References

1 Paul, R. P., *Robot Manipulators*, MIT Press, Cambridge, MA, 1981.
2 Liegois, A., "Automatic Supervisory Control of the Configuration and Behavior of Multibody Mechanisms," *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. SME-7, No. 12, 1977.
3 Renaud, M., "Geometric and Kinematic Models of a Robot Manipulator: Calculation of the Jacobian Matrix and its Inverse," *Proceedings of 11th ISIR*, Tokyo, Oct. 1981, pp. 153–165.
4 Uicker, J. J., Denavit, J., and Hartenberg, R. S., "An Iterative Method for The Displacements Analysis of Spatial Mechanisms," ASME *Journal of Applied Mechanics*, June 1964, pp. 309–314.
5 Waldron, K. J., "Geometrically Based Manipulator Rate Control Algorithms," *Mechanism and Machine Theory*, Vol. 17, No. 6, 1982, pp. 379–385.
6 Konstantinov, M. S., Patarinski, S. P., Zamonov, V. B., and Nenchev, D. N., "A New Approach to the Solution of The Inverse Kinematic Problem for Industrial Robots," *Proceedings of Sixth World Congress on Theory of Machines and Mechanisms*, Vol. 2, New Delhi, Dec. 1983, pp. 970–973.
7 Orin, D. E., and Schrader, W. W., "Efficient Jacobian Determination for Robot Manipulators," *Proceedings of Sixth World Congress on Theory of Machines and Mechanisms*, Vol. 2, New Delhi, Dec. 1983, pp. 994–997.
8 Featherstone, R., "Position and Velocity Transformations Between Robot End-Effector Coordinates and Joint Angles," *Int. J. Robotics Res.*, Vol. 2, No. 2, 1983, pp. 35–45.
9 Hollerbach, J., and Saher, G., "Wrist-Partitioned, Inverse Kinematic Accelerations and Manipulators Dynamics," *Int. J. Robotics Res.*, Vol. 2, No. 4, 1983, pp. 61–76.
10 Gorla, B., "Influence of the Control on the Structure of a Manipulator From a Kinematic Point of View," *Proceedings of 4th Symposium on Theory and Practice of Robots and Manipulators*, Zaborow, Poland, Sept. 1981, pp. 30–46.
11 Whitney, D. E., "The Mathematics of Coordinated Control of Prostheses and Manipulators," ASME *Journal of Dynamic Systems, Measurement, and Control*, Vol. 94, 1972, pp. 303–309.
12 Klein, C. A., and Huang, C. H., "Use of The Pseudoinverse for Control of Kinematically Redundant Manipulators," *IEEE Trans. on Systems, Man, and Cybernetics*, Vol. SME-7, No. 12, 1977.
13 Hunt, K. H., *Kinematic Geometry of Mechanisms*, Oxford, 1978.
14 Tsai, Y. C., and Soni, A. H., "Accessible Region and Synthesis of Robot Arm," ASME *J. of Mechanical Design*, Vol. 103, No. 4, Oct. 1981, pp. 803–811.
15 Waldron, K. J., "Robots and Intelligent Mechanisms: Geometric and Software Design," to appear in *Standard Handbook of Machine Design*, Ed. Shigley, J., and Mischke, C., McGraw-Hill, New York, 1984.
16 Kumar, A., and Waldron, K. J., "The Workspaces of a Mechanical Manipulator," ASME *J. of Mechanical Design*, Vol. 103, No. 3, July 1981, pp. 665–672.
17 Sugimoto, K., and Duffy, J., "Determination of Extreme Distances of a Robot Hand Part 1: A General Theory," ASME *J. of Mechanical Design*, Vol. 103, No. 3, July 1981, pp. 631–636.
18 Waldron, K. J., "The Use of Motors in Spatial Kinematics," *Proceedings of IFToMM Conference on Linkages and Computer Design Methods*, Bucharest, June 1973, Vol. B, pp. 535–545.

DISCUSSION

K. C. Gupta*. The authors have presented a significant paper on the velocity Jacobian of manipulators. The analytical determination of the elements of this Jacobian and its inverse are very important concepts. These derivations have useful applications in manipulator control and in the determination (or avoidance) of all manipulator singularities. Although the authors present only two examples, this discusser will venture to state that, following the authors' methodology, one should be able to obtain similar expressions for all simple configurations of industrial robots for which closed-form positional solutions have been worked out.

Unfortunately, as evidenced by the literature cited, a clear demonstration of how this form of Jacobian is related to the 4×4 matrix velocity equation of the manipulator is lacking in the literature. This derivation constitutes the rest of this discussion. The value of this form of Jacobian in computational and analytical work can then be fully realized.

Let us consider the governing positional equations for an n-jointed serial manipulator in terms of the 4×4 Denavit and Hartenberg matrices $\mathbf{A}_m$ [4]; these should not be confused with the notation $\mathbf{A}$ used in the equation (12) of the paper.

$$\mathbf{A}_H=\mathbf{A}_1\mathbf{A}_2\,\ldots\,\mathbf{A}_n, n\geq 6 \tag{17}$$

Differentiating, we obtain the following matrix velocity relation.

$$\dot{\mathbf{A}}_H=(\dot{\mathbf{A}}_1\mathbf{A}_2\,\ldots\,\mathbf{A}_n)+\,\ldots\,+(\mathbf{A}_1\mathbf{A}_2\,\_\_\_\,\dot{\mathbf{A}}_n) \tag{18}$$

This equation yields a system of 12 dependent velocity equations. Three of these are the generally independent translational equations obtained from the elements (1, 4), (2, 4), and (3, 4). The remaining nine equations are the dependent rotational equations, of which only three are generally independent.

The system of 12 dependent velocity equations (18) can be reduced to a complete system of six generally independent velocity equations by post-multiplying the equation (18) by the matrix A_H^{-1}.

$$\dot{\mathbf{A}}_H\mathbf{A}_H^{-1}=\dot{\mathbf{A}}_1\mathbf{A}_1^{-1}+\mathbf{A}_1(\dot{\mathbf{A}}_2\mathbf{A}_2^{-1})\mathbf{A}_1^{-1}+(\mathbf{A}_1\mathbf{A}_2)(\dot{\mathbf{A}}_3\mathbf{A}_3^{-1})(\mathbf{A}_1\mathbf{A}_2)^{-1}+\,\ldots \tag{19}$$

Let us use the notation $\mathbf{\Omega}_{3\times3}$ for the skew-symmetric angular velocity matrix and $\omega_{3\times1}$ for the angular velocity vector [19]. Likewise, the skew symmetric matrix $\mathbf{W}_{3\times3}$ and the vector $\mathbf{w}_{3\times1}$ represent a joint axis in the base system, and $\mathbf{K}_{3\times3}$ and $\mathbf{k}_{3\times1}=(0, 0, 1)^t$ represent a joint axis in the link system. Then

$$\dot{\mathbf{A}}_H=\begin{bmatrix}\dot{\mathbf{R}} & \mathbf{v}\\ 0 & 0\end{bmatrix},\ \dot{\mathbf{A}}_H\mathbf{A}_H^{-1}=\begin{bmatrix}\mathbf{\Omega} & \boldsymbol{\mu}\\ 0 & 0\end{bmatrix},\ \mathbf{\Omega}=\dot{\mathbf{R}}\,\mathbf{R}^t$$

$$\mathbf{A}_m\mathbf{A}_m^{-1}=\begin{bmatrix}\dot{\theta}_m\mathbf{K} & \dot{s}_m\,\mathbf{k}\\ 0 & 0\end{bmatrix},\ \mathbf{A}_1\mathbf{A}_2\,\ldots\,\mathbf{A}_{m-1}=\begin{bmatrix}\mathbf{R}_m & \mathbf{p}_m\\ 0 & 1\end{bmatrix} \tag{20}$$

Here, $\mathbf{v}$, $\boldsymbol{\mu}$, $\mathbf{w}$ are as defined in the paper, except that $\mathbf{v}$ is specialized to the origin of the hand coordinate system. Matrix $\mathbf{R}_m$ and vector $\mathbf{p}_m$ are the 3×3 and 3×1 parts, respectively, of the indicated product. The equation (19) then becomes

$$\begin{bmatrix}\mathbf{\Omega} & \boldsymbol{\mu}\\ 0 & 0\end{bmatrix}=\sum_m\begin{bmatrix}\dot{\theta}_m\mathbf{R}_m\mathbf{K}\,\mathbf{R}_m{}^t & -\dot{\theta}_m\mathbf{R}_m\mathbf{K}\mathbf{R}_m{}^t\mathbf{p}_m\quad+\dot{s}_m\mathbf{R}_m\mathbf{k}\\ 0 & 0\end{bmatrix} \tag{21}$$

$$=\sum_m\begin{bmatrix}\dot{\theta}_m\mathbf{W}_m & -\dot{\theta}_m\mathbf{W}_m\mathbf{p}_m+\dot{s}_m\mathbf{w}_m\\ 0 & \end{bmatrix} \tag{22}$$

*Professor, Department of Mechanical Engineering, University of Illinois, Chicago, Ill.

In simplifying equations (21) to (22), the following transformations have been used [19]. Also note that $\mathbf{R}_1 = \mathbf{I}$, $\mathbf{p}_1 = \mathbf{0}$.

$$\mathbf{w}_m = \mathbf{R}_m \mathbf{k}, \quad \mathbf{W}_m = \mathbf{R}_m \mathbf{K}\, \mathbf{R}_m{}^t \qquad (23)$$

The equation (22) contains the three generally independent rotational equations in elements (3, 2), (1, 3), (2, 1), and the three generally independent translational equations in elements (1, 4), (2, 4), (3, 4). Interpreting equation (22) in vectorial form, we obtain

$$\omega = \sum_m \dot{\theta}_m \mathbf{w}_m \qquad (24a)$$

$$\mu = \sum_m (\dot{\theta}_m \mathbf{p}_m \times \mathbf{w}_m + \dot{s}_m \mathbf{w}_m) \qquad (24b)$$

Although both $\dot{\theta}_m$ and $\dot{s}_m$ have been carried along so far, only one of these is nonzero for revolute and prismatic joints.

Equations (24) can also be expressed in the dual vector form as follows ($\epsilon^2 = 0$).

$$\hat{\omega} = \sum_m \dot{\theta}_m \hat{\mathbf{w}}_m + \sum_m \dot{s}_m \epsilon\, \hat{\mathbf{w}}_m \qquad (25)$$

Adding $\hat{\omega}^* = \hat{\omega}(-\epsilon)$, we obtain the dual Jacobian shown in equation (26) where only typical revolute and prismatic columns are shown.

$$\begin{bmatrix} \hat{\omega} \\ \hat{\omega} \end{bmatrix} = \begin{bmatrix} & \hat{\mathbf{w}} & & \epsilon\hat{\mathbf{w}} & \\ \text{---} & & \text{---} & & \text{---} \\ & \hat{\mathbf{w}}^* & & -\epsilon\hat{w}^* & \end{bmatrix} \begin{bmatrix} | \\ \dot{\theta} \\ \vdots \\ \dot{s} \\ | \end{bmatrix} \qquad (26)$$

The six independent equations (24) which have been derived from a systematic reduction of the twelve dependent equations (18) are precisely the equations (7) in the paper. These equations, or their minor variations such as equations (3), completely determine the first order differential kinematics of the manipulator. Incidently, equations (3) are obtained from equations (24*a*) and by comparing the elements (1, 4), (2, 4), and (3, 4) of equations (18). It has therefore been shown how both forms of the Jacobian (equations (4) and (8)) can be obtained by starting from the common 4 × 4 matrix approach. The main advantage of this differential formulation (equations (3) or (7)) lies in its great efficiency over the direct differential formulations based upon either of the matrix equations (18) or (19). The computational advantages of using the abovementioned forms of the Jacobian will be reported in the near future [20].

Additional References

19 Bottema, O., and Roth, B., *Theoretical Kinematics*, North-Holland, 1979.

20 Kazerounian, K., "Manipulation and Simulation of General Robots Using the Zero Position Description," Ph.D dissertation, University of Illinois at Chicago, Aug. 1984.