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A Hydrodynamical Model for Calculating the Vertical Temperature Profile in Lakes During Cooling

Jörgen Sahlberg

Swedish Meteorological and Hydrological Institute, Norrköping, Sweden

A one-dimensional hydrodynamical model is used for simulating the vertical temperature profile in a lake during cooling conditions. The vertical mixing rate is calculated by solving the equations for turbulent kinetic energy, k, and dissipation of energy, ε. The heat exchange between the water and atmosphere consists of the radiation fluxes, sensible and latent heat flux. Temperature measurements from Lake Väsman during November-December, 1981, were used in the verification study. The agreement between calculated and measured temperature profiles is very good. This indicates that both the mixing processes and the net heat flux are well described in the model.

Introduction

The purpose of this study is to develop a hydrodynamical model for calculating the temperature profiles in a lake during the cooling period.

The most important processes that control the heat contents and their distribution in a lake during cooling, are the net heat flux through the water surface and the rate of mixing in the lake. The net heat flux mainly consists of three heat fluxes, net long wave radiation, sensible heat flux, and latent heat flux. The short wave radiation (insolation) is of minor importance during the time considered.

The model used is a one-dimensional hydrodynamical model. The mixing rate is calculated by solving the equations for turbulent kinetic energy, k, and dissipation of energy, ε. The main forces effecting these two equations are the wind stress and the buoyancy due to stratification. In the temperature equation the net heat flux enters as a so called boundary condition. Both the surface water temperature and meteorological parameters control the net heat flux. In this study a formulation of the net heat flux is used that needs information on the following meteorological parameters: air temperature, wind velocity, relative humidity, and cloud coverage.

The Model

Basic Assumptions

The study will restrict attention to horizontally homogeneous flows, which means that terms containing gradients in the horizontal plane are neglected. It will further be assumed that no mean vertical velocity is present. Shortwave radiation is known to penetrate the water body, following an exponential decay. During the period considered (late autumn – beginning of the winter) the solar radiation is, however, of minor importance and will therefore be treated as a surface flux of heat. Gravitational effects are assumed to obey the Boussinesq approximation.

Mean Flow Equations

Primarily it is the vertical temperature distribution that is of interest, but a few more variables are needed to describe the problem such as the velocity distribution, since turbulence to a large extent is produced by shear. Within the assumptions made, the equations for these variables read

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_T}\frac{\partial T}{\partial z}\right) + S_T \quad (1)$$

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial z}\left(\nu_T\frac{\partial U}{\partial z}\right) + fV \quad (2)$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial z}\left(\nu_T\frac{\partial V}{\partial z}\right) - fU \quad (3)$$

where z is the vertical space coordinate positive upwards, t time coordinate, f Coriolis' parameter, U and V mean velocities in horizontal directions, and T is the water temperature. Kinematic eddy-viscosity is denoted by ν_T and σ_T is a Prandtl/Schmidt number for temperature. This number depends on the stratification. S_T is a source/sink term in the equation for temperature. For example, if the shortwave radiation (insolation) is important, it is included as a source term in the temperature equation.

Turbulence Model

A fundamental process in mixed layer dynamics is the conversions of energy. Turbulent kinetic energy can be converted into potential energy. A budget equation for k, the turbulent kinetic energy, is needed. With the introduction of a kinematic eddy viscosity, ν_T, a modelled form of the k-equation, see Launder and Spalding (1972) for details, reads

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial z}\right) + \nu_T\left(\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right) + \frac{g}{\rho}\nu_T\frac{\partial \rho}{\partial z} - \varepsilon \tag{4}$$

where σ_k is a Prandtl/Schmidt number, g gravitational acceleration, ϱ the water density and ε dissipation rate of k. Another advantage of solving an equation for k is that a velocity scale, $k^{1/2}$, is available for the determination of ν_T. From physical reasoning and dimensional analysis it is expected that ν_T is the product of a velocity scale and a length scale. Given the velocity scale, a length scale is thus needed. If once again dimensional analysis is employed, it can be shown that a length scale, *1,* can be obtained from k and ε as

$$1 \sim \frac{k^{\frac{3}{2}}}{\varepsilon}$$

Recalling that the velocity scale is $k^{1/2}$, the Prandtl/Kolmogorov relation for ν_T is obtained

$$\nu_T = C_\mu \frac{k^2}{\varepsilon} \tag{5}$$

where C_μ is an empirical constant. It remains to formulate an equation for ε, if ν_T is to be calculated according to Eq. (5).

This equation is the weak point of most turbulence models presented up to date, and some workers argue that a prescribed length scale is as good as any dynamical equation presented. The success met by the ε-equation in predicting a wide range of shear flows, see Rodi (1980) and Singhal and Spalding (1981), does indicate a certain degree of generality, and the form at the ε-equation given by these workers is therefore employed

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}\right) + C_{1\varepsilon}\,\nu_T\frac{\varepsilon}{k}\left(\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right) + C_{3\varepsilon}\frac{g}{\rho}\frac{\varepsilon}{k}\nu_T\frac{\partial \rho}{\partial z} - C_{2\varepsilon}\frac{\varepsilon^2}{k} \tag{6}$$

where σ_ε is a Prandtl/Schmidt number and $C_{1\varepsilon}$, $C_{2\varepsilon}$, and $C_{3\varepsilon}$ are constants.

Boundary Conditions

Surface boundary conditions for mean flów variables are specified according to

$$\frac{\nu_T}{\sigma_T}\frac{\partial T}{\partial z} = \frac{F_N(t)}{\rho\, C_p} \tag{7}$$

$$\nu_{t}\cdot\frac{\partial U}{\partial z} = \frac{\tau_x(t)}{\rho} \tag{8}$$

$$\nu_T\frac{\partial V}{\partial z} = \frac{\tau_y(t)}{\rho} \tag{9}$$

where F_N (t) is the net heat flux, ϱ and C_p are the density and specific heat of water respectively, $\tau_x(t)$ and $\tau_y(t)$ are the wind stress components calculated from the quadratic wind stress law with a drag coefficient of 1.3 10^{-3}. The value of C_p varies with the water temperature, see Gill (1982). The value shown in Table 1 corresponds to a water temperature of approximately 5°C. $F_N(t)$ is the sum of heat fluxes through the air/water interface. These will be discussed in more detail in the next section. At the lower boundary a zero flux condition is used in the temperature equation and in the momentum equations zero velocity is used at the bottom boundary. The turbulence parameters are related to the flux of heat and momentum close to the boundaries.

Equation of State

This equation should reproduce the almost quadratic relation between temperature and density around the point of maximum density. The equation of state employed reads

$$\rho = \rho_0(1-\alpha(T-T_m)^2) \tag{10}$$

where ϱ_o is the maximum density and α is a constant, and T_m is the temperature of maximum density. This equation is a good approximation of the standard equation of state, see Gill (1982). In order to obtain maximum accuracy, the constant α has been chosen with respect to the temperature interval of 0-8°C. Values of all constants which enter the equation of state are in accordance with Farmer and Carmack (1981) and they are shown in Table 1.

Solving the Equations

The empirical constants appearing in the turbulence model are treated as being universal. For complex flows, i.e. recirculating, curved, etc., this universality may be questioned, but for boundary layer flows the standard values, used in the present study and shown in Table 1, may be used with confidence, see Svensson (1978).

Eqs. (1) - (4), (6), and (10) form a closed system and thus constitute the formulation of the mathematical model. This set of equations, in their finite

Table 1 – Model constants used in the calculations.

Constant		Value	Unit
σ_k	Prandtl/Schmitt number	1.4	–
σ_e	Prandtl/Schmitt number	1.3	–
C_μ	constant in the turbulence model	0.09	–
$C_{1\varepsilon}$	constant in the turbulence model	1.44	–
$C_{2\varepsilon}$	constant in the turbulence model	1.92	–
$C_{3\varepsilon}$	constant in the turbulence model	0.8	–
g	gravitational acceleration	9.81	m s^{-2}
α	constant in the equation of state	8.25 10^{-6}	°C^{-2}
ϱ_o	maximum water density	999.975	kg m^{-3}
C_p	specific heat of water	4.200	J kg^{-1}
T_m	temperature of maximum density	3.98	°C
f	Coriolis' parameter	1.3 10^{-4}	s^{-1}

difference form, were integrated forward in time using an implicit scheme and a standard tridiagonal matrix algorithm, see Svensson (1978).

The Net Heat Flux, F_N

The heat contents in a lake is mainly controlled by the heat fluxes through the air/water interface. Those fluxes are, see Fig. 1,

F_s – net short wave radiation,
F_l – net long wave radiation
F_c – sensible heat flux,
F_e – latent heat flux,
F_p – precipitation,

There are two more fluxes that affect the heat contents

F_q – heat flux, due to inlet/outlet from rivers and/or groundwater,
F_b – heat flux to or from sediments,

During the cooling period in basins with an exchange time of the order one year, F_N mainly consists of four heat fluxes; F_l, F_c, F_e, and F_s. However, if the lake is ice covered, F_b, and sometimes F_q, have to be taken into account. Thus, the surface boundary condition for the model during the ice-free cooling period, F_N, can be written as

$$F_N \equiv F_l + F_c + F_e + F_s \qquad (11)$$

Worth to notice is that during spring and summer, when the short wave radiation is

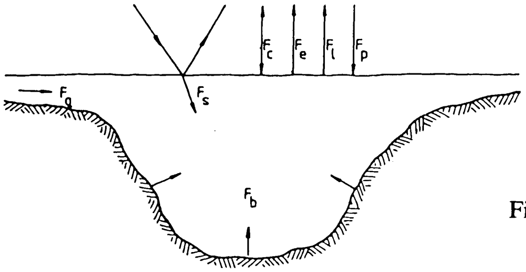

Fig. 1. The heat fluxes that controls the heat contents of a lake.

the dominant heat flux, it should be added as a source term in the temperature Eq. (1), as it penetrates the water surface. As it is of minor importance during cooling, it can be incorporated in the surface boundary condition above.

The formulation of the short wave radiation is in accordance with Bodin (1979). Bulk formulas were used for the sensible and latent heat flux calculations with exchange coefficients C_H and C_E equal to 1.42 10^{-3} and 1.32 10^{-3} respectively, see Friehe and Schmitt (1976). Calculations of the net long wave radiation are in accordance with Washington et al. (1976). They used a Brunt-type formulation modified with a cloud factor.

Model Tests

This chapter describes how the temperature profiles vary in a lake, which is exposed to different meteorological forcing. The lake has a hypsometrical curve according to Fig. 2.

This curve is closed to the mean curve of Swedish lakes found from an investigation of 85 Swedish lakes, see Broman (1982). The calculations are carried out with a constant wind stress, corresponding to wind velocities of 2, 5, or 10 ms^{-1} and a constant net heat flux from the water surface to the atmosphere of 50 or 100 W m^{-2} respectively. The initial temperature profile is 3.98°C and uniform, which is the temperature of maximum density for pure water. The result is shown in Fig. 3.

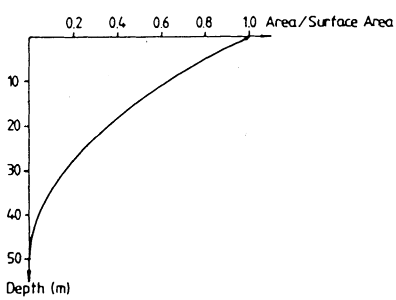

Fig. 2. The hypsometrical curve used in the model tests.

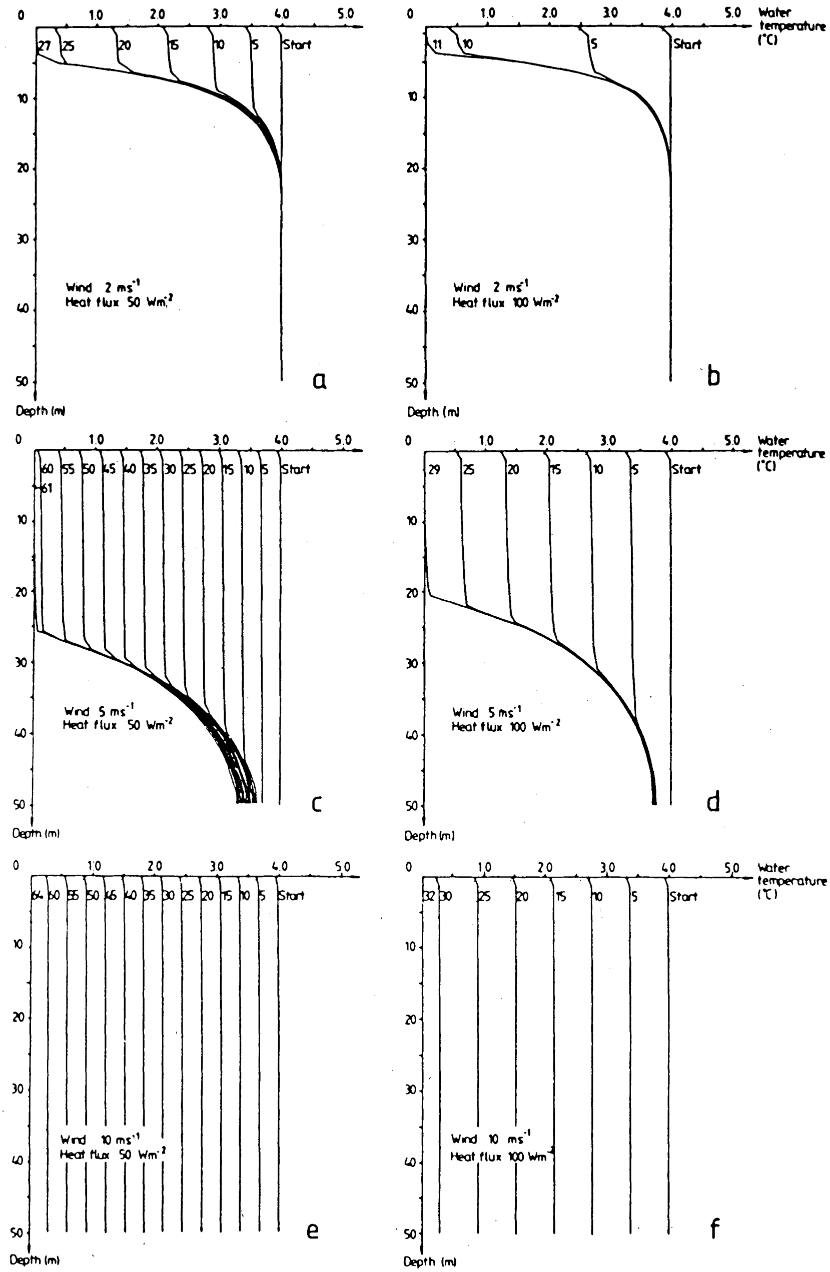

Fig. 3. The temperature development for different meteorological conditions.

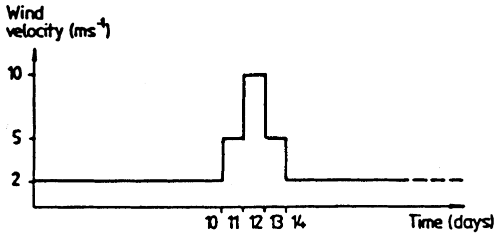

Fig. 4.
The wind variation during one of the cooling calculations.

The number at each temperature profile is the day number calculated from the starting day.

In the one-dimensional model the mixing rate depends on the wind stress and the stability of the water mass. In the equations the stability term can either act to increase or to decrease the production of turbulence. If cooling creates instability, e.g. the density of the surface water is higher than the deep water, convection will occur and produce turbulence. On the other hand, if cooling increases the stability, the turbulence decreases. However, the wind stress will always produce turbulence. As the initial temperature profile is uniform, equal to the temperature of maximum density, cooling will create stability. In all figures, except *e* and *f,* it is shown that the mixed layer depth decreases as cooling proceeds. This is due to the increasing stability. Another interesting point is shown in the figures *c* and *d*, where the only thing that differs is that in *d* the net heat flux is twice the net heat flux in *c*. As the heat flux increases, the stability increases, and that is why the mixed layer depth in *d* is less compared to *c*. Also worth noticing is that when the heat flux is increased with a factor 2, the cooling rate of the mixed layer is more than doubled.

When the wind speed is 10 ms^{-1}, see figures *e* and *f*, the mixing proceeds through the whole water mass, and the whole lake is cooled uniformly from the surface to the bottom.

Of course, neither the wind speed nor the net heat flux are constant under natural conditions. To show the effect of a wind increase, the following test was performed. The net heat flux was constantly 50 W m^{-2}, and the wind velocity varied according to Fig. 4. The result is shown in Fig. 5. The increased wind on day 11 increases the mixed layer depth from 10 m to 20 m. This deepening mixes "warm" deep water with the "cool" surface water, and the result is an increased surface water temperature. The next day the whole lake is completely mixed due to the wind speed of 10 ms^{-1}. Even though the wind decreases on the 13th day, the complete mixing is maintained. Thereafter a new stratification is formed, and the mixed layer depth decreases as cooling continues. Note that the bottom water temperature has decreased from 3.98 to approximately 3.2°C. When the wind speed was constantly 2 ms^{-1}, see Fig. 3*a*, the cooling took approximately 27 days, while the cooling according to Fig. 5 took approximately 32 days.

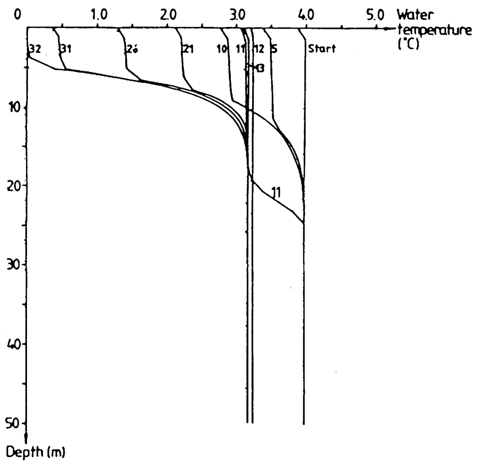

Fig. 5. The figure shows how the temperature profiles are changing when the net heat flux is constant 50 W m^{-2} and the wind velocity varies according to Fig. 4.

Model Application to Lake Väsman

The model calculations will in this chapter be compared to measurements in Lake Väsman. Lake Väsman is situated in the central part of Sweden, see Fig. 6. During the summer 1980 the depth was carefully measured in the south-eastern part of the lake, here called basin A, see Fig. 6. Based on these depth measurements, a hypsometric curve was created, see Fig. 7. The total area for basin A is 19.8 10^6 m^2, and the volume is 36.4 10^7 m^3. There is a river inflow in the north-western part of the lake and an outlet through the town of Ludvika.

The water temperature has been measured by STAL-LAVAL, see Ekberg (1982), in cooperation with ASEA. They used an Anderaa measuring system, which consists of a thermistor chain with 11 thermistors and a data logger. The measuring system was placed in the southeastern part of basin A, see Fig. 6.

The thermistors were placed at the following depths: 3.5, 5.5, 7.5, 9.5, 13.5, 17.5, 21.5, 25.5, 29.5, 31.5, and 33.5 m. Temperature data from the termistor at 33.5 m are excluded, as these data, if they were correct, would create large instabilities in the bottom water. The probable explanation is that this thermistor lies on the bottom or in the upper part of the sediments. Temperature data were sampled every 30 minutes and stored on a magnetic tape. With a carefully calibrated system the relative accuracy of the data is better than 0.05°C. An examination of the data shows that even in profiles, averaged in time over a couple of hours, small instabilities may occur. Therefore, the relative accuracy of these data

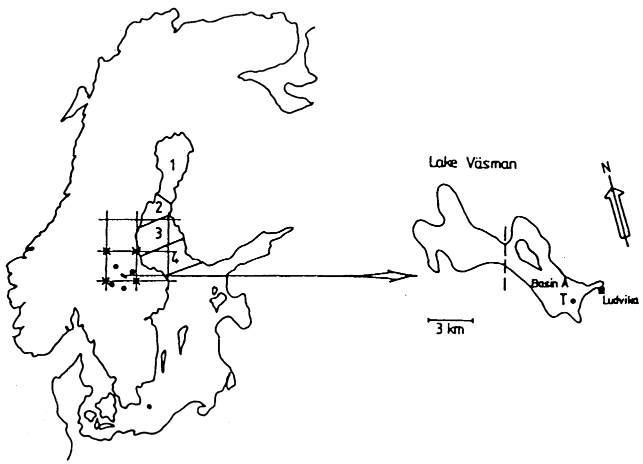

Fig. 6. The air temperature, relative humidity and cloud coverage data were extracted from four synoptic weather stations (·) in the area around Lake Väsman. Air pressure data were extracted according to a 150 km grid net, marked X in the figure. To the right a schematic sketch over Lake Väsman is shown, where T marks the stations where the water temperature was measured.

is probably not better than 0.1°C.

The temperature measurements started in 1980, and the system is still operating. From this large amount of data a cooling period in 1981 has been chosen for the model simulation. The period starts on November 2nd with vertically homogeneous water temperature of 7.7°C. On December 7th the whole lake is ice-covered, except for a lead in the central parts.

Fig. 8 shows the temperature data from three thermistors. During this cooling period the river flow through Lake Väsman was less than 10 $m^3\ s^{-1}$. With this river

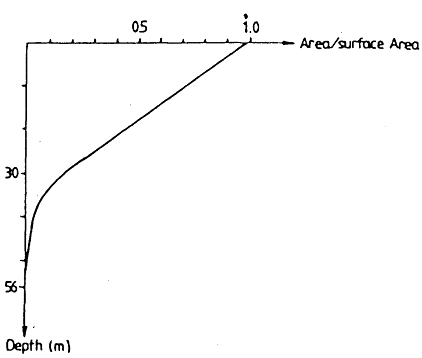

Fig. 7. The hypsometrical curve for basin A.

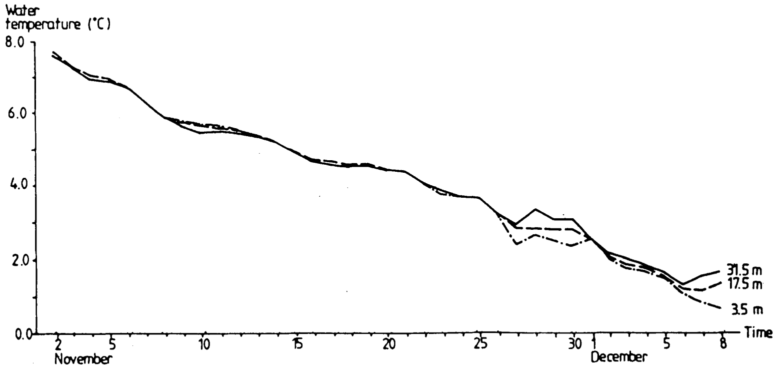

Fig. 8. Water temperature data from three thermistors placed at depths 3.5 m, 17.5 m, and 31.5 m. Worth to mention is that the whole lake is ice covered on December 7th.

flow the exchange time of Lake Väsman will be larger than one year, and therefore F_q is ignored in the following calculations.

The two driving forces in the model are the net heat flux and the wind stress, see Eqs. (7) – (9). On the basis of the air temperature, the relative humidity, the amount of clouds, and the wind velocity calculations of the net heat flux were performed according to Eq. (11) and the wind stress according to a quadratic wind stress law.

The meteorological data, except the wind velocity, have been extracted every 6th hour from four synoptic weather stations situated in the area around Lake Väsman, see Fig. 6. Areal mean values were calculated on the basis of all four stations and assumed to be the representative values over Väsman. From the analysed surface air pressure the wind velocity was calculated every 6th hour according to a 150 km grid net, see Fig. 6. Both grid net and analysis are the same as those used in the Swedish weather forecasting model. A wind velocity calculated from the balance between the pressure gradient and the Coriolis' force is called a geostrophic wind velocity. Depending on the stability of the atmospheric boundary layer and the friction to the ground, the geostrophic wind velocity has to be reduced. In this study this reduction factor is put constant to 0.5. This is, of course, a very rough estimate, since the atmospheric stability often shows large diurnal variations. The constant value of 0.5 was found to be the mean value, when the geostrophic wind was compared to wind measurements at the airports of Västerås and Uppsala. All meteorological data are shown in Fig. 9.

The model calculations start on November 2nd with an initial water temperature profile of 7.7°C in the whole vertical. This profile is assumed to be the representative water temperature in basin A. In order to make realistic cooling calculations of the basin, the hypsometric curve shown in Fig. 7 is incorporated in

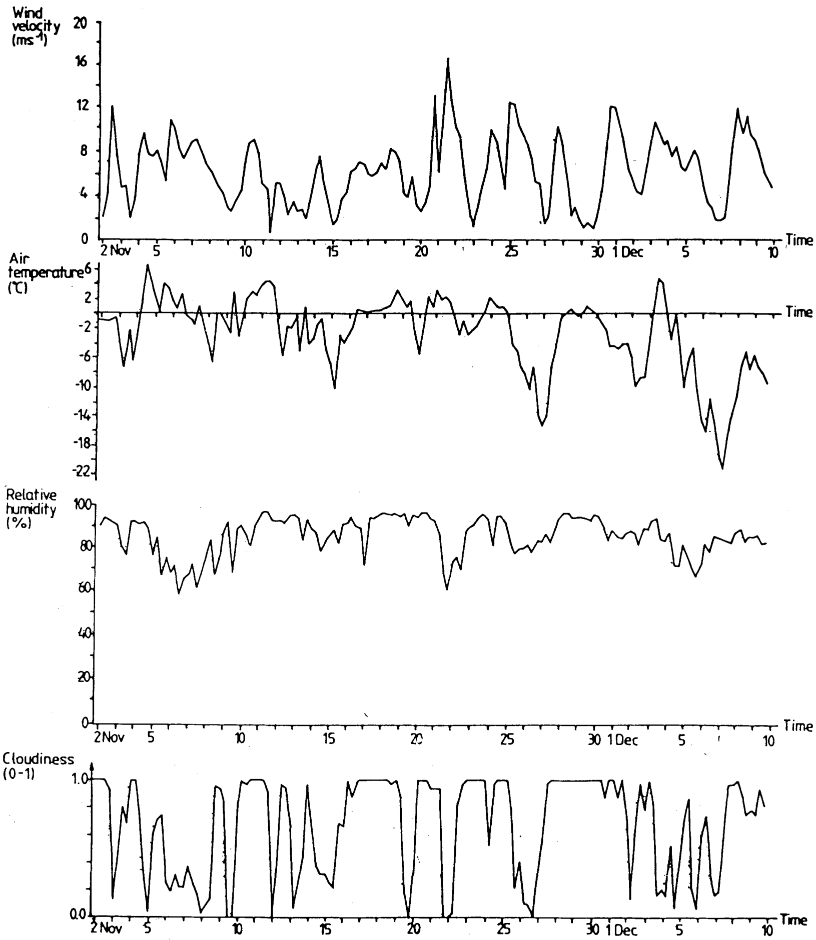

Fig. 9. All meteorological data used in the model calculations are shown in the figure.

the model. A time step of 10 minutes is used, and the net heat flux and the wind stress are changed every 6th hour. As the calculated surface water temperature has decreased to just below 0°C, ice formation is assumed to start, and therefore all calculations are terminated.

The result is presented in two different ways, see Figs. 10 and 11. Fig. 10 shows both calculated and measured temperature profiles every fifth day from November 2nd at 00 hours to December 7th at 00 hours. The calculations stop at 04 hours on December 7th, which by the definition above is the calculated ice formation time. This last profile is also shown in the figure, and it only differs in the upper 5 m from the profile on December 7th at 00 hours. This ice formation date agrees very well with the observed data, and this implies that both the treatment of the

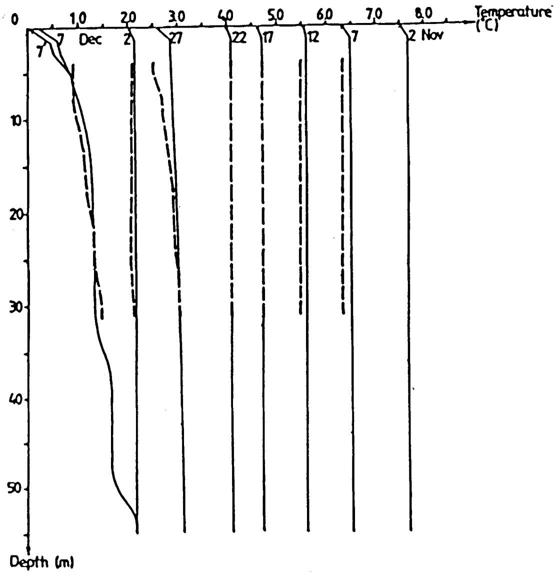

Fig. 10. The calculated (–) and measured (---) temperature profiles every fifth day from November 2nd at 00 hours to December 7th at 00 hours. The last calculated profile, marked 7', is calculated at 04 hours on December 7th.

driving forces and the physics in the model are well described. During almost the whole period the basin is well mixed, except on November 27th and December 7th, when the temperature profiles show a weak stratification. The discrepancy between measured and calculated temperature profile on November 27th is probably due to an overestimated calculated mixing, which is due to the assumption of a constant wind factor. If the atmosphere is strongly stratified, the wind factor should decrease. The probable explanation is therefore that the atmosphere is strongly stratified on November 27th. This is supported by the meteorological data shown in Fig. 9, where the air temperature near the ground is close to −15°C and the cloud coverage is close to zero.

Another way to present the result is to show how the heat contents vary with time, see Fig. 11. The heat contents per square metre are calculated from the surface down to a depth of 32 m. To get the total heat contents in the basin one has to multiply the values with the total surface area. Observe that the water temperature is given in degrees centigrade. Even the small fluctuations in the measured heat contents are found in the calculated curve. This means that the net heat flux is correctly calculated during the whole cooling period. Worth to notice is that after December 7th the heat contents are almost constant. The explanation is that as ice has formed, it prevents wind mixing and heat loss to the temperature.

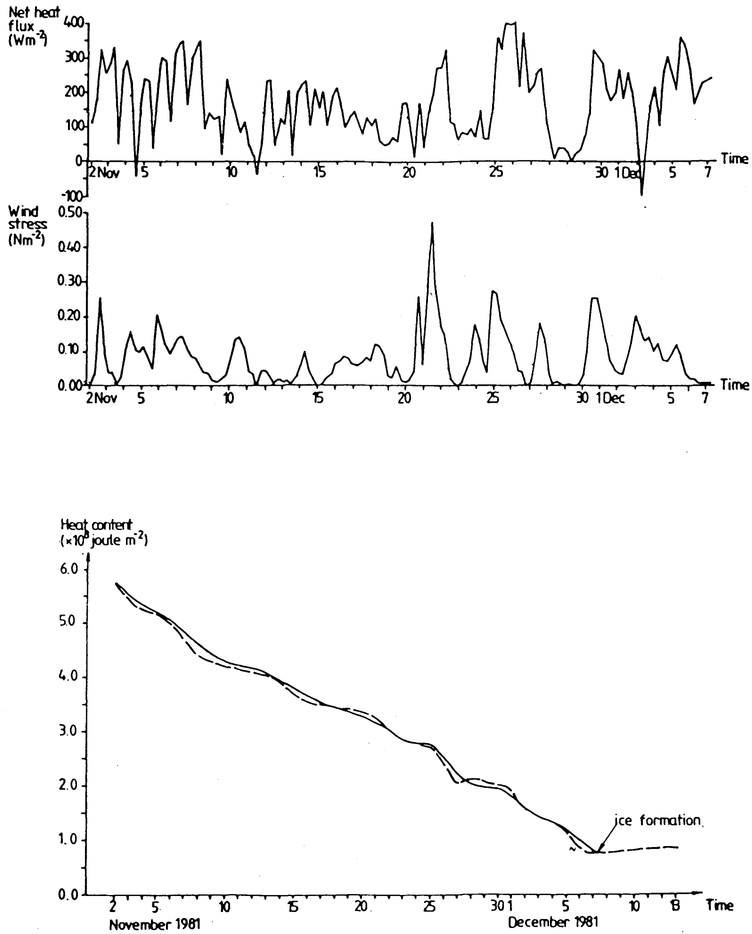

Fig. 11. The calculated (–) heat contents per square metre. Both are calculated from the surface to a depth of 32 m. The upper part of the figure shows how the net heat flux and the wind stress varied during the whole cooling period. Observe that positive values of the net heat flux means that the heat flux is going from the water surface to the atmosphere.

Conclusions

The procedure for the net heat flux calculation and the treatment of meteorological input data is, in this study, the same as used in two other cooling studies, "A study of the large scale cooling in the Bay of Bothnia", see Sahlberg and Törnevik (1980) and "Vertical mixing and restratification in the Bay of Bothnia during cooling", see Omstedt and Sahlberg (1982) and Omstedt, Sahlberg and Svensson (1983). These studies verify the net heat flux program, and they show that it is possible to use "large scale" meteorological input data in the model calculations.

The study shows that even the weak stratification that occurs at the end of the cooling period in Lake Väsman, see Fig. 10, is simulated by the one-dimensional model. This indicates that the mixing processes are well described by this kind of model.

Finally the conclusion is that the model is very useful for heat contents calculating and its vertical distribution, in both lakes and brackish waters during the cooling period.

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Address:
Swedish Meteorological and
Hydrological Institute,
Box 923,
S-601 19, Norrköping,
Sweden.