

## References

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## DISCUSSION

**K. C. Gupta\***. The authors have presented a significant paper on the velocity Jacobian of manipulators. The analytical determination of the elements of this Jacobian and its inverse are very important concepts. These derivations have useful applications in manipulator control and in the determination (or avoidance) of all manipulator singularities. Although the authors present only two examples, this discussor will venture to state that, following the authors' methodology, one should be able to obtain similar expressions for all simple configurations of industrial robots for which closed-form positional solutions have been worked out.

Unfortunately, as evidenced by the literature cited, a clear demonstration of how this form of Jacobian is related to the  $4 \times 4$  matrix velocity equation of the manipulator is lacking in the literature. This derivation constitutes the rest of this discussion. The value of this form of Jacobian in computational and analytical work can then be fully realized.

Let us consider the governing positional equations for an  $n$ -jointed serial manipulator in terms of the  $4 \times 4$  Denavit and Hartenberg matrices  $\mathbf{A}_m$  [4]; these should not be confused with the notation  $\mathbf{A}$  used in the equation (12) of the paper.

$$\mathbf{A}_H = \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_n, n \geq 6 \quad (17)$$

Differentiating, we obtain the following matrix velocity relation.

$$\dot{\mathbf{A}}_H = (\dot{\mathbf{A}}_1 \mathbf{A}_2 \dots \mathbf{A}_n) + \dots + (\mathbf{A}_1 \mathbf{A}_2 \dots \dot{\mathbf{A}}_n) \quad (18)$$

This equation yields a system of 12 dependent velocity equations. Three of these are the generally independent translational equations obtained from the elements (1, 4), (2, 4), and (3, 4). The remaining nine equations are the dependent rotational equations, of which only three are generally independent.

The system of 12 dependent velocity equations (18) can be

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reduced to a complete system of six generally independent velocity equations by post-multiplying the equation (18) by the matrix  $\mathbf{A}_H^{-1}$ .

$$\begin{aligned} \dot{\mathbf{A}}_H \mathbf{A}_H^{-1} &= \dot{\mathbf{A}}_1 \mathbf{A}_1^{-1} + \mathbf{A}_1 (\dot{\mathbf{A}}_2 \mathbf{A}_2^{-1}) \mathbf{A}_1^{-1} \\ &+ (\mathbf{A}_1 \mathbf{A}_2) (\dot{\mathbf{A}}_3 \mathbf{A}_3^{-1}) (\mathbf{A}_1 \mathbf{A}_2)^{-1} + \dots \end{aligned} \quad (19)$$

Let us use the notation  $\Omega_{3 \times 3}$  for the skew-symmetric angular velocity matrix and  $\omega_{3 \times 1}$  for the angular velocity vector [19]. Likewise, the skew symmetric matrix  $\mathbf{W}_{3 \times 3}$  and the vector  $\mathbf{w}_{3 \times 1}$  represent a joint axis in the base system, and  $\mathbf{K}_{3 \times 3}$  and  $\mathbf{k}_{3 \times 1} = (0, 0, 1)^T$  represent a joint axis in the link system. Then

$$\begin{aligned} \dot{\mathbf{A}}_H &= \begin{bmatrix} \dot{\mathbf{R}} & \mathbf{v} \\ \mathbf{0} & \dot{H} \end{bmatrix}, \dot{\mathbf{A}}_H \mathbf{A}_H^{-1} = \begin{bmatrix} \Omega & \boldsymbol{\mu} \\ \mathbf{0} & \dot{0} \end{bmatrix}, \Omega = \dot{\mathbf{R}} \mathbf{R}^T \\ \mathbf{A}_m \mathbf{A}_m^{-1} &= \begin{bmatrix} \dot{\theta}_m \mathbf{K} & \dot{s}_m \mathbf{k} \\ \mathbf{0} & \dot{0} \end{bmatrix}, \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_{m-1} = \begin{bmatrix} \mathbf{R}_m & \mathbf{p}_m \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned} \quad (20)$$

Here,  $\mathbf{v}$ ,  $\boldsymbol{\mu}$ ,  $\mathbf{w}$  are as defined in the paper, except that  $\mathbf{v}$  is specialized to the origin of the hand coordinate system. Matrix  $\mathbf{R}_m$  and vector  $\mathbf{p}_m$  are the  $3 \times 3$  and  $3 \times 1$  parts, respectively, of the indicated product. The equation (19) then becomes

$$\begin{bmatrix} \Omega & \boldsymbol{\mu} \\ \mathbf{0} & \dot{0} \end{bmatrix} = \sum_m \begin{bmatrix} \dot{\theta}_m \mathbf{R}_m \mathbf{K} \mathbf{R}_m^T & -\dot{\theta}_m \mathbf{R}_m \mathbf{K} \mathbf{R}_m^T \mathbf{p}_m & +\dot{s}_m \mathbf{R}_m \mathbf{k} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (21)$$

$$= \sum_m \begin{bmatrix} \dot{\theta}_m \mathbf{W}_m & -\dot{\theta}_m \mathbf{W}_m \mathbf{p}_m + \dot{s}_m \mathbf{w}_m \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (22)$$

In simplifying equations (21) to (22), the following transformations have been used [19]. Also note that  $\mathbf{R}_1 = \mathbf{I}$ ,  $\mathbf{p}_1 = \mathbf{0}$ .

$$\mathbf{w}_m = \mathbf{R}_m \mathbf{k}, \quad \mathbf{W}_m = \mathbf{R}_m \mathbf{K} \mathbf{R}_m^t \quad (23)$$

The equation (22) contains the three generally independent rotational equations in elements (3, 2), (1, 3), (2, 1), and the three generally independent translational equations in elements (1, 4), (2, 4), (3, 4). Interpreting equation (22) in vectorial form, we obtain

$$\boldsymbol{\omega} = \sum_m \dot{\theta}_m \mathbf{w}_m \quad (24a)$$

$$\boldsymbol{\mu} = \sum_m (\dot{\theta}_m \mathbf{p}_m \times \mathbf{w}_m + \dot{s}_m \mathbf{w}_m) \quad (24b)$$

Although both  $\dot{\theta}_m$  and  $\dot{s}_m$  have been carried along so far, only one of these is nonzero for revolute and prismatic joints.

Equations (24) can also be expressed in the dual vector form as follows ( $\epsilon^2 = 0$ ).

$$\hat{\boldsymbol{\omega}} = \sum_m \dot{\theta}_m \hat{\mathbf{w}}_m + \sum_m \dot{s}_m \epsilon \hat{\mathbf{w}}_m \quad (25)$$

Adding  $\hat{\boldsymbol{\omega}}^* = \hat{\boldsymbol{\omega}} (-\epsilon)$ , we obtain the dual Jacobian shown in equation (26) where only typical revolute and prismatic columns are shown.

$$\begin{bmatrix} \hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{w}} & \epsilon \hat{\mathbf{w}} \\ \dots & \dots \\ \hat{\mathbf{w}}^* & -\epsilon \hat{\mathbf{w}}^* \\ \dots & \dots \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \vdots \\ \dot{s} \\ \vdots \end{bmatrix} \quad (26)$$

The six independent equations (24) which have been derived from a systematic reduction of the twelve dependent equations (18) are precisely the equations (7) in the paper. These equations, or their minor variations such as equations (3), completely determine the first order differential kinematics of the manipulator. Incidentally, equations (3) are obtained from equations (24a) and by comparing the elements (1, 4), (2, 4), and (3, 4) of equations (18). It has therefore been shown how both forms of the Jacobian (equations (4) and (8)) can be obtained by starting from the common  $4 \times 4$  matrix approach. The main advantage of this differential formulation (equations (3) or (7)) lies in its great efficiency over the direct differential formulations based upon either of the matrix equations (18) or (19). The computational advantages of using the abovementioned forms of the Jacobian will be reported in the near future [20].

#### Additional References

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