Propagation of Shock Waves in Inhomogeneous Gases. II

—Hydromagnetic Shock—

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Using the previous method, the propagation of a hydromagnetic shock wave in inhomogeneous magnetic field is considered. The existence of magnetic field has the effect of restraining the growth of shock strength. When the magnetic pressure is large compared with gas pressure, even the diminution of shock strength can occur. As an application, the propagation of shock wave in solar chromosphere is discussed.

§ 1. Introduction

In the previous paper,¹ we considered the shock propagation in an inhomogeneous medium as the first approximation. The same method can be applied to the case of a hydromagnetic shock, i.e. a shock wave propagating through a plasma of infinite conductivity in an inhomogeneous magnetic field. This will be important in treating interstellar gasdynamics and stellar surface phenomena.

Although the method can be applied to general hydromagnetic shocks, including oblique or parallel shocks, it would be very cumbersome to deal with the general cases because of the necessity of taking a suitable coordinate system for each layer of matter. So we here treat only the case of the normal shock, where the direction of magnetic lines of force is perpendicular to that of the propagation velocity.

In § 2, we derive the general formulae for the normal shock, which are applied to special cases in § 3. Especially, the propagation in the solar chromosphere is discussed.

§ 2. General formulae

Corresponding to the formulae (2·5) ~ (2·8) in "I", we have the following relations for the case of the normal hydromagnetic shock:²

\[ z_{12} = \frac{1 - \xi y_{12} - (1 - 3\xi^2)(1 - 1/y_{12})P_{m1}/P_{1}\ast}{y_{12} - \xi^2} , \]  

(2·1)

* This paper is hereafter referred to as Paper "I".

++ Notation is the same as that in "I", except for those newly introduced.
and

\[ U = u_1 \pm \phi(z_{12}, p_{1*}, p_{m1}, \tau_1), \]

(2.3)

where

\[ \phi(z_{12}, p_{1*}, p_{m1}, \tau_1) = \sqrt{\frac{p_{1*}}{\tau_1}} \sqrt{\left(\frac{p_{m1}}{p_{1*}}\right)} \frac{1}{(1 - y_{12})}, \]

(2.4)

\[ \phi(z_{12}, p_{1*}, p_{m1}, \tau_1) = \phi(z_{12}, p_{1*}, p_{m1}, \tau_1) \frac{1}{(1 - y_{12})} \]

(2.5)

\( \tau \) is the specific volume, i.e. \( \tau_1 = 1/p_1 \), and \( y_{12} = \tau_2/\tau_1 = \rho_1/\rho_2 \). \( p^* \) is the total pressure, which is the sum of gas pressure \( p_g \) and magnetic pressure \( p_m = H^2/8\pi \) (\( H \) is the magnetic field strength, whose direction is always parallel to the shock front):

\[ p^* = p_g + p_m = p_g + H^2/8\pi. \]

(2.6)

\( z_{12} \) here means the ratio of total pressures ahead and behind the shock wave:

\[ z_{12} = \frac{p_{1*}}{p_{1*}}. \]

In the function \( \phi \) and \( \psi \), \( y_{12} \) is related implicitly to \( z_{12} \) by the Rankine-Hugoniot relation (2.1) for this case.\(^*\)

Finally, in the relation (2.1), it is assumed that the magnetic flux is "frozen" in the material throughout the process, because of the infinite conductivity of the plasma concerned. This condition is expressed as

\[ H_s/p_1 = H_s/p_2, \]

or

\[ p_{m1} \tau_1^2 = p_{m2} \tau_2^2. \]

(2.7)

Using this relation and the definition (2.6), we obtain the restriction imposed on \( y_{12} \) and \( z_{12} \) as

\[ z_{12} y_{12} \geq \frac{p_{m1}}{p_{1*}}. \]

(2.8)

Of course, the following inequality is valid in general:

\[ p_{m}/p^* \leq 1. \]

(2.9)

\(^*\) Solving the relation (2.1) for \( y_{12} \), we get

\[ y_{12} = \frac{1 + \sqrt{\frac{1}{\tau_2} + \left(\frac{3}{\tau_1}\right)}}{\frac{1}{\tau_2} + \left(\frac{3}{\tau_1}\right)}, \]

(2.1')

which reduces to (2.1) of "I" when \( p_{m1} \to 0 \). From this equation we get the limiting relations

\[ y = \frac{1 + \sqrt{1 - 3(1 - 1/\tau_1)}}{1 + \sqrt{1 - 3(1 - 1/\tau_1)}}, \]

for \( \tau \to 1/\tau_1 \)

and

\[ y = 1 - \frac{1 - \sqrt{1 - 3(1 - 1/\tau_1)}}{1 + \sqrt{1 - 3(1 - 1/\tau_1)}}, \]

for \( \tau = 1 \).
Now we treat the problem of shock propagation in the plasma of variable density or inhomogeneous magnetic field strength. The conditions (2·10) and (3·4) in “I” hold, except that $p^*$ is taken instead of $p$. Following the same procedure as in “I”, we finally get the equation of the first approximation,

$$\frac{d \ln \rho}{dz} - \frac{d \ln p^*}{dz} \left(1+2 \sqrt{\frac{(z-1) y (1-\lambda^2)}{z (1-y) \left[1+\lambda^2+(1-3\lambda^2) p_m/y^2 p^* z\right]}}\right)
+ \frac{(1-3\lambda^2)}{y \left[z+\lambda^2+(1-3\lambda^2) p_m/y^2 p^* \right]} \frac{d}{dz} \left(\frac{p_m}{p^*}\right)
= \frac{1}{z-1} + \frac{y-\lambda^2}{(1-y) \left[z+\lambda^2+(1-3\lambda^2) p_m/y^2 p^* \right]} + 2 \sqrt{\frac{y (1-\lambda^2)}{z (z-1) (1-y) \left[1+\lambda^2+(1-3\lambda^2) p_m/y^2 p^* z\right]}}. \tag{2·10}$$

To this equation are added the following conditions:

$$zy-1+\lambda^2 (y-z) + (1-3\lambda^2) (1-1/y) p_m/p^* = 0,$$
$$zy^2 \geq p_m/p^*, \quad \tag{2·11}$$

and

$$p_m/p^* \leq 1.$$

To solve Eq. (2·10) in general, the initial distribution of the magnetic pressure $p_m$ should be a function of density $\rho$. For example, $p_m \propto \rho^{k_m}$, where $k_m$ can be called “magnetic polytropic index”, though it is not always necessary to take this form (Case i)). The initial distribution of $p_o$ is assumed always as

$$p_o \propto \rho^k.$$

When the magnetic field is absent, (2·10) reduces to (3·7) in “I”.

In order to see the effect of the magnetic field explicitly, we take the limiting cases. In the limit of strong shock, i.e. $z \gg 1/\lambda^2$, we get

$$z \propto (p_o/p^*)^{\alpha \propto (\gamma, k)}, \tag{2·12}$$

where $\alpha(\gamma, k)$ is defined by (4·5) in “I”. We see the increase of the shock strength as the decrease of gas pressure is restrained by the factor of $p_o/p^* < 1$ due to the existence of a magnetic field. Conversely, in the limit of weak shock, i.e. $z-1 \leq 1$, we get

$$z-1 \propto (p_o/p^*)^{\beta \propto (\gamma, k)} \left[1 + \frac{(1-3\lambda^2)/(1+\lambda^2)}{p_m/p^*}\right]^{1/4} \cdot p_o^{-\mu \propto (\gamma, k)}, \tag{2·13}$$

where $\beta(\gamma, k)$ is defined by (4·7) in “I”. Since the factor multiplied by $p_o^{-\mu \propto (\gamma, k)}$ is always less than unity, here also exists the restraining effect of the magnetic field.*

* Note that the restraining factors of (2·12) and (2·13) do not depend on $k$, but only on the ratio of gas and magnetic pressures. These factors would perhaps be valid also in the second approximation.
Thus, the existence of the magnetic field has in general the effect of restraining the growth of shock strength as the decrease of gas pressure. Moreover, it is noted that this so to speak "magneto-restraining effect" is more remarkable for strong shocks, and the shock strength can even be diminished when the magnetic field is sufficiently large compared with gas pressure. This latter effect will be seen in the solar chromosphere as will be shown in the next section.

§ 3. Special cases

Now we apply the formulae obtained in § 2 to the following two special cases:

Case i). Total pressure \( p^* \) is initially constant everywhere

This corresponds to both a laboratory system and interstellar space, since the gravity is neglected.*

Noting that \( dp^* = 0 \) and \( dp_g = -dp_m \),

for this case, we can reduce the left-hand side of (2·10) to

\[
\frac{d}{dz} \left( \frac{p_m}{p^*} \right) \left[ -\frac{1}{k(1-p_m/p^*)} + \frac{(1-3\beta^2)}{y\{x + \beta^2 + (1-3\beta^2)p_m/p^*}\} \right].
\] (3·1)

In Fig. 1, the calculated relations between shock strength and magnetic pressure are shown for \( \beta = 1/4 (\gamma = 5/3) \) and \( k = 1 \) (isothermal). Initial shock strength \( z_0 = 2.0 \) is taken at \( p_m/p^* = 0.9 \) and 0.98 respectively. We see the shock strength approaches to a finite value when the magnetic pressure becomes negligible.

This behaviour of shock strength is due to that of sound velocity. As is known, the square of sound velocity is expressed as the sum of squares:

\[
c^2 = \gamma \rho_g/\rho + H^2/4\pi \rho.
\] (3·2)

In our case, this can be reduced to

\[
c^2 \propto p^*(2-1)^k/k \times \{1 + (2-\gamma)/(\gamma \cdot p_m/p^*)\}/(1-p_m/p^*)^{2/k}
\]
or

\[
c^2 \propto (1 + p_m/5p^*)/(1-p_m/p^*) \quad \text{(for } \gamma = 5/3 \text{ and } k = 1)\).
\]

* Interstellar matter is usually in violent motion. Thus it would be necessary to treat the propagation in moving gases.
This is also plotted in Fig. 1. The rapid increase of sound speed at large $\frac{p_m}{p^*}$ corresponds to the rapid decrease of shock strength. For the limit $\frac{p_m}{p^*}\rightarrow 0$, the sound speed approaches a finite value owing to the sound velocity $\frac{\gamma p_0}{\rho}$, and the shock strength is also saturated to a finite value. This tendency is a special representation of the effects of the magnetic field restraining the shock from growing up, as mentioned in § 2.

Case ii). Constant magnetic field $p_m=\text{constant in the gravitational field}$

This is applied to the phenomena in the solar chromosphere. In this case, the lefthand side of (2·10) is reduced to

$$\frac{d \ln \rho}{dz} \left[ 1 - k \frac{p_0}{p^*} \right] \left[ 1 + \frac{(1 - 3\gamma) p_m}{\gamma (1 + \gamma^2 + (1 - 3\gamma) p_m/\gamma^2 p^*)} \right] + 2 \sqrt{\frac{(x - 1) y (1 - \gamma)}{x (1 - y) \left[ 1 + \gamma^2 + (1 - 3\gamma) p_m/\gamma^2 p^* \right]}} \right]$$

(3·3)

Now we investigate the propagation of a shock wave which is generated at the bottom of the solar chromosphere and propagates outwards. For simplicity, we assume that the chromosphere is in the isothermal state and so the parameter $k$, the polytropic index of gas, is taken equal to unity. For the other physical parameter $p_m$, we take as $p_m=0.4$ dyne/cm$^2$, i.e. $H=\sqrt{10}$ gauss which corresponds to the general solar magnetic field. The magnetic field is assumed to be constant throughout the whole chromosphere. Taking the initial value of the total pressure and shock strength respectively as $p_0^*=10^6$ dyne/cm$^2$ and $z_0=2.2$, Eq. (2·10) supplemented by (3·3) and (2·11) is integrated by the numerical method. The result of integration, the variation of shock strength versus the variation of total pressure, is shown in Fig. 2.

It shows a remarkable feature of shock strength coming up to a maximum value at the point where the magnetic pressure is nearly equal to gas pressure. Though the height which corresponds to the point of maximum shock depends on the physical parameter of the chromosphere, it is about 2000 km above the photosphere for the adopted value of $p_m$. Beyond the maximum point, the mag-
netic pressure exceeds the gas pressure and the shock strength decreases rapidly. This tendency is also a special representation of "magneto-restraining effect" mentioned in § 2.

Note here that we treat the propagation of a shock wave of constant power-supply, which corresponds to the continuous pushing of a piston. Thus the growing or decreasing of shock strength comes from the reflection of rarefactive or compressive wavelets by the inhomogeneities. When the shock weakens by a magnetic field, the compressive reflection-wavelets grow up to a shock wave of reflection. Thus the magnetic field takes the rôle of a reflecting wall.*

Although the shock waves really considered in general are not of constant power-supply, and so the propagation of a shock-pulse must be treated by taking account of the energy dissipation, it is still interesting to speculate that the whole feature of propagation, i.e. the increase of shock strength in early phase and rapid decrease (reflection) in later phase, has some connection with the formation and the disappearance of solar fine mottles and the spicules.3) As for the contribution of shock waves to the energy balance and the heating of chromosphere or corona, it is further necessary to estimate the dissipation of the energy by shock waves, which will be treated in another paper.

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References


* In this sense, it would be better to say the "magnetic-wall effect" rather than the "magneto-restraining effect" in this case.