Excitations in a High Density Electron Gas. II

—Diamagnetism—

Emiko FUJITA and Tunemaru USUI

College of General Education, University of Tokyo
Meguro-ku, Tokyo

(Received December 22, 1959)

As another application of the method of describing a high density fermion gas in terms of electron-hole pairs, diamagnetism of the gas is investigated. The following two corrections to Landau diamagnetism obtained come from the screened interaction of exchange type. It is intended to lay a stress on the effectiveness of the physical concept of electron-hole pair excitation.

§ 1. Introduction

In a previous paper1) under the same title we presented a general formalism which describes a system of fermions in terms of bosons of a certain type. The intention was to point out the importance of Sawada's2) or Wentzel's3) exciton, i.e. electron-hole pair as a basic concept in dealing with the electron gas at high density. The formalism described there is such that it enables us to investigate the behaviour of the gas entirely on the basis of the pair picture. This is considered particularly effective when one tries to study the properties of the gas down from the high density limit. However, the only one application presented there was a derivation of the dispersion relation of the plasma oscillation modified by exchange effect of the electrons. To clarify the point further, we shall present another application in this paper. The subject which we are going to deal with is diamagnetism of the electron gas.

This line of investigation has already been tried by Wentzel,4) who obtained Landau diamagnetism while he took only the Coulomb part of the interaction into account. Hence our present task is to investigate the effect of the exchange part. The effect will be, as a matter of course, of higher order in the mean distance of the electrons. However, the point is to show that the Hamiltonian of the excitons is appropriate to the description of the gas in a successive way from the high density limit.

§ 2. Unperturbed Hamiltonian and its diagonalization

The "exciton" representation of the Hamiltonian of an electron gas in a positive charge background is given by Eq. (18) of the paper I to the second order.
in creation or annihilation operators. This can be written in the following form:

\[ \mathcal{H} = \sum_q \mathcal{H}_q \]

\[ \mathcal{H}_q = \frac{1}{2} \sum_p \sum_{p'} (B_{pp'}^{(q)} \varphi_{p,q} \varphi_{-p',-q} + A_{pp'}^{(q)} \varphi_{p,q} \varphi_{-p',-q}). \]  

(2.1)

\[ \varphi, \pi \text{'s are coordinates and conjugate momenta of the oscillators:} \]

\[ \varphi_{p,q} = \frac{1}{\sqrt{2}} (c_{p,q} + c_{-p,-q}^*) = \varphi_{-p,-q}^* \]

\[ \pi_{p,q} = \frac{i}{\sqrt{2}} (c_{p,q}^* - c_{-p,-q}) = \pi_{-p,-q}^* \]  

(2.2)

and the coefficients are given by

\[ A_{pp'}^{(q)} = A_{p'-p,-q}^{(-q)} = \omega_{p,q} \delta_{pp'} + (V_q - V_{p-p'}) + (V_{q'-p'} - V_{p-p'}) \]

\[ B_{pp'}^{(q)} = B_{-p,-p'-q}^{(-q)} = \omega_{p,q} \delta_{pp'} + V_{p+p'-q} - V_{p-p'} \]  

(2.3)

where

\[ \omega_{p,q} = \sum_{p'} \left( V_{p-p'} - V_{p+p'-q} \right) \]

and \( V_q \) is a \( q \) Fourier component of the electronic interaction: \( V_q = 4\pi \alpha^2 / \Omega q^2 \).

Refer to the paper I for detailed description of the notation.

\( \varphi, \pi \text{'s satisfy the standard commutation relations; in particular,} \)

\[ [\varphi_{p,q}, \pi_{p',q'}] = i \delta_{pp'} \delta_{q q'} \]  

(2.5)

and, according to Eq. (1), their equations of motion are given by

\[ \frac{d\varphi_{p,q}}{dt} = \sum_{p'} B_{pp'}^{(q)} \varphi_{p',q} \]

\[ \frac{d\pi_{-p,-q}}{dt} = \sum_{p'} A_{pp'}^{(q)} \varphi_{p',q}. \]  

(2.6)

Let us consider the corresponding eigen-value problem:

\[ -i\nu \hat{\varphi}_{p,q} = \sum_{p'} B_{pp'}^{(q)} \hat{\varphi}_{p',q} \]

\[ + i\nu \hat{\pi}_{q,p} = \sum_{p'} A_{pp'}^{(q)} \hat{\varphi}_{p',q}. \]  

(2.7)

This set gives a secular equation of the following form:

\[ \begin{vmatrix} B & -i\nu \\ +i\nu & A \end{vmatrix} = 0, \]  

(2.8)

from the form of which one can see that \( -\nu \) is also an eigen-value if \( \nu \) is one.
However, we shall agree to consider only the positive eigen-values, the number of which is the same as that of \( p \)'s in that the beret shaped region of the Fermi sphere which can be moved out of the sphere by a displacement \( q \).

From Eq. (7) and the symmetric properties of the matrices \( A \) and \( B \), one can derive, with an appropriate normalization, the relations:
\[
\sum_p (\xi_{p,q}^v \gamma_{p,q}^v - \xi_{p,q}^v \gamma_{p,q}^v) = i \delta_{v,v'} \tag{2.9}
\]
and
\[
\sum_p (\xi_{p,q}^v \gamma_{p,q}^v - \xi_{p,q}^v \gamma_{p,q}^v) = 0. \tag{2.10}
\]

Introduce then a new set of creation annihilation operators by
\[
b_{q}^v = \frac{1}{i} \sum_p (\gamma_{p,q}^v \phi_{p,q}^v - \xi_{p,q}^v \pi_{p,q}^v),
\]
\[
b_{q}^v = -\frac{1}{i} \sum_p (\gamma_{p,q}^v \phi_{p,q}^v - \xi_{p,q}^v \pi_{p,q}^v). \tag{2.11}
\]
Here we shall assume that \( \xi_{p,q}^v = \xi_{p,-q}^v, \gamma_{p,q}^v = \gamma_{p,-q}^v \). Then the relations (9) and (10) ensure that
\[
[b_{q}^v, b_{q'}^{v*}] = \delta_{qq'} \delta_{vv'}. \tag{2.12}
\]

The inverse relations to Eq. (11) are
\[
\phi_{p,q} = \sum_v (\xi_{p,q}^v b_{q}^v + \xi_{p,q}^v b_{q}^{v*}),
\]
\[
\pi_{p,q} = \sum_v (\gamma_{p,q}^v b_{q}^v + \gamma_{p,q}^v b_{q}^{v*}), \tag{2.13}
\]
from which one can derive
\[
\sum_v (\xi_{p,q}^v \gamma_{p,q}^{v*} - \xi_{p,q}^v \gamma_{p,q}^{v*}) = i \delta_{p,p'}, \tag{2.14}
\]
\[
\sum_v (\xi_{p,q}^v \xi_{p,q}^{v*} - \xi_{p,q}^v \xi_{p,q}^{v*}) = 0, \tag{2.15}
\]
and
\[
\sum_v (\gamma_{p,q}^v \gamma_{p,q}^{v*} - \gamma_{p,q}^v \gamma_{p,q}^{v*}) = 0. \tag{2.16}
\]

One can easily see that this new set of creation annihilation operators brings the Hamiltonian (1) to a diagonal form:
\[
\mathcal{H}_q + \mathcal{H}_{-q} = \sum_v \nu + \sum_v \nu (b_{q}^{v*} b_{q}^v + b_{q}^{v*} b_{q}^{v*}). \tag{2.17}
\]

\section{3. The Zeeman energy}

Let the vector potential of the magnetic field be
\[
A(x) = A_k e^{-ikx} + A_{-k} e^{ikx}, \quad \text{with} \quad A_{-k} = A_k^*, \tag{3.1}
\]
so that the strength of the field is given by
Then the Zeeman energy, or the magnetic energy, is \( \mathcal{H}' + \mathcal{H}'' \) with

\[
\mathcal{H}' = \frac{e}{2mc} \left\{ \left( \sum_x (2\kappa - k) a^{*}_x a_x \right) A_k + \left( \sum_x (2\kappa + k) a^{*}_{x+k} a_x \right) A_{-k} \right\} \\
\mathcal{H}'' = \frac{e^2 n}{mc^2} A_k \cdot A_{-k}
\]

where we retained only a constant term in \( \mathcal{H}'' \), the magnetic energy of second order in the vector potential. The electronic charge and mass are denoted by \(-e\) and \(m\), respectively, and \(c\) is the light velocity. The number density of the electrons is \(n\).

Transforming \( \mathcal{H}' \) into the "exciton" representation by means of \(U\) and \(\tilde{U}\), Eqs. (4) and (7) of the paper I gives

\[
\mathcal{H}' = -\frac{1}{c} (j_k A_k + j_{-k} A_{-k})
\]

with

\[
j_k = -\frac{e}{m} \sum_p \left( p + \frac{k}{2} \right) (c_{p,k} - c^{*}_{p,-k}),
\]

we retained here only terms linear in the "exciton" creation or annihilation operators. This approximation is consistent with our Hamiltonian (1) truncated beyond the third order in \(c\) or \(c^{*}\), as will be seen from the following argument.

The fluctuating number density of the electrons,

\[
\rho(x) = \sum_x \sum_{x'} a^{*}_x a_{x'} e^{-i(x-x') \cdot x},
\]

is represented, after transformation by means of \(U\) and \(\tilde{U}\), to the first order in \(c\) or \(c^{*}\) by

\[
\rho(x) = \sum_{p,q} (c_{p,q} e^{iq \cdot x} + c^{*}_{p,q} e^{-iq \cdot x}).
\]

The Hamiltonian (2·1) then causes motion of \(\rho(x)\) according to

\[
\dot{\rho}(x) = \frac{1}{i} \sum_q \sum_p \sum_{q'} (c_{p,q} e^{ip \cdot x} + c^{*}_{p,q} e^{-ip \cdot x}) B^{pq}_{q'} (c_{p',q'} - c^{*}_{p',-q'}),
\]

where we have neglected the effect of the ordering operator, which will be justifiable because of the smallness of the overlapping region \(r\), specified in the paper I. The right-hand side of this equation can be further simplified thanks to the equality:

\[
\sum_{p'} B^{pq}_{q'} = \delta_{pq},
\]

which is always valid for any \(V_q\) that is a function of the absolute magnitude of \(q\). Thus one obtains a continuity equation,
Excitations in a High Density Electron Gas. II

\[ (-e) \dot{\rho}(x) = -F \cdot j(x), \]

with the following expression for the current density:

\[ j(x) = \sum_y j_y e^{iq_y x}, \]

with the expression (3.5) for \( j_y \).

Now that the expression for the magnetic energy has been determined, we shall proceed to calculate the magnetic susceptibility. To do this, we have to find out a canonical transformation which will cancel out \( \mathcal{K}' \), which is linear in the vector potential. This is easy enough, however, because the unperturbed Hamiltonian has been brought to a diagonal form (2·17). Define a hermitic operator by

\[ S = -\frac{e}{\sqrt{2mc}} \sum \sum (2p + k) \frac{1}{\nu} \left\{ \eta_{\nu, h}^* (b_{\nu, h} A_h - b_{\nu, h}^* A_{-h}) + \eta_{\nu, h}^* (b_{\nu, h} A_{-h} - b_{\nu, h}^* A_h) \right\}. \]

(3·7)

Then, the canonical transformation \( \exp(iS) \) just fulfills the requirements, as can readily be seen. Thus the transformed Hamiltonian, to the second order in \( A \), is just the unperturbed Hamiltonian plus a constant, which is given by \( \mathcal{K}'' + (1/2)[iS, \mathcal{K}'] \), or explicitly,

\[ \frac{1}{2} [iS, \mathcal{K}'] = -\frac{e^2}{m c^2} \sum \sum (p + \frac{k}{2}) \left( p' + \frac{k}{2} \right) \times \frac{1}{\nu} (\eta_{\nu, h}^* \eta_{\nu, h}^* + \eta_{\nu, h}^* \eta_{\nu, h}^* \gamma_{\nu, h}^* \gamma_{\nu, h}^*) : (A_h A_{-h} + A_{-h} A_h). \]

(3·8)

The expression involving the transformation coefficients in this equation, however, has a neat relation with the matrix \( B \):

\[ \sum B_{pp'} \left( \sum \frac{1}{\nu} (\eta_{\nu, h}^* \eta_{\nu, h}^* + \eta_{\nu, h}^* \eta_{\nu, h}^*) : (A_h A_{-h} + A_{-h} A_h) \right) = \delta_{pp'} . \]

(3·9)

One can prove this by multiplying the first of Eq. (2·7) by \( \eta^* \) and adding the resultant to its complex conjugate. The relation (2·9), then, brings about the result (3·9).

Thus, the final expression for the magnetic energy can be written in the following compact form:

\[ \frac{1}{2} [iS, \mathcal{K}'] = -\frac{e^2}{m c^2} \sum \sum (p + \frac{k}{2}) \left( p' + \frac{k}{2} \right) (B^{-1})_{pp'} : (A_h A_{-h} + A_{-h} A_h). \]

(3·10)

Before evaluating the coefficient, or the susceptibility, we shall prove that the result will be gauge-invariant. The \( k \) Fourier component of the current density can be derived from the expression for the magnetic energy by taking variation of the vector potential. Thus,

\[ \langle j_k \rangle = K(k) \cdot A_{-h} \]

(3·11)
with

\[ K(k) = -e^2 \left\{ \frac{e^2}{mc^2} n - \frac{e^2}{2m^2 c^2} \sum_{p} \sum_{p'} (2p+k) (2p+k') (B^{-1})_{pp} \right\}. \]

Since the tensor \( K \) is clearly symmetric, the requirement of gauge-invariance is equivalent to that of continuity of the current. However, we shall explicitly show here that \( k \cdot K(k) \) vanishes, at least, up to the order of \( k^3 \). First we have the equality:

\[ \sum_{p'} (B^{-1})_{pp'} \alpha_{p'k} = 1. \]

This can be obtained from the defining equation

\[ 1 = \sum_{p} \sum_{p'} (B^{-1})_{pp'} B_{p'p}, \]

by making use of the equality (3.6). The expression \( k \cdot K(k) \), then, reduces to

\[ k \cdot K(k) = -\frac{e^2}{mc} \left\{ nk - \sum_{p} (2p+k) \right\}, \]

which may now be evaluated by an elementary integration. The result is

\[ k \cdot K(k) = -\frac{e^2}{mc} \left\{ nk - \frac{Q}{(2\pi)^3} \left( \frac{2}{3} p_s^3 + \frac{1}{4} k^2 + \cdots \right) k \right\}. \]

The part of contribution linear in \( k \) from the paramagnetic current just cancels that from the London diamagnetic current because

\[ n = \frac{Q}{(2\pi)^3} \frac{4\pi}{3} p_s^3 \]

and thus the remainder is of order \( k^4 \), as required for the gauge-invariance of the susceptibility for homogeneous field limit, \( i.e. \ k \to 0 \).

Now we shall proceed to evaluate the susceptibility. First one has to know the matrix \( B^{-1} \) explicitly. Writing \( B \) as a sum of the diagonal kinetic energy part \( C \) and of the potential energy part \( V \), which is proportional to \( e^2 \), one can get \( B^{-1} \) in a series form:

\[ B^{-1} = (\omega + W)^{-1} = \omega^{-1} (1 + W \omega^{-1})^{-1} = \omega^{-1} - \omega^{-1} W \omega^{-1} + \cdots. \]

* If one writes for \( B^{-1} \)

\[ B^{-1} = \omega^{-1} - \omega^{-1} (\hat{\Omega} \omega^{-1}), \]

in terms of an effective potential part \( \hat{\Omega} \), then one easily sees that \( \hat{\Omega} \) should satisfy the equation

\[ (1 + \omega^{-1}) (\hat{\Omega} \omega^{-1}) = W. \]

We have not succeeded to solve this equation, nor even to work out the quantitative characteristics of \( \hat{\Omega}_{pp'} \) when \( p \to p' \). However, it can be argued that the solution \( \hat{\Omega}_{pp'} \) may have lost its singularity, which the original \( W_{pp'} \) had at \( p = p' \). This is the weakest point in our calculation which gives a trace of ambiguity to the final result. Instead of going into this problem, we shall believe in the common statement that the screening of the Coulomb potential arises from virtual excitation of the plasma oscillation. See the next section.
Excitations in a High Density Electron Gas. II

Hence to the order $e^2$,

$$(B^{-1})_{p\nu} = \frac{1}{\omega_{ph}} \delta_{p\nu} - \frac{1}{\omega_{ph}^2} \sum_{p'} \left( V_{p-p'} - V_{p-k+p'} \right) \delta_{p\nu} + \frac{1}{\omega_{ph} \omega_{ph'} k} (V_{p-p'} - V_{p+k+p'}) \quad (3.14)$$

With this approximation of $B^{-1}$ one can readily go into the formula (3.10) and get

$$K(k) = -\frac{e^2}{mc} n + \frac{2e^2}{m^2 c} \left\{ \sum_{p} \frac{P_s^2}{\omega_{ph}} + \frac{1}{2} \sum_{p'} \sum_{p''} \left( \frac{P_s}{\omega_{ph}} - \frac{P_s'}{\omega_{ph'}} \right)^2 (V_{p+k+p'} - V_{p-p'}) \right\} \quad (3.15)$$

Here we assumed that $k$ is in the $z$-direction, and $A$, the vector potential, is in the $x$-direction.

Evaluation of this last sum is rather tedious, but very elementary. Assuming the $V_{p-p'}$ is of the form

$$V_{p-p'} = \frac{4\pi e^2}{\Omega} \frac{1}{|p-p'|^2 + \xi^2} \quad (3.16)$$

where we have introduced a parameter for the screening effect for the sake of convenience for the following discussions, we shall just write down the result:

$$\frac{1}{2} \sum_{p} \sum_{p'} \left( \frac{P_s}{\omega_{ph}} - \frac{P_s'}{\omega_{ph'}} \right)^2 (V_{p+k+p'} - V_{p-p'}) = q^2 \frac{e^2 m^2 \Omega}{\pi^2} \frac{1}{18} \left\{ -2 + \ln 2 - \ln \frac{\xi}{\rho} \right\} \quad (3.17)$$

This expression for the susceptibility clearly diverges for $\xi = 0$, i.e. for the bare Coulomb interaction, so long as we assume the first approximation (3.15). Note that $V_{p+k+p'}$ in Eq. (3.15) makes no trouble with its bare Coulomb form.

Thus for the main correction to the Landau diamagnetism, one waits for evaluation of the screening constant $\xi$. We shall do this in the next section.

§ 4. The screening effect

The screening of the exchange interaction (3.16), which was found essential for the evaluation of the main correction to the Landau diamagnetism, will be tried in this section. As mentioned in connection with Eq. (3.13), there is probably another screening effect which has never been studied before, but we shall not go into this investigation. Instead of this, we shall show how the well-known polarization screening is incorporated in our formalism. The perturbation which gives rise to this effect is nothing but that three-exciton Hamiltonian, given as Eq. (3.8) in the paper I:

$$\mathcal{H}_3 = \sum_p \sum_{p'} \left( V_{p-p} - V_{p'-p} \right) \left\{ c_{p+P'}^* c_{p-p} c_{p'} + c_{p+P'}^* c_{p-p} c_{p'}^\dagger \right\}$$

$$- \sum_p \sum_{p'} \left( V_{p-p} - V_{p'-p} \right) \left\{ c_{p+P'}^* c_{p-p} c_{p'} + c_{p+P'}^* c_{p-p} c_{p'}^\dagger \right\} \quad (4.1)$$

Downloaded from https://academic.oup.com/ptp/article-abstract/23/5/799/1865167 by guest on 01 January 2019
This is an especially satisfactory feature of our formalism, because this part is the next term to the main harmonic one (2·1) in our series expansion of our exciton Hamiltonian.

It was first shown by Bohm and Pines that electrons are organized by the long range part of the Coulomb interactions to perform collective oscillations with long wavelength but high frequency so that long wavelength density fluctuations are strongly inhibited. This was found to mean, in particular, that the individual electrons, with correspondingly reduced number of degrees of freedom, are interacting via screened Coulomb forces with a range of the order of the inter-electronic distance.

Hubbard regarded the electron gas as a dielectric medium and thought of the electrons as interacting with one another like particles in this medium, their interaction being therefore screened. From this viewpoint the plasma oscillations are thought of as being the polarization waves in the medium.

In order to get this polarization correction, we only have to eliminate the three-exciton Hamiltonian (4·1) by a canonical transformation \( \exp(iS) \)

\[
\frac{iS = -i \lim_{\epsilon \to 0} \int_0^\infty \exp(-\epsilon t) \exp(i\mathcal{K}_t) \mathcal{K}_s \exp(-i\mathcal{K}_t) dt.}{(4\cdot2)}
\]

Then the transformed Hamiltonian to the second order in \( \mathcal{K}_s \) is given by \( \mathcal{K}_{s\epsilon} = \mathcal{K} + (1/2)[iS, \mathcal{K}_s] \). This is to involve a modified coefficient \( V_{p-p'} \) of the type (3·16). Its exact calculation, however, is very troublesome because it involves various minor processes other than the relevant polarization. Before clarifying what we have to do for retaining only this process, we shall analyze the structure of \( \mathcal{K}_s \).

If one drops the exchange coefficients, one easily sees that the three-exciton Hamiltonian (4·1) can be written in the form:

\[
\mathcal{K}_s = \sum_q V_q \Omega_q \sum_p (c^2_{p,q} + c^2_{p,-q})
\]

with

\[
\Omega_q = \sum_{p'} \sum_{q'} (c^*_{p',q'+q} - c^*_{p'-q'+q}) c_{p',q'} = \Omega^*_q.
\]

As is apparent from the attached indices, \( \Omega_q \), here is an operator which causes scattering of electrons or holes with a momentum change \( q \), while the other factor is the \( q \)-Fourier component of the density fluctuation.

Consider the motion of the operators due to the unperturbed Hamiltonian (2·1). While, as is well known, the density fluctuation of long wavelength, \( i.e. \) of small \( q \), involves a high frequency (namely, the plasma frequency), the operator \( \Omega_q \) changes with time comparatively slowly. This can be seen, for example, by calculating the time variation of \( \Omega_q \):

\[
i \frac{d}{dt} \Omega_q = [\Omega_q, \mathcal{K}] = - \sum_{p'} \sum_{q'} \left( \frac{q' \cdot q}{m} + \frac{q' \cdot q + q^2}{2m} \right) c^*_{p',q'+q} c_{p',q'}
\]
Excitations in a High Density Electron Gas. II

If one assumes that \( q \) and \( q' \) be small in comparison with \( p' \), which is of the order of the Fermi momentum, one can see that the frequencies involved are much smaller than the plasma frequency. On this basis we are allowed to eliminate the fluctuating density, treating the operators \( \Omega_q \) constant with time.

Then, by writing the density fluctuation in terms of \( b \)-operators according to (2·13), one can readily perform the integration in Eq. (4·2) and thus gets

\[
\frac{\sqrt{2 \pi}}{2} \int d q \int d q' \frac{1}{\nu} \left( \xi_{p,q}^{*} b_{q} - \xi_{p,q} b_{q}^{*} \right).
\]

(4·5)

This canonical transformation gives, with only the commutators pertaining to the density fluctuations retained in compliance with the above statement,

\[
\frac{1}{2} \left\langle i \mathcal{S}, \mathcal{H}_{q} \right\rangle = - \sum_{p} \sum_{q} V_{q}^{2} \Omega_{q} \Omega_{-q} \sum_{p'} \sum_{q'} \frac{1}{\nu} \left( \xi_{p,q}^{*} \xi_{p',q'}^{*} + \xi_{p,q} \xi_{p',q'} \right).
\]

(4·6)

Rearranging the operators \( \Omega_{q} \Omega_{-q} \) into normal products and retaining binary terms in \( c \)-operators give

\[
\Omega_{q} \Omega_{-q} = \sum_{p'} \sum_{q'} \left( \Gamma_{p',q'-q}^{*} + \Gamma_{p'+q,q'-q}^{*} \right) c_{p'}^{*} c_{p'} - \sum_{p'} \sum_{q'}^{*} c_{p'-q,q'}^{*} c_{p',q'}^{*} - \sum_{p'}^{*} \sum_{q'}^{*} c_{p'+q,q'}^{*} c_{p',q'},
\]

(4·7)

where \( \Gamma_{p,q} \) is defined as follows:

\[
\Gamma_{p,q} = \begin{cases} 1, & \text{if } |p| < p_{F} \text{ and } |p+q| > p_{F} \\ 0, & \text{otherwise}. \end{cases}
\]

Now, if one neglects the exchange coefficients in the harmonic Hamiltonian (2·1), one can derive the following relation from the second equation of (2·7):

\[
\sum_{q} \xi_{p,q}^{*} = - \sum_{p} \frac{1}{\omega_{p,q}} \xi_{p,q}^{*}.
\]

(4·8)

Then the relation (2·14) gives

\[
\sum_{q} \sum_{p'} \sum_{q'} \frac{1}{\nu} \left( \xi_{p,q}^{*} \xi_{p',q'}^{*} + \xi_{p,q} \xi_{p',q'} \right) = \sum_{p} \frac{1}{\omega_{p,q}} \xi_{p,q}^{*}.
\]

(4·9)

Taking this result, Eq. (4·7) with (4·9), into account, we finally obtain an effective Hamiltonian with the same form as (2·1), but with renormalized coefficients. These are all free from the singularity met with in the bare Coulomb interaction. Thus the \( V_{p-p'} \) which is appearing explicitly in (2·3) is replaced by \( \mathcal{U}_{p-p'} \), where the renormalized coefficient \( \mathcal{U}_{q} \) is given by
\[ \gamma_q = \frac{V_q}{1 + 2V_q \sum_p \frac{1}{a_{pq}} - \frac{4}{q^2 + \frac{2}{\pi} \epsilon^2 m_p} \frac{1}{\epsilon^2 m_p} } \]  

(4.10)

which is of the same form as (3·16). The \( \gamma_{p,q} \) is also modified to the extent

\[ \gamma_{p,q} \rightarrow \frac{1}{2} \left\{ \sum_{p'} \gamma_{p'-p-q} + \sum_{p''} \gamma_{p''-p-q} - \sum_{p'} V_{p'-p-q} - \sum_{p''} V_{p''-p-q} \right\} . \]

As one may recognize, the resulting Hamiltonian does not strictly meet the gauge-invariance argument of Section 3. To dispose of this lack of invariance formally, one has also to replace the \( V_{p+q+p'} \)'s, those appearing explicitly and those involved in \( \gamma_{p,q} \) as well, by renormalized ones of the type (4·10). This somewhat arbitrary replacement will be allowed because, to the present approximation, the matter with which we are concerned is the extent of cutoff of the singularity of \( V_{p+q+p} \) involved in Eq. (3·15). That is, the cutoff value of \( V_{p+q+p'} \) does not affect the result within our approximation. Hence, we shall refrain from giving any detailed analysis of this point.

Now that we have got the appropriate value of the parameter \( \epsilon \), we can write down the expression for the susceptibility according to (3·11) and (3·17). The result may conveniently be written down as its ratio to the Landau susceptibility:

\[ \frac{\chi}{\chi_0} = 1 - \frac{\alpha r_s}{6\pi} \left( -4 + \ln \frac{2\pi}{\alpha r_s} \right) \]  

(4·11)

with

\[ \alpha = (2/9\pi)^{1/3} \]

where \( \chi_0 \), the Landau susceptibility, is given by

\[ \chi_0 = -\frac{1}{12\pi^2} \frac{1}{\alpha r_s} \frac{1}{m^2 c^2 a_r^2} \]

This result may be compared with that of the collective description of electron interactions, mentioned in Pines' review article:

\[ \frac{\chi}{\chi_0} = 1 - \frac{\alpha r_s}{6\pi} \left\{ -4 + \ln \frac{2\pi}{\alpha r_s} + \ln \frac{2\alpha r_s}{\pi (p_c/p_r)^2} \right\} , \]

where \( p_c \) is the cutoff momentum for plasma oscillation. Formally, therefore, our result corresponds to the choice

\[ p_c/p_r = \sqrt{2\alpha/\pi r_s^{1/2}} \approx 0.353 r_s^{1/2} . \]

This should be compared with the value 0.353 \( r_s^{1/2} \), which was

* In this self-energy \( \gamma_{p,q} \) we have thus taken into account only the plasmon exciting processes which is connected with the scattering of electrons or holes (Feynman diagram Fig. 1). This is evoked by the three-"exciton" interaction.
obtained by approximately minimizing the long-range correlation energy with respect to \( p_n \).

\section*{§ 5. Concluding remarks}

Our primary result is Eq. (3.11) for \( \langle j_k \rangle \), which is general enough so long as we limit ourselves to the harmonic approximation for the Hamiltonian (2.1). The point in this connection is that only the exchange coefficients come into the expression. However, we have not succeeded yet in evaluating the inverse matrix \( B^{-1} \), although it can be well expected that this may give a finite expression for the susceptibility.

We left this unsolved and turned to base our discussion on the approximation (3.14) for \( B^{-1} \). The ensuing integral expression (3.15) is, in point of fact, an average over the Fermi ground state of the operator expression for \( j_k \) which is obtained after the two canonical transformations to eliminate the magnetic energy term linear in \( A \) and then to eliminate also the resulting cross energy involving the Coulomb interaction and \( A \). This, however, gives a diverging result for the static susceptibility.

It is readily suggested by the expression (3.15) that we must take the screening effect into account to get the susceptibility finite. It was shown in Section 4 that this type of screening arises partly from the three-"exciton" interaction. The final result is (4.11) for the susceptibility, which corresponds to that of the collective description of electron interaction with the cutoff momentum somewhat larger than that of Pines. Formally, therefore, the result is just that approximation stated in the preceding paragraph but with the screening Coulombic interaction.

It should, however, be noted again that this result is based on the approximation (3.14) for \( B^{-1} \). Hence it cannot be said anyway that (4.11) were final. We are intending to analyze the problem along the line mentioned in the footnote for (3.13).

Finally, we have not justified formally that replacement of \( V_{p+q+r} \) by the screened one on the present elementary approach. Before doing this, we shall have to solve that problem of \( B^{-1} \), which may be possible to give a finite result within the simple gauge-invariant scheme of Section 3. This will settle the effectiveness of the simple "exciton" picture, because any need of more and more "exciton" interaction will invalidate the effectiveness.

The authors wish to thank Prof. M. Nogami, Mr. N. Matsudaira and Mr. H. Ezawa, with whom they enjoyed various discussions.

\section*{References}

1) T. Usui, Prog. Theor. Phys. 23 (1960). This will be referred to as I.
4) D. Bohm and D. Pines, Phys. Rev. 92 (1953), 609.