Pion-Pion Interaction and Pion-Nucleon Scattering

Kin-ichi ISHIDA,* Atsushi TAKAHASHI and Yoshiaki UEDA

*Faculty of Liberal Arts and Science, Yamagata University, Yamagata
Department of Physics, Tohoku University, Sendai

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Pion-pion interaction is analyzed by using the dispersion relation for pion-nucleon scattering obtained by keeping the momentum transfer between an initial pion and a final nucleon constant. In order to take into account the singularity of two-pion threshold in the dispersion relation, the dispersion relation may be regarded as an integral equation for pion-nucleon scattering amplitude with the kernel of a pion-pion scattering amplitude. When this solution is compared with experiments on pion-nucleon scattering, it is found that the unknown quantity in the dispersion relation is only a pion-pion scattering amplitude. Therefore, if pion-pion amplitude is expressed in terms of an unknown parameter such as scattering length, then this value can be determined via dispersion relation. Thus we are led to the following conclusion: In the isotopic spin state $I=0$ (S-wave) of pion-pion system pion-pion interaction is attractive and the scattering length is of the order of one pion Compton wavelength, while in the isotopic spin state $I=1$ (P-wave) of pion-pion state a definite conclusion could not be obtained. Finally the possibility of explaining the momentum dependence of pion-nucleon phase shift $\delta_{13}$ related to the pion-pion interaction is briefly discussed.

§ 1. Introduction

There are some phenomena which might be considered to show the existence of the pion-pion interaction. For instance, the cross section for the multiple production of pions in pion-nucleon scattering is of the order of $(\hbar/\mu c)^2$, where $\mu$ is a pion mass. This fact suggests that the pion-pion interaction plays an important role in the multiple production of pions, since this large radius can be understood only in terms of a pion cloud. The role of the pion cloud in the pion-nucleon interaction was stressed already by Dyson and Takeda several years ago. In the problem of electromagnetic structure of nucleon too, the pion-pion interaction might be important. This was pointed out by Federbush et al.

Recently, Goebel and Chew and Low have suggested that one could obtain the knowledge on the pion-pion interaction from the analysis of the process $\pi+N\rightarrow N+\pi+\pi$. The result follows only from a plausible assumption about the location and residues of poles in the $S$ matrix, but unfortunately this kind of experiment needs highly accurate measurement and the present experiment does not seem to be precise enough to determine the cross section for the pion-pion interaction.
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Sato and two of us (A.T., Y.U.) have analyzed the effective pion-pion interaction from the dispersion relations for the pion-nucleon scattering, obtained by keeping the momentum transfer between an initial pion and a final nucleon constant. In order to estimate the contributions of the pion-pion interaction to the pion-nucleon scattering, in these dispersion relations one must, however, integrate the pion-pion scattering amplitude multiplied by a certain matrix element similar to the pion-nucleon scattering amplitude. Here "similar" means that this amplitude is not physical, i.e. the sign of the four-momentum of one pion is reversed. In Ref. I this amplitude was expressed by the one analytically continued from the scattering amplitude given by Chew et al., as was done by Federbush et al. and Chew et al. in the problem of the structure of nucleon.

When we regard the pion-nucleon scattering amplitude as the function of \( \xi \), the square of the momentum transfer between initial and final pions with reversed sign, holding another variable \( \sigma^2 \) fixed (see (2.7)), \( \xi = 4\mu^2 \) may become the branch point which is the threshold of the production of two pions, if the effective pion-pion interaction exists (see (2.12)). For this reason it is not clear that the previous analytic continuation from the region \( \xi < 4\mu^2 \) to the region \( \xi > 4\mu^2 \) (the required amplitude appear in this region) is right. As our aim is to analyze the pion-pion interaction from the pion-nucleon scattering, it is desirable to treat the pion-nucleon scattering amplitude more properly by taking into account the above singularity.

In this paper we shall regard the dispersion relations as the integral equations satisfied by the pion-nucleon scattering amplitudes where the kernel is the pion-pion scattering amplitude. If we can solve these integral equations, we shall be able to know the correct behaviour of the pion-nucleon scattering amplitude in \( \xi > 4\mu^2 \) as well as in \( \xi < 4\mu^2 \). This is indeed possible under some assumptions. Instead of using the expression continued analytically, we shall use these solutions as the pion-nucleon scattering amplitudes. Hereafter, we proceed with a method similar to that of Ref. I, namely we expand the pion-pion scattering amplitude in partial waves and limit these waves to S- and P-waves, for which scattering length approximation is used. In the dispersion relation all quantities other than those unknown scattering lengths can be expressed in terms of phase shifts for pion-nucleon scattering and of its coupling constant. Therefore, comparing this identity with experiment, we can estimate the measure of the scattering length of pion-pion interaction.

In § 2, dispersion relations are derived. This section is added for completeness and for explaining our notation. In § 3 the integral equations are derived and solved. In § 4 numerical results and discussions are presented. The conclusions are as follows: (1) There may exists S-wave pion-pion interaction with the isotopic spin \( I=0 \), which is attractive with the scattering length of the order of one pion Compton wavelength, (2) about the P-wave pion-pion interaction with \( I=1 \), on the other hand, we could not obtain a definite conclusion. This situation is discussed

* Hereafter referred to as Ref. I.
in § 4 in connection with the result of Ref. 1. Further, the possibility of explaining the energy dependence of pion-nucleon phase shift $\delta_{13}$ related to the pion-pion interaction is briefly discussed in § 4.

§ 2. Dispersion relation

In this section, we shall write down the dispersion relations in the representation where the momentum transfer between the incident pion and the final nucleon is kept constant. Let the four-momenta of the incident and the outgoing pions be $k_1$ and $k_2$, respectively, while those of the initial and the final nucleon are $p_1$ and $p_2$. Then the pion-nucleon scattering amplitude can be given in the form*

$$S = \delta_{\alpha\beta} \delta(p_1 - p_2) \delta(k_1 - k_2)$$

$$- i (2\pi)^4 \langle p_1 + k_2 - k_1 - k_2 \rangle \langle m^2 / p_2^\alpha p_1^\beta \rangle^{1/2} \langle 1/4 k_2^\alpha k_1^\beta \rangle^{1/2} u_\beta u_\alpha,$$

(2.1)

where the retarded matrix element $M_{\alpha\beta}$ is given by

$$\langle m^2 / 2 k_2^\alpha p_1^\beta \rangle^{1/2} M_{\alpha\beta} u_\beta = (-i) \int d^4z \exp[-iK \cdot z] \langle k_2^\beta | T f(z/2), j_\alpha (-z/2) | p_1 \rangle,$$

(2.2)

$f(z/2)$ and $j_\alpha (-z/2)$ are the nucleon and pion current operators, respectively, and Greek subscripts $\alpha$ and $\beta$ are the isotopic spin indices of the incident and the outgoing pions, respectively, and $K = (p_1 + k_2) / 2$. We have omitted the contribution from the equal time commutator for simplicity. Here, we define the Feynman amplitude $F_{\alpha\beta}$ for the latter purpose,

$$\langle m^2 / 2 k_2^\alpha p_1^\beta \rangle^{1/2} F_{\alpha\beta} u_\beta = (-i) \int d^4z \exp[-iK \cdot z] \langle k_2^\beta | T f(z/2), j_\alpha (-z/2) | p_1 \rangle.$$

(2.3)

We separate the $M_{\alpha\beta}$ into dispersive part $(M_{\alpha\beta})^D$ and absorptive part $(M_{\alpha\beta})^A$:

$$\langle m^2 / 2 k_2^\alpha p_1^\beta \rangle^{1/2} (M_{\alpha\beta})^D u_\beta = (-i/2) \int d^4z \exp[-iK \cdot z] \langle k_2^\beta | T f(z/2), j_\alpha (-z/2) | p_1 \rangle,$$

$$\langle m^2 / 2 k_2^\alpha p_1^\beta \rangle^{1/2} (M_{\alpha\beta})^A u_\beta = (-1/2) \int d^4z \exp[-iK \cdot z] \langle k_2^\beta | T f(z/2), j_\alpha (-z/2) | p_1 \rangle.$$

(2.4)

In the same way $F_{\alpha\beta}$ is written as $(F_{\alpha\beta})^D + i(F_{\alpha\beta})^A$. Then we have

$$(F_{\alpha\beta})^D = (M_{\alpha\beta})^D,$$

$$\langle m^2 / 2 k_2^\alpha p_1^\beta \rangle^{1/2} (F_{\alpha\beta})^A u_\beta = (-1/2) \int d^4z \exp[-iK \cdot z] \langle k_2^\beta | T f(z/2), j_\alpha (-z/2) | p_1 \rangle.$$

(2.5)

* Hereafter, we set $\hbar = c = 1$ and the normalization volume of system = 1.
$M_{\beta\alpha}$ can be expressed by the four-invariant functions in the form

$$M_{\beta\alpha} = M^{(+)} \delta_{\beta\alpha} + [\tau_\beta, \tau_\alpha] M^{(-)/2}, \quad M^{(\pm)} = -A^{(\pm)} + i\gamma^1 Q B^{(\pm)}; \tag{2.6}$$

where, $Q = (k_1 + k_2) / 2$. $(M_{\beta\alpha})^p$ and $(M_{\beta\alpha})^a$ can be expressed in the same form as in (2.6). Then from the invariance under time reversal it can be proved that the invariant functions defined for the $(M_{\beta\alpha})^p$ and $(M_{\beta\alpha})^a$ in (2.6) are the real and the imaginary parts of the invariant functions defined for the $M_{\beta\alpha}$, respectively. $M_{\beta\alpha}$ can be expressed as the function of the following two variables:

$$\xi = -(k_2 - k_1) \gamma, \quad \sigma^2 = -(p_3 + k_1)^2, \tag{2.7}$$

At some stages of calculation, however, it is more convenient to use variables $(W^2, \sigma^2)$ rather than $(\xi, \sigma^2)$, where $W$ is the total c.m. energy of a pion and a nucleon. $W^2$ is expressed by $\xi$ and $\sigma^2$ as

$$W^2 = -\xi + \sigma^2. \tag{2.8}$$

Now we expect that the pion-nucleon scattering amplitude is analytic in the upper-half of the complex $-\xi$ or $W^2$ plane for fixed values of $\sigma^2$. Then the dispersion relations will be derived in the same way as done in Ref. 1. First $(M_{\beta\alpha})^a$ is divided into two parts:

$$(m/2k^2 p_{10})^{1/2} (M_{\beta\alpha})_1 u_1 = -(1/2) \int d^4 z \exp[-iK \cdot z] \langle k_{20} | f(z/2) j_\alpha (-z/2) | p_1 \rangle,

(m/2k^2 p_{10})^{1/2} (M_{\beta\alpha})_2 u_2 = +(1/2) \int d^4 z \exp[-iK \cdot z] \langle k_{20} | j_\alpha (-z/2) f(z/2) | p_1 \rangle. \tag{2.9}$$

Then, from (2.5), we obtain

$$(F_{\beta\alpha})^a = (M_{\beta\alpha})_1 - (M_{\beta\alpha})_2. \tag{2.10}$$

In (2.9) we may expand the matrix element of operators into the product of the matrix element of an operator by inserting a complete set of states labeled by the quantum number $n$. Then, in $(M_{\beta\alpha})_1$, the term coming from the one-nucleon state becomes the "renormalized Born term," and the term coming from the one-nucleon and one-pion state becomes the imaginary part of the pion-nucleon scattering amplitude. Here contributions from the terms of higher mass spectra such as one nucleon and two pion, etc., are neglected since these neglected terms might not be important in the pion-nucleon phenomena at low energies. While in $(M_{\beta\alpha})_2$, the lowest mass state is a two-pion state. The contribution from this state can be expressed by the product of the pion-pion scattering amplitude and the unphysical pion-nucleon scattering amplitude. Here contributions coming from the states

* The relationship between these two variables and $(\omega, \sigma^2)$ defined in Ref. 1 is

$\omega$ (defined in Ref. 1) = $(1/4) \sigma^2$ (defined here) $- (1/2) \xi$,

$\sigma^2$ (defined in Ref. 1) = $(1/4) \sigma^4$ (defined here).
higher than the four-pion state are neglected by the same reason as stated for \((M_{\pi\pi})_1\). Dispersion relation should be subtracted once at least, since, for instance, the contribution of the equal time commutator to \(M_{\pi\pi}\) does not contain the variable \(\xi\). In fact we subtracted the dispersion relation once. Here we introduce the notation \(\partial A^{(\pm)}\) and \(\partial B^{(\pm)}\) for the contributions of pion-pion scattering to the \(M_\pi\), namely

\[
(M_{\pi\pi})_2 = \partial_{\pi\pi}[-\delta A^{(\pm)} + i\gamma \cdot Q \partial B^{(\pm)}] + \frac{\tau_\pi}{2} [\tau_\pi - \delta A^{(\pm)} + i\gamma \cdot Q \partial B^{(\pm)}].
\] (2.11)

Then the dispersion relations in question may be written as

\[
A^{(\pm)}(\xi, \sigma^2) = A^{(\pm)}(0, \sigma^2) - \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} A^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2) (W'^2 - \sigma^2 + \xi - i\epsilon)}
\]

\[
- \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} A^{(\pm)}(\xi', \sigma^2)}{\xi' (\xi' - \xi - i\epsilon)},
\]

\[
B^{(\pm)}(\xi, \sigma^2) = B^{(\pm)}(0, \sigma^2) - \frac{G^2}{(m^2 - \sigma^2)(m^2 - \sigma^2 + \xi)}
\]

\[
- \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} B^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2) (W'^2 - \sigma^2 + \xi - i\epsilon)} - \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} B^{(\pm)}(\xi', \sigma^2)}{\xi' (\xi' - \xi - i\epsilon)}.
\] (2.12)

where \((G^2/4\pi) = (2m/\mu)^2 f^2\), and \(f^2 = 0.08\). \(A^{(\pm)}\) and \(B^{(\pm)}\) are regarded as the Feynman amplitudes. The explicit forms of \(A^{(\pm)}(0, \sigma^2)\) and \(B^{(\pm)}(0, \sigma^2)\) will be given in the next section. As they are forward pion-nucleon scattering amplitudes, they will be expressed easily by using the dispersion relations given by Goldberger, Miyazawa and Oehme.\(^7\) Therefore, the right-hand sides, except the last terms, of these dispersion relations may be considered to be known, if we express these by using the experimental data of the pion-nucleon scattering and its coupling constant \(f\). In this sense we introduce \(A_1^{(\pm)}\) and \(B_1^{(\pm)}\) corresponding to the equations of \(A^{(\pm)}(\xi, \sigma^2)\) and \(B^{(\pm)}(\xi, \sigma^2)\), respectively, as

\[
A_1^{(\pm)}(\xi, \sigma^2) = A^{(\pm)}(0, \sigma^2) - \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} A^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2) (W'^2 - \sigma^2 + \xi - i\epsilon)},
\]

\[
B_1^{(\pm)}(\xi, \sigma^2) = B^{(\pm)}(0, \sigma^2) - \frac{G^2}{(m^2 - \sigma^2)(m^2 - \sigma^2 + \xi)}
\]

\[
- \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} B^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2) (W'^2 - \sigma^2 + \xi - i\epsilon)} - \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} B^{(\pm)}(\xi', \sigma^2)}{\xi' (\xi' - \xi - i\epsilon)}.
\] (2.13)

Then dispersion relations may be written compactly as

\[
A^{(\pm)}(\xi, \sigma^2) = A_1^{(\pm)}(\xi, \sigma^2) - \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} A^{(\pm)}(W'^2, \sigma^2)}{(W'^2 - \sigma^2) (W'^2 - \sigma^2 + \xi - i\epsilon)}
\]

\[
- \frac{\xi}{\pi^2} \int_{(m + p)^2}^\infty d(W'^2) \frac{\text{Im} A^{(\pm)}(\xi', \sigma^2)}{\xi' (\xi' - \xi - i\epsilon)},
\]
First, we shall calculate the contribution of the pion-pion scattering to \((M_{pa})_2\). The expression for \((M_{pa})_2\) is

\[ (m/2k^0) \frac{1}{\sqrt{2}} \left[ \frac{\partial B^{(\pm)}(\vec{\varepsilon}, \sigma^a)}{\partial \vec{\varepsilon}'} \right] \frac{\delta B^{(\pm)}(\vec{\varepsilon}', \sigma^a)}{\varepsilon' - \varepsilon' - i\varepsilon}. \]  

(2·14)

where \(t_1\) and \(t_2\) are the four-momenta of the ingoing or outgoing pions, and Greek subscripts \(a'\) and \(\beta'\) are the isotopic spin indices of these pions. The integral over intermediate states should be understood to be the average of the integral over "in" states and the one over "out" states. The matrix element of \(j_a(0)\) in (2·15) is the pion-pion scattering amplitude. If we limit the partial waves of pion-pion scattering amplitude to the \(S\)- and \(P\)-waves, this matrix element may be written by the requirement of Lorentz invariance in the form:

\[ \langle k_{20} | j_a(0) | t_{1a} \rangle = 4n(\frac{8k^0 l_0^2 t_0^2}{4\pi})^{-1/2} \left( \frac{\varepsilon}{4} - \mu^2 \right)^{-1/2} \]

\[ \times \{ \exp[-i\delta_S(\varepsilon)] \sin \delta_S(\varepsilon) (1/3) \delta_{ab} \delta_{a'b'} \]

\[ + 3 \exp[-i\delta_P(\varepsilon)] \sin \delta_P(\varepsilon) (1/4)[(t_2 - t_1)(k_1 + k_2)][(\varepsilon/4) - \mu^2]^{-1} \]

\[ \times (1/2) [\delta_{aa'} \delta_{bb'} - \delta_{aa'} \delta_{bb'}] \}, \]  

(2·16)

where \(\delta_S\) and \(\delta_P\) are the \(S\)- and the \(P\)-wave pion-pion phase shifts, which correspond to the total isotopic spin state of the pion-pion system \(I=0\) and \(I=1\), respectively. (It should be noted that the \(I=2\) state in the pion-pion system does not contribute to nucleon-antinucleon scattering.)

As the matrix element \(\bar{u}(p_2) \langle t_{1a} t_{2a'} | f(0) | p_1 \rangle\) is equivalent to the matrix element of the process \(N(p_1) + \pi(-t_{1a}) \rightarrow \pi(t_{2a'}) + N(p_2)\), this may be written as

\[ \langle t_{1a} t_{2a'} | f(0) | p_1 \rangle = - (m/p_1^0)^{1/2} (1/4k_1^0 k_2^0)^{1/2} [\delta_S(\varepsilon, \sigma^a) + i\gamma \cdot Q' B(\varepsilon, \sigma^a)], \]  

(2·17)

where \(\varepsilon = -(t_1 + t_2)^2\), \(\sigma^a = -(p_2 - t_1)^2\) and \(Q' = (t_2 - t_1)/2\), corresponding to the reversed sign of the energy of an initial pion. It should be noticed that this \(\varepsilon\) is identical with \(\varepsilon = -(k_1 - k_2)^2\) defined previously due to the energy momentum conservation \(t_1 + t_2 = k_2 - k_1\). Therefore the latter calculation becomes simple. This is the reason why we choose \(\varepsilon\) as a variable.

Substituting (2·16) and (2·17) in (2·15), and performing the integration over the variables of the intermediate state, \(M_4\) may be written in the form

\[ (M_{pa})_2 = \delta_{pa} \text{Re}[\exp[-i\delta_S(\varepsilon)] \sin \delta_S(\varepsilon) a^{(s)}(\varepsilon)] \]

\[ + \langle \tau_{a'}, \tau_{a}/2 \rangle \{ \text{Re}[\exp[-i\delta_P(\varepsilon)] \sin \delta_P(\varepsilon) a^{(-)}(\varepsilon, \sigma^a)] \]

\[ - i\gamma \cdot Q \text{Re}[\exp[-i\delta_P(\varepsilon)] \sin \delta_P(\varepsilon) b^{(-)}(\varepsilon)] \}, \]  

(2·18)
where $a^{(+)}(\xi)$, $a^{(-)}(\xi, \sigma^2)$ and $b^{(-)}(\xi)$ are written as follows:

\[
a^{(+)}(\xi) = \int d\Omega \left[ A^{(+)}(\xi, \sigma^2) + \frac{(m/4)}{(\xi/4) - m^2} (p_1 + p_2, t_2-t_1) B^{(+)}(\xi, \sigma^2) \right],
\]

\[
b^{(-)}(\xi) = \int d\Omega' \frac{3}{4} \frac{(t_2-t_1, k_1+k_2)}{(\xi/4) - \mu^2} \left[ \frac{(\xi/2) - \sigma^2 + m^2 + \mu^2}{2} \right]^2
\]

\[
- 4 \left\{ \frac{(\xi/4) - m^2}{} \right\} \{ (\xi/4) - \mu^2 \}^{-1}
\]

\[
\times \left[ (p_1 \cdot k_1 + k_2) (p_1 + p_2, t_2-t_1)/2 - \{ (\xi/4) - m^2 \} (k_1 + k_2, t_2-t_1) \right] B^{(-)}(\xi, \sigma^2),
\]

\[
a^{(-)}(\xi, \sigma^2) = \int d\Omega' \frac{3}{4} \frac{(t_2-t_1, k_1+k_2)}{(\xi/4) - \mu^2} \left[ A^{(-)}(\xi, \sigma^2) + m \left\{ \frac{(\xi/2) - \sigma^2 + m^2 + \mu^2}{2} \right\}^2
\]

\[
- 4 \left\{ \frac{(\xi/4) - m^2}{} \right\} \{ (\xi/4) - \mu^2 \}^{-1}
\]

\[
\times \left[ (p_1 \cdot k_1 + k_2) (k_1 + k_2, t_2-t_1)/2 - \{ (\xi/4) - \mu^2 \} (p_1 + p_2, t_2-t_1) \right] B^{(-)}(\xi, \sigma^2),
\]

(2.19)

where the following abbreviation is used,

\[
\int d\Omega' = (\xi'^{1/2}/\pi) [(\xi/4) - \mu^2]^{-1/2} \int dt_1 dt_2 \delta(p_1 - p_2 - t_1 - t_2) \delta(t_2^2 + \mu^2) \delta(t_1^2 + \mu^2) \theta(t_1) \theta(t_2).
\]

Choosing the Lorentz frame characterized by the condition $p_1 - p_2 = 0*$ we further reduce to the following angle integrals:

\[
a^{(+)}(\xi) = (1/2) \int dx \left[ A^{(+)}(\xi, \sigma^2) + x \frac{(\xi/4) - \mu^2}{\mu \cdot q} \right] m B^{(+)}(\xi, \sigma^2),
\]

\[
b^{(-)}(\xi) = (3/4) \int dx (1 - x^2) B^{(-)}(\xi, \sigma^2),
\]

\[
a^{(-)}(\xi, \sigma^2) = (3/2) \{ - (\sigma^2/2) + (\xi/4) + (m^2 + \mu^2)/2 \} a^{(-)}(\xi),
\]

\[
a^{(-)}(\xi) = \frac{1}{p \cdot q} \int dx x A^{(-)}(\xi, \sigma^2) + m \frac{1}{\mu^2} \int dx \{ (3x^2/2) - 1/2 \} B^{(-)}(\xi, \sigma^2),
\]

(2.20)

where

\[
p = ((\xi/4) - m^2)^{1/2}, \quad q = ((\xi/4) - \mu^2)^{1/2} \quad \text{and} \quad \sigma^2 = m^2 + \mu^2 + \xi/2 - 2p \cdot qx.
\]

By identifying (2.11) and (2.18), the explicit expression of $\delta A^{(\pm)}$ and $\delta B^{(\pm)}$ may be written as

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* This may be possible because of $\xi' > 4\mu^2$. See the footnote of Ref. I.
\( \delta A^{(+)}(\xi, \sigma^2) = -\text{Re}[\exp[-i\delta_s(\xi)] \sin \delta_s(\xi) a^{(+)}(\xi)] \)
\( \delta B^{(+)}(\xi, \sigma^2) = 0 \)
\( \delta A^{(-)}(\xi, \sigma^2) = -\text{Re}[\exp[-i\delta_p(\xi)] \sin \delta_p(\xi) a^{(-)}(\xi, \sigma^2)] \)
\( \delta B^{(-)}(\xi, \sigma^2) = -\text{Re}[\exp[-i\delta_p(\xi)] \sin \delta_p(\xi) b^{(-)}(\xi)]. \)  

(2.21)

Hence if we express \( \delta A^{(\pm)} \) and \( \delta B^{(\pm)} \) explicitly, we can write the dispersion relations in the form
\[
A^{(+)}(\xi, \sigma^2) = A_1^{(+)}(\xi, \sigma^2) + \left( \xi/\pi \right) \int_{4\pi^2}^{\infty} d\xi' \text{Re}[\exp[-i\delta_s(\xi')] \sin \delta_s(\xi') a^{(+)}(\xi')] \frac{1}{\xi' - \xi - i\epsilon},
\]
\[
B^{(+)}(\xi, \sigma^2) = B_1^{(+)}(\xi, \sigma^2), \tag{2.22a}
\]
\[
A^{(-)}(\xi, \sigma^2) = A_1^{(-)}(\xi, \sigma^2) + \left( \xi/\pi \right) \int_{4\pi^2}^{\infty} d\xi' \text{Re}[\exp[-i\delta_p(\xi')] \sin \delta_p(\xi') a^{(-)}(\xi', \sigma^2)] \frac{1}{\xi' - \xi - i\epsilon},
\]
\[
B^{(-)}(\xi, \sigma^2) = B_1^{(-)}(\xi, \sigma^2) + \left( \xi/\pi \right) \int_{4\pi^2}^{\infty} d\xi' \text{Re}[\exp[-i\delta_p(\xi')] \sin \delta_p(\xi') b^{(-)}(\xi')] \frac{1}{\xi' - \xi - i\epsilon}. \tag{2.22b}
\]

§ 3. Integral equations

Now we shall derive the integral equations satisfied by \( a^{(+)}(\xi) \), \( b^{(-)}(\xi) \) and \( a^{(-)}(\xi) \). We begin with \( a^{(+)}(\xi) \), (2.20).

Although \( A^{(+)}(\xi, \sigma^2) \) and \( B^{(+)}(\xi, \sigma^2) \) in the expression for \( a^{(+)}(\xi) \), (2.20), are unphysical scattering amplitudes, they may be obtained by continuing analytically (2.22a). Then \( a^{(+)}(\xi) \) becomes
\[
a^{(+)}(\xi) = a_1^{(+)}(\xi) + \left( \xi/\pi \right) \int_{4\pi^2}^{\infty} d\xi' \text{Re}[\exp[-i\delta_s(\xi')] \sin \delta_s(\xi') a^{(+)}(\xi')] \frac{1}{\xi' - \xi - i\epsilon}, \tag{3.1}
\]
where
\[
a_1^{(+)}(\xi) = \frac{1}{2} \int_{\xi_1}^{\xi} dx \{ A_1^{(+)}(\xi, \sigma^2) + x(\xi/4 - \mu^2) (pq)^{-1} mB_1^{(+)}(\xi, \sigma^2) \}. \tag{3.2}
\]

Since \( a_1^{(+)}(\xi) \) is considered to be a known function, this is an integral equations satisfied by \( a^{(+)}(\xi) \), where the kernel is an S-wave pion-pion scattering amplitude.

Similar equations may be derived for \( b^{(-)}(\xi) \) and \( a^{(-)}(\xi) \). The integral equation satisfied by \( b^{(-)}(\xi) \) is
\[
b^{(-)}(\xi) = b_1^{(-)}(\xi) + \left( \xi/\pi \right) \int_{4\pi^2}^{\infty} d\xi' \text{Re}[\exp[-i\delta_p(\xi')] \sin \delta_p(\xi') b^{(-)}(\xi')] \frac{1}{\xi' - \xi - i\epsilon}, \tag{3.3}
\]
where
\[ b_1^{(-)(\xi)} = (3/4) \int_{-1}^{1} dx (1 - x^2) B_1^{(-)(\xi, \sigma^2)}, \]  
and the one satisfied by \( a^{(-)(\xi)} \) is
\[ a^{(-)(\xi)} = a_1^{(-)(\xi)} + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \Re \left[ \exp \left\{ -i\theta(\xi') \right\} \sin \delta(\xi') a^{(-)(\xi')} \right] \frac{1}{\xi' (\xi' - \xi - i\epsilon)} \]  
where
\[ a_1^{(-)(\xi)} = (1/pq) \int_{-1}^{1} dx x A_1^{(-)(\xi, \sigma^2)} + (m/p^2) \int_{-1}^{1} dx \left[ (3x^2 - 1)/2 \right] B_1^{(-)(\xi, \sigma^2)}. \]  

In general the form of these integral equations is as follows;
\[ f(\xi) = f_1(\xi) + (\xi/\pi) \int_{4\mu^2}^{\infty} d\xi' \Re \left[ \exp \left\{ -i\theta(\xi') \right\} \sin \delta(\xi') f(\xi') \right] \frac{1}{\xi' (\xi' - \xi - i\epsilon)} \]  
According to Omnes\(^9\) the solution of this integral equation is given by
\[ f(\xi) = \begin{cases} 
  f_1(\xi) + \exp[\rho(\xi)] \left( \xi/\pi \right) \int_{4\mu^2}^{\infty} d\xi' \exp \left[ -\rho(\xi') \right] \sin \delta(\xi') f_1(\xi'), & \text{for } \xi < 4\mu^2 \\
  f_1(\xi) + \exp[\rho(\xi)] \exp[i\theta(\xi)] \left( \xi/\pi \right) \int_{4\mu^2}^{\infty} d\xi' \exp \left[ -\rho(\xi') \right] \sin \delta(\xi') f_1(\xi'), & \text{for } \xi > 4\mu^2 
\end{cases} \]  
where
\[ \rho(\xi) = (\xi/\pi) P \int_{4\mu^2}^{\infty} d\xi' \delta(\xi') [\xi' (\xi' - \xi)]^{-1}. \]

Here we have assumed that \( f_1(\xi) \) is real and that the function in question behaves properly at infinity. In order that the method is useful, the functional form of the phase shift of the pion-pion interaction must be known. Now we have no knowledge about the behaviour of the phase shifts, but since dispersion integrals may contribute at low energies owing to the energy denominator, we may expect that the results are insensitive to the behavior of the phase shift at high energies, so we assume that the energy dependence of phase shifts is given by a scattering length approximation.
\[ \tan \delta_\alpha(\xi) = (\alpha_\alpha/2) (\xi - 4\mu^2)^{1/2}, \quad \tan \delta_\pi(\xi) = (\alpha_\pi/2) (\xi - 4\mu^2)^{3/2} \]
Then after elementary calculation, \( \exp[\rho(\xi)] \), except for constant factors independent of (3·8), is given by:

(for \( S \)-wave)

\[
\exp[\rho(\xi)] = \begin{cases} 
1 + \left( |\alpha_0|/2 \right) (4\mu^2 - \xi)^{1/2} \xi \left( \xi - 4\mu^2 \right)^{1/2} & \text{for } \xi < 4\mu^2, \\
\left[ 1 + \left( \alpha_0/2 \right)^2 (\xi - 4\mu^2) \right]^{1/2} & \text{for } \xi > 4\mu^2,
\end{cases}
\]

(for \( P \)-wave)

\[
\exp[\rho(\xi)] = \begin{cases} 
\left[ 1 - \left( \alpha_0/2 \right)^2 (4\mu^2 - \xi) \right] / \left[ 1 - \left( \alpha_0/2 \right)^2 (4\mu^2 - \xi)^{3/2} \right] \xi \left( \xi - 4\mu^2 \right)^{1/2} & \text{for } \xi < 4\mu^2, \\
\left[ 1 + \left( \alpha_0/2 \right)^2 (\xi - 4\mu^2) \right] / \left[ 1 + \left( \alpha_0/2 \right)^2 (\xi - 4\mu^2)^{3/2} \right]^{1/2} & \text{for } \xi > 4\mu^2,
\end{cases}
\]

where

\[
\xi(x) = \begin{cases} 
1 & \text{for } x > 0, \\
-1 & \text{for } x < 0.
\end{cases}
\]

From (3·8) and (3·11), the final expressions of \( A^{(\pm)}(\xi, \sigma^2) \), \( B^{(\pm)}(\xi, \sigma^2) \) and \( A^{(-)}(\xi, \sigma^2) \) for \( \xi < 4\mu^2 \) and for positive scattering lengths, for simplicity, are given by

\[
A^{(\pm)}(\xi, \sigma^2) = A_1^{(\pm)}(\xi, \sigma^2) + \frac{\alpha_0}{\left( \alpha_0/2 \right)^2 (4\mu^2 - \xi)^{1/2}} \frac{\xi}{\pi} \int d\xi' \frac{(\xi' - 4\mu^2)^{1/2}}{\xi' (\xi' - \xi)},
\]

\[
B^{(\pm)}(\xi, \sigma^2) = B_1^{(\pm)}(\xi, \sigma^2)
\]

\[
+ \frac{1 - \left( \alpha_0/2 \right)^2 (4\mu^2 - \xi)}{1 - \left( \alpha_0/2 \right)^2 (4\mu^2 - \xi)^{3/2}} \frac{\xi}{\pi} \int d\xi' \frac{(\alpha_0/2)^3 (\xi' - 4\mu^2)^{3/2} \xi'}{\xi' (\xi' - \xi) \left[ 1 + \left( \alpha_0/2 \right)^2 (\xi' - 4\mu^2) \right]},
\]

\[
A^{(-)}(\xi, \sigma^2) = A_1^{(-)}(\xi, \sigma^2) + \frac{1 - \left( \alpha_0/2 \right)^2 (4\mu^2 - \xi)}{1 - \left( \alpha_0/2 \right)^2 (4\mu^2 - \xi)^{3/2}} \frac{\xi}{\pi} \int d\xi' \frac{(\alpha_0/2)^3 (\xi' - 4\mu^2)^{3/2} (3/2) \left( m^2 + \mu^2 \right)^{1/2} + \frac{\xi / 4 - \sigma^2 / 2 + a_{1}^{(-)}(\xi')}{\xi' (\xi' - \xi) \left[ 1 + \left( \alpha_0/2 \right)^2 (\xi' - 4\mu^2) \right]}},
\]

(3·12)

In order to apply these relations practically, the case for \( \xi < 4\mu^2 \) is useful, because only this region contains the physical pion-nucleon scattering. In these relations all the quantities other than scattering lengths are expressed in terms of the experimental phase shifts of the pion-nucleon scattering and the coupling constant \( f^2 \). Therefore we can estimate the magnitudes of the scattering lengths from these dispersion relations. Similar relations might be formally written down for negative scattering lengths.

Now we shall give explicit expressions for \( A_1^{(\pm)}(\xi, \sigma^2) \), \( B_1^{(\pm)}(\xi, \sigma^2) \), \( a_{1}^{(\pm)}(\xi) \) and \( b_{1}^{(\pm)}(\xi) \). As \( A^{(\pm)}(0, \sigma^2) \) and \( B^{(\pm)}(0, \sigma^2) \) in (2·12) are forward pion-nucleon scattering amplitudes at the c.m. energy \( W = (\sigma^2)^{1/2} \), these may be expressed in
terms of the scattering amplitudes given by Goldberger, Miyazawa and Oehme, as was stated previously. That is,

\[ A^{(\pm)}(0, \sigma^2) = A^{(\pm)}(0, (m + \mu)^2) + \left( \frac{1}{\pi} \right) \int_{(m + \mu)^2}^{m^2} d(W^2) \text{Im} A^{(\pm)}(W^2, 0) \]

\[ \times \left\{ \frac{1}{(W^2 - \sigma^2)} \pm \frac{1}{(W^2 + \sigma^2 - 2(m^2 + \mu^2))} \right\} \]

\[ - \left\{ 1/(W^2 - (m + \mu)^2) \pm 1/(W^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right\} \}, \]

and

\[ B^{(\pm)}(0, \sigma^2) = B^{(\pm)}(0, (m + \mu)^2) + G^2 \left\{ \frac{1}{(m^2 - \sigma^2)} \pm \frac{1}{(m^2 + \sigma^2 - 2(m^2 + \mu^2))} \right\} \]

\[ + \left\{ 1/(m^2 - (m + \mu)^2) \pm 1/(m^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right\} \}, \quad (3.13) \]

Introducing \( \text{Im} A^{(\pm)}(W^2, \xi) \) and \( \text{Im} B^{(\pm)}(W^2, \xi) \) and taking into account the relation

\[ \cos \theta = 1 + \frac{\xi}{2q^2} = 1 + \frac{(\sigma^2 - W^2)}{(2q^2)}, \quad (3.14) \]

where \( \theta \) and \( q \) are the scattering angle and momentum in the c.m. system, respectively, we can rewrite \( A^{(\pm)}(\xi, \sigma^2) \) and \( B^{(\pm)}(\xi, \sigma^2) \) in the following form,

\[ A^{(\pm)}(\xi, \sigma^2) = A^{(\pm)}(0, (m + \mu)^2) \]

\[ + \left( \frac{1}{\pi} \right) \int_{(m + \mu)^2}^{m^2} d(W^2) \text{Im} A^{(\pm)}(W^2, \xi) \left\{ \frac{1}{(W^2 - \sigma^2 - \xi - i\epsilon)} \right\} \]

\[ \pm \text{Im} A^{(\pm)}(W^2, 0) \left\{ \frac{1}{(W^2 + \sigma^2 - 2(m^2 + \mu^2))} \right\} \]

\[ - \text{Im} A^{(\pm)}(W^2, 0) \left\{ 1/(W^2 - (m + \mu)^2) \pm 1/(W^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right\} \}, \]

\[ B^{(\pm)}(\xi, \sigma^2) = B^{(\pm)}(0, (m + \mu)^2) \]

\[ + G^2 \left\{ \frac{1}{(m^2 - \sigma^2 + \xi)} \pm \frac{1}{(m^2 + \sigma^2 - 2(m^2 + \mu^2))} \right\} \]

\[ - \left\{ 1/(m^2 - (m + \mu)^2) \pm 1/(m^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right\} \}, \]

\[ + \left( \frac{1}{\pi} \right) \int_{(m + \mu)^2}^{m^2} d(W^2) \text{Im} B^{(\pm)}(W^2, \xi) \left\{ \frac{1}{(W^2 - \sigma^2 + \xi - i\epsilon)} \right\} \]
\[\pm \text{Im} B^{(\pm)}(W^2, 0) / (W^2 + \sigma^2 - 2(m^2 + \mu^2)) \]

\[- \text{Im} B^{(\pm)}(W^2, 0) \left[ 1/(W^2 - (m + \mu)^2) \mp 1/(W^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right].\]

(3.15)

It is convenient to compare these forms with the dispersion relations given by Chew et al. \(^5\) Im\(A^{(\pm)}(W^2, \xi)\) and Im\(B^{(\pm)}(W^2, \xi)\) or Im\(A^{(\pm)}(W^2, \sigma)\) and Im\(B^{(\pm)}(W^2, \sigma)\) are assumed to be able to expand in partial waves and particularly to dominate \(S\)- and \(P\)-wave amplitudes.

Substituting \(A_{1}^{(\pm)}(\xi, \sigma)\) and \(B_{1}^{(\pm)}(\xi, \sigma)\) in (3.2), (3.4) and (3.6) and performing the integration over angle \(\varphi\), we obtain the final expressions for \(a_{i}^{(\pm)}(\xi)\), \(b_{i}^{(\pm)}(\xi)\) and \(a_{i}^{(-)}(\xi)\) given by

\[a_{i}^{(\pm)}(\xi) = A^{(\pm)}(0, (m + \mu)^2)\]

\[+ (1/\pi) \int_{(m + \mu)^2}^{\infty} \text{d}(W^2) J_{1}(W^2, \xi) \left[ \text{Im} A^{(\pm)}(W^2, \xi) + \text{Im} A^{(\pm)}(W^2, 0) \right] \]

\[- \left[ 1/(W^2 - (m + \mu)^2) \mp 1/(W^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right] \text{Im} A^{(\pm)}(W^2, 0) \}

\[- \left( m/(2m^2 - \xi^2) \right) \left[ (1/\pi) \int_{(m + \mu)^2}^{\infty} \text{d}(W^2) J_{1}(W^2, \xi) \left[ \text{Im} B^{(\pm)}(W^2, \xi) \right] \right. \]

\[+ \text{Im} B^{(\pm)}(W^2, 0) \left] + 2G^2 J_{1}(m^2, \xi) \right],\]

\[b_{i}^{(-)}(\xi) = B^{(-)}(0, (m + \mu)^2)\]

\[+ (1/\pi) \int_{(m + \mu)^2}^{\infty} \text{d}(W^2) J_{1}(W^2, \xi) \left[ \text{Im} B^{(-)}(W^2, \xi) + \text{Im} B^{(-)}(W^2, 0) \right] \]

\[- \text{Im} B^{(-)}(W^2, 0) \left[ 1/(W^2 - (m + \mu)^2) \mp 1/(W^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right] \}

\[+ G^2 \left[ 2J_{1}(m^2, \xi) - \left[ 1/(m^2 - (m + \mu)^2) + 1/(m^2 + (m + \mu)^2 - 2(m^2 + \mu^2)) \right] \right];\]

\[a_{i}^{(-)}(\xi) = - (1/\pi) \int_{(m + \mu)^2}^{\infty} \text{d}(W^2) J_{1}(W^2, \xi) \left[ \text{Im} A^{(-)}(W^2, \xi) + \text{Im} A^{(-)}(W^2, 0) \right] \]

\[- \left( 4m/(4m^2 - \xi^2) \right) \left[ 2G^2 J_{1}(m^2, \xi) \right] \]

\[+ (1/\pi) \int_{(m + \mu)^2}^{\infty} \text{d}(W^2) J_{1}(W^2, \xi) \left[ \text{Im} B^{(-)}(W^2, \xi) + \text{Im} B^{(-)}(W^2, 0) \right] \}

(3.16)
Pion-Pion Interaction and Pion-Nucleon Scattering

where

\[ J_1(W^2, \xi) = \frac{2}{3} \tan^{-1}(\beta/\alpha) \]
\[ J_2(W^2, \xi) = 2 \left\{ 1 - (\alpha/\beta) \tan^{-1}(\beta/\alpha) \right\} \]
\[ J_3(W^2, \xi) = \frac{3}{\beta^2} \left\{ - (\alpha/\beta) + (1 + (\alpha/\beta)^2) \tan^{-1}(\beta/\alpha) \right\} \]
\[ J_4(W^2, \xi) = \frac{16}{\beta^3} \left\{ 1 - (\alpha/\beta) \tan^{-1}(\beta/\alpha) \right\} \]
\[ J_5(W^2, \xi) = \frac{2}{\beta} \left\{ (3/\beta) - (1 + 3(\alpha/\beta)^2) \tan^{-1}(\beta/\alpha) \right\} , \]

and

\[ \alpha = 2W^2 - 2(m^2 + p^2) + \xi \]
\[ \beta = (4m^2 - \xi)^{1/2} (\xi - 4\mu^2)^{1/2} . \]

\( J_i(m^2, \xi) \) is obtained by exchanging \( \alpha \) for \( -2\mu^2 + \xi \) in \( J_i(W^2, \xi) \). Finally we add one remark that \( a_i^{(+)}(\xi) \), \( b_i^{(-)}(\xi) \) and \( a_i^{(-)}(\xi) \) are real.

§ 4. Numerical results and discussions

At first we shall estimate \( S \)- and \( P \)-wave scattering lengths for pion-pion scattering by comparing dispersion relations with experiment. Practically, the following two methods are used: I. Regarded as the functions of \( \xi \) and \( \sigma^2 \), scattering amplitudes are differentiated with respect to \( \sigma^2 \) for fixed \( \sigma^2 \). They are evaluated at \( \xi = 0 \) and \( \sigma^2 = (m + \mu)^2 \), i.e. for forward pion-nucleon scattering at zero kinetic energy. II. Regarded as the functions of \( \xi \) and \( W^2 \), scattering amplitudes are differentiated with respect to \( \xi \) for fixed \( W^2 \). They are evaluated at \( \xi = 0 \) and \( W^2 = (m + \mu)^2 \), i.e. for forward pion-nucleon scattering at zero kinetic energy. These choices enable us to simplify the principal part integration and it turns out that convergence of integrals is good. Now, in order to compare \( \partial A^{(\pm)}/\partial \xi \) and \( \partial B^{(\pm)}/\partial \xi \) thus obtained with experiment, one must express these \( \partial A^{(\pm)}/\partial \xi \) and \( \partial B^{(\pm)}/\partial \xi \) by the experimental pion-nucleon phase shifts. Unfortunately, however, these extremely depend on the \( D \)-wave pion-nucleon phase shifts in both cases, and on the \( P \)-wave effective ranges in case I. Therefore we do not know to what extent the neglect of these contributions, assumed in Ref. I, can be trusted. Though we shall discuss later on this point, we can avoid this difficulty in the following way: We begin with the combined dispersion relations \( A^{(\pm)} + (W - m)B^{(\pm)} \), then the contribution from \( D \)-wave phase shifts and \( P \)-wave effective ranges are eliminated exactly; namely differentiating the identity

\[ A^{(\pm)} + (W - m)B^{(\pm)} = 4\pi(2W)f_1^{(\pm)}/(E + m) , \] (4.1)

with respect to \( \xi \) for fixed \( \sigma^2 \) or for fixed \( W^2 \) corresponding to cases I or II, respectively, we obtain the following identities,

\[ (\partial/\partial \xi) A^{(\pm)}(\xi, \sigma^2) + (W - m)(\partial/\partial \xi) B^{(\pm)}(\xi, \sigma^2) \]
\[ = (1/2W)B^{(\pm)}(\xi, \sigma^2) - (\partial/\partial W^2)(4\pi(2W)f_1^{(\pm)}/(E + m)) \quad \text{for I}, \] (4.2)
(\partial/\partial \bar{\xi}) A^{(\pm)}(W^2, \bar{\xi}) + (W - m) (\partial/\partial \bar{\xi}) B^{(\pm)}(W^2, \bar{\xi}) = (\partial/\partial \bar{\xi}) (4\pi(2W)f_{i}^{(\pm)}/(E+m)) \quad \text{for II}, \quad (4\cdot3)

where we follow the notation given by Chew et al.\(^9\)

In the real part of (4·2) or (4·3), the contribution of S-wave pion-nucleon phase shifts is estimated to be negligible in case I or is zero in case II respectively. Therefore it is sufficient to consider the contribution of P-wave pion-nucleon scattering lengths in the real part of the relation.

The numerical values of \((\partial/\partial \bar{\xi}) \text{Re} A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) \text{Re} B^{(\pm)}\) and \((\partial/\partial \bar{\xi}) A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) B^{(\pm)}\) are shown in Table I for both cases I and II. Here we used experimental values reported by Puppi at CERN Conference\(^9\) for the pion-nucleon scattering lengths in \((\partial/\partial \bar{\xi}) \text{Re} A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) \text{Re} B^{(\pm)}\), and use the empirical formula given by Anderson\(^9\) for the \((3,3)\) phase shift and pion-nucleon coupling constant \(f^2\) is taken as \(f^2 = 0.08\) in \((\partial/\partial \bar{\xi}) A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) B^{(\pm)}\).

Table I. Numerical values of \((\partial/\partial \bar{\xi}) \text{Re} A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) \text{Re} B^{(\pm)}\) and \((\partial/\partial \bar{\xi}) A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) B^{(\pm)}\). Here, subscript A or B indicates the contribution from A\(^{(\pm)}\) or B\(^{(\pm)}\) respectively. Errors are inserted. The pion-nucleon coupling constant is taken as \(f^2 = 0.08\).

<table>
<thead>
<tr>
<th>Charge state (+)</th>
<th>((\partial/\partial \bar{\xi}) \text{Re} A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) \text{Re} B^{(\pm)}) ((\mu^{-3}))</th>
<th>((\partial/\partial \bar{\xi}) A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) B^{(\pm)}) ((\mu^{-3}))</th>
<th>Born</th>
<th>Dispersion Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>+0.38±0.2</td>
<td>-0.87</td>
<td>(-0.70)(_A^{(\pm)}) + (1.00)(_n^{(\pm)}) = +0.30±0.1</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>+3.10±0.4</td>
<td>+1.17</td>
<td>(1.00)(_A^{(\pm)}) + (0.01)(_n^{(\pm)}) = 1.01±0.1</td>
<td></td>
</tr>
<tr>
<td>Charge state (-)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>-0.51±0.2</td>
<td>-0.87</td>
<td>(0.35)(_A^{(\pm)}) + (-0.50)(_n^{(\pm)}) = -0.15±0.1</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>-1.97±0.4</td>
<td>-1.17</td>
<td>(-0.65)(_A^{(\pm)}) + (-0.07)(_n^{(\pm)}) = -0.72±0.1</td>
<td></td>
</tr>
</tbody>
</table>

The error in the real part is due to inaccuracy of the pion-nucleon scattering lengths. In case I, the scattering lengths \(a_{11}, a_{33}, a_{13}\) and \(a_{31}\) should be expressed by experimental data. Because of inaccuracy of \(a_{33}\) and \(a_{31}\), particularly, the real part may have the error of about 50% of the absolute value in case I. In case II, on the other hand, only two scattering lengths, \(a_{13}\) and \(a_{33}\), are contained in the real part, so the error may be relatively small, i.e. about 10∼20%. From these considerations, we get the values of \((\partial/\partial \bar{\xi}) \text{Re} A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) \text{Re} B^{(\pm)}\) as shown in Table I.

The error due to inaccuracy of evaluating the dispersion integral in \(\partial/\partial \bar{\xi}A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) B^{(\pm)}\) is assumed to be 10%, the inaccurate small phase shifts and the unknown behaviours of the scattering amplitude at high energies over the \(\partial_{33}\) dominant energy region being taken into account (although, numerically, the dispersion integrals converge very rapidly under the assumption of \(\partial_{33}\) dominance in both cases I and II). From these considerations we get the values of \((\partial/\partial \bar{\xi}) A^{(\pm)} + \mu(\partial/\partial \bar{\xi}) B^{(\pm)}\) as indicated in Table I.
Using the values in Table I, we write the numerical values of the following quantity as

\[
\begin{align*}
&\frac{\partial}{\partial \tilde{\xi}} \Re A^{(+)} + \mu \frac{\partial}{\partial \tilde{\xi}} \Re B^{(+)} - \frac{\partial}{\partial \tilde{\xi}} A_1^{(+)} - \mu \frac{\partial}{\partial \tilde{\xi}} B_1^{(+)} \\
&= \left[ \left( 0.38 \pm 0.2 \right) R + \left( 0.87 \right) R - \left( 0.30 \pm 0.1 \right) I \right] \mu^{-3} = (0.95 \pm 0.3) \mu^{-3}, \quad I \\
&\left[ \left( 3.1 \pm 0.4 \right) R - \left( 1.17 \right) R - \left( 1.01 \pm 0.1 \right) I \right] \mu^{-3} = (0.92 \pm 0.5) \mu^{-3}, \quad II
\end{align*}
\]

\[
\begin{align*}
&\frac{\partial}{\partial \tilde{\xi}} \Re A^{(-)} + \mu \frac{\partial}{\partial \tilde{\xi}} \Re B^{(-)} - \frac{\partial}{\partial \tilde{\xi}} A_1^{(-)} - \mu \frac{\partial}{\partial \tilde{\xi}} B_1^{(-)} \\
&= \left[ \left( -0.51 \pm 0.2 \right) R + \left( 0.87 \right) R + \left( 0.15 \pm 0.1 \right) I \right] \mu^{-3} = (0.51 \pm 0.3) \mu^{-3}, \quad I \\
&\left[ \left( -1.97 \pm 0.4 \right) R + \left( 1.17 \right) R + \left( 0.72 \pm 0.1 \right) I \right] \mu^{-3} = (-0.08 \pm 0.5) \mu^{-3}, \quad II
\end{align*}
\]

where the subscripts \( R, B \) and \( I \) mean the contribution from the real part, Born term and dispersion integral, respectively.

On the other hand, if we differentiated the pion-pion part of the combined dispersion relation (see (3.12)) with respect to \( \tilde{\xi} \) corresponding to I and II, we would find that both results are the same as far as the once-subtracted dispersion relation is correct. By virtue of the dispersion relation, therefore, the contributions from pion-pion scattering are numerically given by

\[
\begin{align*}
\frac{(\alpha_s/2)}{1 + (\alpha_s/2) 2\mu^2} \frac{1}{\pi} \int d\tilde{\xi}\left( \tilde{\xi}^2 - 4\mu^2 \right)^{1/2} \alpha_i^{(+)}(\tilde{\xi}) &
= \left\{ \begin{array}{l}
0.95 \pm 0.3 \mu^{-3} \quad \text{for I,} \\
0.92 \pm 0.5 \mu^{-3} \quad \text{for II,}
\end{array} \right.
\quad \text{(4.6)}
\\
\frac{1 - (\alpha_F/2)^2 4\mu^2}{1 - (\alpha_F/2)^2 (4\mu^2)^{3/2}} \frac{1}{\pi} \int d\tilde{\xi}\left( \tilde{\xi}^2 - 4\mu^2 \right)^{3/2} \left[ \left[ \mu b_1^{(-)}(\tilde{\xi}) - (3/2) m \mu a_i^{(-)}(\tilde{\xi}) \right] \right]
\frac{1}{\tilde{\xi}^2 \left[ 1 + \left( \alpha_F/2 \right)^2 \left( \tilde{\xi}^2 - 4\mu^2 \right) \right]} &
= \left\{ \begin{array}{l}
0.51 \pm 0.3 \mu^{-3} \quad \text{for I,} \\
-0.08 \pm 0.5 \mu^{-3} \quad \text{for II,}
\end{array} \right.
\quad \text{(4.7)}
\end{align*}
\]

where the left-hand side is written for positive scattering length for simplicity. Thus the difference between I and II should be considered to be due to the errors occurring in our approach.

At first we consider \( S \)-wave \((I=0)\) interaction. In Fig. 1 we plot numerical values of the left-hand side of (4.6) for particular \( \alpha_s \). The dotted lines show the values of the right-hand side of (4.6) for I and II. Dispersion relations for negative scattering length being not written till now, these results are added in Fig. 1. From Fig. 1, we obtain, irrespective of our choice of I or II, \( \alpha_s \approx 1/\mu \).

The result is interpreted as follows: Pion-pion interaction is attractive in the \( S \)-state with the scattering length of the order of one pion Compton wavelength. This result agrees qualitatively with that of Ref. I in which the contribution of \( D \)-waves for the pion-nucleon scattering is neglected.
About the $P$-wave pion-pion interaction, on the other hand, the situation is more complex. The right-hand side of the $P$-wave pion-pion dispersion relation $(4·7)$ depends upon whether the result of case I or that of case II is taken, though the results might be considered to be consistent within the limit of errors. According to case II, the right-hand side of the dispersion relation, accordingly the pion-pion part in the left-hand side, becomes zero. In fact, if we calculate a pion-pion part of the left-hand side in the static theory, the integrand of this term becomes zero. (This is advantageous for case II, but case I is not excluded, because the pion-pion part may be able to have a small value due to the relativistic effect.) Here it should be noticed that vanishing of the pion-pion part does not always mean $\alpha_r = 0$; it means only that $\alpha_r$ cannot be well determined from a combined dispersion relation $\left( \frac{\partial}{\partial \xi} \right) A^{(-)} + (W - m) \left( \frac{\partial}{\partial \xi} \right) B^{(-)}$. Not only the difference between case I and case II which might be considered as an error of the right-hand is large, but also the pion-pion part in the left-hand side cannot be estimated precisely, since the integral involving $a^{(-)}(\xi)$ does not rapidly converge.

Anyway, $\alpha_r$ could not be well determined from the dispersion relation $\left( \frac{\partial}{\partial \xi} \right) A^{(-)} + (W - m) \left( \frac{\partial}{\partial \xi} \right) B^{(-)}$ which was combined in order to cancel $D$-wave pion-nucleon phase shifts. If $D$-wave pion-nucleon phase shifts are negligibly small or experimentally determined in future, then $\alpha_r$ can be estimated by using the relation for $B^{(-)}$ only of which integrals converge rapidly.
In Ref. I, under the assumption that D-wave should be neglected, $\alpha^P$ is estimated by using $B^{(-)}$ relation. Validity of this approximation, however, is doubtful as was mentioned previously, so we wish to add the contribution from D-wave. If we take for this the value of D-wave phase shifts estimated by Chew et al. as the first trial, then the conclusion of Ref. I does not change, i.e.

$$\alpha^P \approx 0.4 \mu^{-1}.$$ 

The conclusion of Ref. I that P-wave pion-pion interaction exists may be considered to be favorable, but in any case this conclusion is not definite since the adopted D-wave phase shifts are not accurate enough to be trusted. Thus we could not obtain a definite conclusion about the P-wave pion-pion interaction.

Now we should like to make a remark about the behaviour of the pion-nucleon scattering phase shift $\delta_{13}$. According to the previous results, it may be allowed to take the value of contributions of pion-pion scattering to affect the pion-nucleon scattering amplitudes as

$$[\left(\partial/\partial \xi\right) A^{(e)}]_{\pi-p} = 1.0 \mu^{-a}$$

$$[\left(\partial/\partial \xi\right) B^{(e)}]_{\pi-p} = 0$$

$$[\left(\partial/\partial \xi\right) A^{(-)} + \mu \left(\partial/\partial \xi\right) B^{(-)}]_{\pi-p} = 0.$$ 

This last choice is insensitive to the tendency of $\delta_{13}$. Then, the contribution of pion-pion S-wave scattering to $\delta_{13}$ and $\delta_{33}$ are

$$(\delta_{13})_{\pi-p} = (\delta_{33})_{\pi-p} = (1/6\pi) q^3 [(E + m) / (2W)] [(\partial/\partial \xi) A^{(e)}]_{\pi-p} \approx 0.05 q^3 / \mu^3.$$ 

![Fig. 2. Pion-nucleon scattering phase shift versus momentum](https://academic.oup.com/ptp/article-abstract/23/4/731/1850630)

The dotted curve represents the contribution only of the Born term to $\delta_{13}$ and the solid curve represents the sum of the contribution of the Born term and that of S-wave pion-pion interaction.
The dominant contribution except that from pion-pion to the $\delta_{13}$ may be "Born term contribution" according to the Chew-Low static theory.

$$(\delta_{13})_{\text{Born}} = -(2/3) f^2 q^3 / \omega l^2 \approx -0.05 q^3 / \omega l^2$$

Combining these results, we obtain $\delta_{13}$ as

$$\delta_{13} \approx 0.05(q^3 / \mu^2 - q^3 / \omega l^2) \approx 3q(q^3 / \mu^2 - q^3 / \omega l^2).$$

This rough estimate shows that $\delta_{13} \approx 0$ at low energies, and that $\delta_{13}$ increases positively as the energy of pion increases (see Fig. 2). This behaviour may be consistent with the recent experiment.

Finally, we comment on our treatment. $\text{Im} A$ and $\text{Im} B$ in the inhomogeneous term are assumed to be able to be expanded in partial waves, and the integrals involving these are assumed to converge properly. In fact, this may be open to question. But this is beyond our present consideration. Besides, our dispersion relation does not satisfy the property of crossing symmetry. On this point, further more systematic study, for instance, by Mandelstam's double representation, should be necessary.

We mention here that Frazer and Fulco independently applied a similar method to ours to the problem of electromagnetic structure of nucleon.

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