdot12b)
\end{align*}
\]

and from using (4\cdot11), that

* \( U^+ \) is not the Hermitian conjugate of \( U \).
\[ U(\lambda, \lambda_0) = \exp(i/\hbar \cdot \mathcal{H}_0 \lambda) \exp(-i/\hbar \cdot \mathcal{H}(\lambda - \lambda_0)) \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda_0). \] (4·13)

Thus we have
\[ U(\lambda, \lambda') U(\lambda', \lambda_0) = U(\lambda, \lambda_0). \] (4·14)

If we set \( \lambda = \lambda_0 \) in (4·14), we obtain
\[ U(\lambda_0, \lambda) U(\lambda, \lambda_0) = 1. \] (4·15)

From (4·8) and (4·15) we get
\[ U^{-1}(\lambda, \lambda_0) = U^+(\lambda, \lambda_0) = U(\lambda_0, \lambda). \] (4·16)

Gell-Mann and Low\(^{14}\) have shown that
\[ \mathcal{T}_n = C_n U(0, \pm \infty) \varphi_n/(\varphi_n, U(0, \pm \infty) \varphi_n), \] (4·17)

where \( \varphi_n \) and \( \varphi_n \) are the normalized eigenfunctions of the total Hamiltonian and the unperturbed Hamiltonian respectively. \( C_n \) is the normalization constant. For a typical order \( \lambda_1 > \lambda_2 \) we have
\[
\hat{G}(x_1, x_2) = i \sum_{\mathcal{N}} \sum_{\varphi_n} \langle \mathcal{T}_n \rangle \exp(-\beta\mathcal{H}) \phi(x_1) \bar{\phi}(x_2) \varphi_n + \mathcal{N} \rangle / \sum_{\mathcal{N}} \sum_{\varphi_n} \langle \mathcal{T}_n \rangle \exp(-\beta\mathcal{H}) \varphi_n + \mathcal{N} \rangle \\
= i \sum_{\mathcal{N}} \sum_{\varphi_n} \langle \mathcal{T}_n \rangle \exp(-\beta\mathcal{H}) \phi(x_1) \bar{\phi}(x_2) \varphi_n + \mathcal{N} \rangle / \sum_{\mathcal{N}} \sum_{\varphi_n} \langle \mathcal{T}_n \rangle \varphi_n + \mathcal{N} \rangle \\
\times \exp(-\beta\mathcal{H}_0) U(-i\hbar \beta, 0) U(0, -\infty) \varphi_n + \mathcal{N} \rangle \rangle \} \\
\times \exp(-\beta\mathcal{H}_0) U(-i\hbar \beta, 0) U(0, -\infty) \varphi_n + \mathcal{N} \rangle \rangle \} \langle U(-i\hbar \beta, 0) \rangle \langle U(0, -\infty) \rangle. \] (4·18)

Here
\[ \phi(x) = \exp(i/\hbar \cdot \mathcal{H}_0 \lambda) \phi(x) \exp(-i/\hbar \cdot \mathcal{H}_0 \lambda). \] (4·19)

Using the relation
\[ \exp(\beta\mathcal{H}_0) U(\infty, 0) \exp(-\beta\mathcal{H}_0) = U(\infty - i\hbar \beta, -i\hbar \beta), \] (4·20)

we rewrite (4·18) as
\[
\hat{G}(x_1, x_2) = i \text{Tr} \exp(-\beta\mathcal{H}_0) U(\infty - i\hbar \beta, -i\hbar \beta) U(-i\hbar \beta, 0) U(0, \lambda_1) \\
\times \exp(-\beta\mathcal{H}_0) U(\infty - i\hbar \beta, -i\hbar \beta) U(-i\hbar \beta, 0) U(0, -\infty) / \text{Tr} \\
= i \text{Tr} \exp(-\beta\mathcal{H}_0) \hat{T} \{ \phi(x_1) \bar{\phi}(x_2) U(\infty - i\hbar \beta, -\infty) \} / \text{Tr} U(\infty - i\hbar \beta, -\infty) \\
= i \langle \hat{T} \{ \phi(x_1) \bar{\phi}(x_2) U(\infty - i\hbar \beta, -\infty) \} \rangle \langle U(\infty - i\hbar \beta, -\infty) \rangle. \] (4·21)

The above equation holds also for \( \lambda_1 \leq \lambda_2 \). We have a similar expression for the two-particle Green function.

The proof of the fact that the average of a normal product vanishes is quite the same as that given by Matsubara\(^8\) and Thouless\(^{19}\) for the temperature-dependent Green functions. We shall briefly recapitulate the proof. We decompose \( \bar{\phi}(x) \) and \( \phi(x) \) into two parts.
Green Function Method for Electron Gas. I

\[ \bar{\phi}(x) = \bar{\phi}_+(x) + \bar{\phi}_-(x), \]
\[ \phi(x) = \phi_+(x) + \phi_-(x), \] where

\[ \bar{\phi}_+(x) = V^{-1/2} \sum_k (1 - g_k) a_k^* \exp \{-i k \cdot r + i (\varepsilon_0(k) - \mu) \lambda \}, \]
\[ \bar{\phi}_-(x) = V^{-1/2} \sum_k g_k a_k^* \exp \{-i k \cdot r + i (\varepsilon_0(k) - \mu) \lambda \}, \]
\[ \phi_+(x) = V^{-1/2} \sum_k g_k a_k \exp \{i k \cdot r - i (\varepsilon_0(k) - \mu) \lambda \}, \]
\[ \phi_-(x) = V^{-1/2} \sum_k (1 - g_k) a_k \exp \{i k \cdot r - i (\varepsilon_0(k) - \mu) \lambda \}. \] (4.23)

The normal product of \( F(\bar{\phi}, \phi) \), where \( F \) is a functional of \( \bar{\phi} \) and \( \phi \), is defined by making the substitution (4.22) in \( F \), and reordering each term so that \( \bar{\phi}_- \) and \( \phi_- \) come first, followed by \( \bar{\phi}_+ \) and \( \phi_+ \), with a sign of \( \pm \) according to whether the permutation is even or odd. The contraction \( \phi^i \bar{\phi}^j \) is given by

\[ \langle \phi(x) \bar{\phi}(x') \rangle = \bar{T} \left[ \phi(x) \bar{\phi}(x') \right] - N \left[ \phi(x) \bar{\phi}(x') \right] \]
\[ = \left[ \phi_-(x), \bar{\phi}_+(x') \right] + \left[ \phi_+(x), \bar{\phi}_-(x') \right]. \] (4.24)

If \( g_k^2 = 1 - f_k \), we find that

\[ \langle \phi(x) \bar{\phi}(x') \rangle = \sum_k (1 - f_k) \exp \{i k \cdot (r - r') - i (\varepsilon_0(k) - \mu) (\lambda - \lambda') \} \] if \( \lambda > \lambda' \)
\[ - \sum_k f_k \exp \{i k \cdot (r - r') - i (\varepsilon_0(k) - \mu) (\lambda - \lambda') \} \] if \( \lambda \leq \lambda' \)
\[ = -i G_0(x, x') \] (4.25)

and

\[ \langle N[\phi(x) \bar{\phi}(x')] \rangle_0 = \sum_k \left\{ 2 g_k (1 - f_k) (1 - f_k) + (1 - g_k)^2 (1 - f_k) - g_k^2 f_k \right\} \]
\[ \times \exp \{i k \cdot (r - r') - i (\varepsilon_0(k) - \mu) (\lambda - \lambda') \} = 0. \] (4.26)

Further, it can be shown that normal products of higher order vanish. We can also prove the “linked cluster expansion” in the same way as in the case of the time-dependent Green functions or temperature-dependent Green functions. Therefore we may apply the same techniques as in § 2 and § 3 to the calculation of the Green functions (4.1) and (4.2), using Feynman diagrams.

The Green functions (4.1) and (4.2) would be useful for the calculation of transport quantities. The expressions for the transport quantities are given by a formalism developed by Kubo. In the case that the external electromagnetic field is represented by a vector potential \( A(r, t) \), the response current is given by

\[ i^z(r, t) = \text{Tr} \rho j^z(r, t) - \frac{1}{i \hbar c} \sum_{\beta=1}^3 \text{Tr} \int_{-\infty}^{0} d\tau' \int d^3r' \rho [j_0^z(r), j_0^z(r', t')] A^3(r', t + \tau'), \] (4.27)
where
\[ p = \exp(-\beta \mathcal{H}) / \text{Tr} \exp(-\beta \mathcal{H}), \tag{4.28} \]
and
\[ j_0(r) = e^2 / (2m) \left[ -\hbar / i \cdot \psi^*(r) \cdot \psi(r) + \phi^*(r) \phi(r) \right], \tag{4.29a} \]

\[ j_1(r, t) = (-e^2 / mc) A(r, t) \phi^*(r) \phi(r), \tag{4.29b} \]

and
\[ j_0(r, t) = e^2 / (i \hbar \cdot \mathcal{H}_0) j_0(r) \exp(-i / \hbar \cdot \mathcal{H}_0 t). \tag{4.30} \]

In the case that the external electric field is represented by a scalar potential \( \varphi(r, t) \), the response current is given by
\[ i^s(r, t) = \text{Tr} \int_0^{+ \infty} d\tau \int_0^{+ \infty} d\tau' \int d^3 \rho \int d^3 \rho' \text{div} j_0(r', -i\hbar \beta') j_0^*(r, \tau) \varphi(r', t + \tau) \]
\[ = \text{Tr} \int_0^{+ \infty} d\tau \int_0^{+ \infty} d\tau' \int d^3 \rho \int d^3 \rho' j_0(r, \tau - i\hbar \beta) \text{div} j_0(r', -i\hbar \beta') \varphi(r', t + \tau). \tag{4.31} \]

The right-hand sides of (4.27) and (4.31) can be expressed in terms of Green functions as follows,
\[ i^s(r, t) = \frac{ie^2}{mc} A^s(r, t) \tilde{G}(r0, r0) - \frac{1}{i\hbar c} \int_0^{+ \infty} dt' \int d^3 \rho \int d^3 \rho' \sum_{\beta=1}^3 \left[ \frac{e^2 \hbar^2}{4m^2} \cdot \lim_{\beta, \beta' \to \beta''} \frac{1}{(F_{\beta} - F_{\beta'})} (F_{\beta'} - F_{\beta''}) \tilde{G}(r0, r0'; r0', 0, r0'', r0', t + t') - c.c. \right] A^b(r', t + t'), \tag{4.27'} \]

\[ i^s(r, t) = \frac{e^2 \hbar^2}{4m^2} \sum_{\beta=1}^3 \int_0^{+ \infty} dt' \int d^3 \rho' \int d^3 \rho' \lim_{\beta, \beta' \to \beta''} \frac{1}{(F_{\beta} - F_{\beta'})} (F_{\beta'} - F_{\beta''}) \cdot \tilde{G}(r, \tau - i\hbar \beta, r', -i\hbar \beta'; r'', (\tau - i\hbar \beta'), r''', (-i\hbar \beta'), \varphi(r', t + \tau). \tag{4.31'} \]

where \( \lambda_+ = \lambda + \delta (\delta \to 0) \).

For a classical gas we obtain the Drude conductivity, if we set \( \tilde{G} \) (already defined in (4.31), where \( \tilde{G}_0 \) is given by (4.24)).

The Green function \( \tilde{G}(x_1, x_2) \) is defined only in regions I and III shown in Fig. 5. To make a Fourier
transform of \( \hat{G}_0(x_1, x_2) \) it is expedient that we extend the definition of \( \hat{G}_0(x_1, x_2) \) to regions II and IV as follows: In regions I and II

\[
\hat{G}_0(k, \lambda_1 - \lambda_2) = i(1 - f_\lambda) \exp \{-i(\varepsilon_\lambda(k) - \mu)(\lambda_1 - \lambda_2)\}
\]

and in regions III and IV

\[
\hat{G}_0(k, \lambda_1 - \lambda_2) = -i f_\lambda \exp \{-i(\varepsilon_\lambda(k) - \mu)(\lambda_1 - \lambda_2)\}.
\]

Then the Fourier transform of \( \hat{G}_0(x_1, x_2) \) is given by

\[
\hat{G}_0(k, \varepsilon, \omega_n) = \frac{1}{2\pi i \beta} \frac{1}{i\omega_n - (\varepsilon_\lambda(k) - \mu)} \delta(\varepsilon - \varepsilon_\lambda(k)).
\]

It is desirable that we define an ordering symbol, which is different from \( \hat{T} \), so that the definition of \( G \) covers the whole region. Details will be published later.

**Appendix**

We derive the normalization condition for \( f_{\lambda \psi}(p) \). We make use of the \( \hat{f} \)-sum rule for the oscillator strength:

\[
\sum_p f_{\lambda \psi}(k) = N, \tag{A·1}
\]

where

\[
f_{\lambda \psi}(k) = \frac{2m}{\hbar^2} \omega_n \langle \Psi_\lambda, \rho_\psi \Psi_n \rangle^2 \left( \rho_\psi = \frac{1}{V} \sum_p a_{\lambda \psi}^* a_p \right), \tag{A·2}
\]

which may be rewritten as

\[
\frac{\hbar^2}{2m} N = \omega_n(k) \left| \sum_p f_{\lambda \psi}(p) \right|^2 + \sum_q \omega_q \left| \sum_p f_{\lambda \psi}(p) \right|^2. \tag{A·3}
\]

Here \( \omega_q = \{\varepsilon_q(q + k/2) - \varepsilon_q(q - k/2)\}/\hbar \). The scattering state amplitude \( f_{\lambda \psi}(p) \) is determined by the equation

\[
f_{\lambda \psi}(p) = \delta_{pq} \left[ n_p(q + k/2) (1 - n_p(q + k/2)) \right]^{1/2}
- \frac{v(k)}{\hbar} \frac{n_p(p + k/2) - n_p(p - k/2)}{\omega_q - \omega_p - i\delta \{ n(p + k/2) - n_p(p - k/2) \}} \sum_{p'} f_{\lambda \psi}(p'). \tag{A·4}
\]

Inserting the explicit expression for \( f_{\lambda \psi}(p) \) into (A·3) and rearranging terms, we get

\[
\frac{\hbar^2}{2m} N = \omega_n(k) \left| \sum_p f_{\lambda \psi}(p) \right|^2 + \sum_q \omega_q n_q(q - k/2) (1 - n_q(q + k/2))
- \sum_q \frac{1}{1 + f((\omega_q + i\delta)^2)} \left[ \frac{1}{\omega_p - \omega_q - i\delta} + \frac{1}{\omega_p + \omega_q + i\delta} \right], \tag{A·5}
\]

where

\[
f((\omega_q + i\delta)) = v(k) \sum_p n_p(p) (1 - n_p(p + k)) \left[ \frac{1}{\omega_p + \omega_q + i\delta} + \frac{1}{\omega_p - \omega_q - i\delta} \right]. \tag{A·6}
\]
The direct calculation shows that the second term of the right-hand side in (A·5) = $\hbar k^2 N/2m$. Making use of the techniques of Sawada et al., we get

\[
\frac{1}{v(k)} \int_C dz \cdot z \left\{ \frac{1}{1+f(z)} - f(z) \right\}, \quad (A\cdot7)
\]

where the contour $C_1$ is shown in Fig. 5. Changing the contour $C_1$ to $C_2$ and $C_3$ (Fig. 6) and calculating the residue, we find

\[
|\sum_p f_k^{\text{out}}(p)|^2 = \frac{\hbar}{v(k)} e^2 \frac{\partial \omega_{pd}(k)}{\partial e^2}. \quad (A\cdot8)
\]

It is easy to rewrite the above expression as

\[
\sum_p |f_k^{\text{out}}(p)|^2 \left[ n_0(p-k/2) - n_0(p+k/2) \right] = 1. \quad (A\cdot9)
\]

The correlation energy can be written in terms of $f_k^{\text{out}}(p)$ and $f_k^{\text{in}}(p)$:

\[
E_{\text{corr.}} = \frac{1}{2V} \int_0^\pi \frac{dy}{\sqrt{\sum_k \left| \sum_p f_k^{\text{out}}(p) \right|^2 + \sum_k \left| \sum_p f_k^{\text{in}}(p) \right|^2}} - \frac{N}{2} v(0). \quad (A\cdot10)
\]

It is easy to show that, using the above normalization condition for $f_k^{\text{out}}(p)$, we are led to GB's result.
Green Function Method for Electron Gas. I

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