Coupling of Dirac Particle and Gravitational Field

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(Received April 19, 1960)

By using legs connection we investigate the general relativistic Dirac equation which is invariant under the general coordinate transformations and spin transformations. We obtain the four-dimensional representation for the generalized Dirac matrices by introducing the six legs and the concept of sedenion conjugate. It is shown that Fermi-Fermi interaction term must be added to the original Lagrangian density, if the legs connection is regarded as independent field quantity. The electromagnetic field is also introduced in connection with the complex legs. Finally, we consider the relation between the spin transformation and Pauli-Gürsey type transformation.

§ 1. Introduction and summary

In the last few years many people have made attempt to unify all the elementary particles and their interactions. In connection with this, there remains an important problem of investigating the internal correlation between the space-time and iso-spin space. The fact that parity and strangeness are conserved in the strong interaction and that neither of them are conserved in the weak interaction may be considered to suggest that the space-time and iso-spin space are not completely independent. The author conjectures that iso-spin space may be amalgamated in space-time structure at least in the case of lepton where iso-spin has no definite meaning in contrast to the case of baryon. Originally, the reality of iso-spin itself is not a matter, but the symmetrical properties derived from this space play an important role. Therefore the role of the iso-space may be reduced to the algebraic manipulation. In fact, Heisenberg and his collaborators have attempted to introduce the same freedom as iso-spin from the standpoint of the spinor unified theory by postulating that their non-linear equation is invariant under the Pauli-Gürsey transformation. The non-linearity of the spinor equation will have some connection with the space-time structure as the non-linearity appears in the general relativity. As an example, Gürsey has obtained the non-linear spinor equation different from Heisenberg's one on the assumption that the equation is invariant under the conformal transformation. The aim of this paper is to investigate the relationship among the non-linear character of the spinor equation, Pauli-Gürsey transformation and the general covariance or spin transformation. We also speculate that our legs connection will play the role of a link between the weak
interactions and gravitational interactions.

We shall start with the generalized Dirac equation in curved space-time

$$\gamma^\mu (\partial^\mu - \Gamma_\mu^\nu) \psi = 0 \quad (1.1)$$

where $\Gamma_\mu^\nu$ is the spin-affine connection and its general form is given by Green except for its spur. Green has investigated this equation in order to assert that the observed failure of parity conservation is reconciled with the invariance properties postulated by general relativity. Further he identified some terms of $\Gamma_\mu^\nu$ with the electromagnetic potential, etc. As will be mentioned in § 2, it seems to us that there is no necessary reason for such identification. Furthermore, to carry out this identification we must introduce the complex legs and add the electromagnetic interaction term artificially. Since the introduction of the complex legs complicates the problem, we use the real legs in § 3 and § 4.

Under a general transformation of coordinates $x_r \rightarrow x_r'$, Eq. (1.1) is invariant if Dirac spinor $\psi$ and Dirac matrices $\gamma^\mu$ transform as

$$\phi \rightarrow \phi' = \left| \frac{\partial x}{\partial x'} \right|^{1/2} \psi, \quad \gamma^\mu \rightarrow \gamma'^\mu = \gamma^\nu \frac{\partial' x^\nu}{\partial x'^\mu}. \quad (1.2)$$

In addition, another type of invariance is possible for what may be called spin transformation which leaves the functional dependence of the metric tensor on the coordinates unaltered,

$$\phi \rightarrow \phi' = S^{-1} \psi, \quad \gamma^\mu \rightarrow \gamma'^\mu = S^{-1} \gamma^\mu S. \quad (1.3)$$

Spin transformation is not necessarily related to a change of coordinates. For the special coordinate transformations which do not change the functional dependence of the metric tensor on the coordinates, i.e. such that $\gamma'^\nu (x') = g'^\nu(x')$, we can find a corresponding spin transformation so that the two transformations together leave the Dirac matrices unchanged

$$\phi(x) \rightarrow S \phi(x) \left| \frac{\partial x}{\partial x'} \right|^{1/2}, \quad \gamma^\mu(x) \rightarrow \gamma'^\mu(x) = S \gamma^\nu(x) S^{-1} \frac{\partial' x^\nu}{\partial x'^\mu}.$$

Such transformations play an important role in the case of special relativity. On the contrary, in the curved space-time, the spin transformation (1.3) has no connection with the coordinate transformation. The requirement of invariancy under the spin transformations will then introduce the new entities. The general form of spin transformations $S$ can be expressed in terms of constant Dirac matrices $\gamma_{(a)}$ ($a=1, 2, 3, 4, 5$)

$$S = S_A + S_B$$

where

* In what follows, the small Latin and small Greek indices represent quantities with respect to the space-time coordinates system and to the legs system respectively. We use the units $\hbar = c = 1$. 
\[ S_A = 1 + \frac{1}{2} \sum_{\alpha, \beta = 1}^{4} \psi^{\alpha\beta} \left[ \gamma_{(\alpha)} \gamma_{(\beta)} \right] + \psi^{5} \gamma_{(5)}, \]

\[ S_B = \sum_{\alpha = 1}^{4} \psi^{\alpha} \gamma_{(\alpha)} + \sum_{\alpha = 1}^{4} \psi^{\alpha} \gamma_{(\alpha)} \tag{1.4} \]

in which the coefficients \( \psi^{\alpha} \) and \( \psi^{\alpha\alpha} \) depend on the coordinates and \( \gamma_{(\alpha)} \) satisfy \( \gamma_{(\alpha)}^{2} = \gamma_{(\beta)}^{2} = \gamma_{(\gamma)}^{2} = -\gamma_{(\delta)}^{2} = \gamma_{(\epsilon)}^{2} = 1 \). The indices in brackets are used to indicate explicitly quantities with respect to the legs system. The form of \( S \) shows that the spin transformation under which the theory is invariant is isomorphic with the group of rotations in an abstract 6-dimensional space. This corresponds to the result of Finkelstein who showed that the group of generally covariant coordinate transformations may be mapped on the orthogonal group in six dimensions by introducing the line coordinates. Corresponding to the 6-dimensional abstract space, in § 3 we attach six legs at each point of the curved space-time. If Eq. (1.1) is obtained from the requirement that the action integral is invariant under the general 6-dimensional rotation, \( I_{\nu} \) should be of the form:

\[ I_{\nu} = \frac{1}{2} \sum_{\mu, \nu = 1}^{6} \psi^{(\mu\nu)} T^{(\mu\nu)} \tag{1.5} \]

where \( T^{(\mu\nu)} \) is the representation of the generator of the six-dimensional rotation for \( \phi \). The \( \psi^{(\mu\nu)} \) has the meaning of the legs connection and depends the coordinates. Since we start with the massless Dirac equation, it should be sufficient to use the four-dimensional representation. In § 3, using the legs connection \( \psi^{(\mu\nu)} \) and the concept of sedenion conjugate, we obtain a four-dimensional representation of Eq. (1.1).

In our formalism the legs connection \( \psi^{(\mu\nu)} \) plays an important role. Therefore, it is natural to consider \( \psi^{(\mu\nu)} \) as independent field variables. In § 4, we extend Weyl's four legs formalism to the six legs one. If we vary the Lagrangian with respect to \( \psi^{(\mu\nu)} \), we obtain an equation which is different from the one defining the legs connection. To circumvent this unsatisfactory situation we must alter the definition of \( \psi^{(\mu\nu)} \), and also add the Fermi-Fermi interaction term to the Lagrangian in order to satisfy the integrability condition for six legs. Our added non-linear term has more general form than those of Heisenberg or Kita. It is remarkable that the non-linear term is obtained naturally by taking \( \psi^{(\mu\nu)} \)'s as independent field quantities. In order to introduce the coupling constant larger than the gravitation constant, we attempt to introduce the vector field expressed in terms of six legs \( h_{(\mu)}^{(\nu)} \) and \( u^{(\mu\nu)} \).

As already mentioned, if we want to have the imaginary spur part in \( I_{\nu} \), we must use complex six legs. We see then that the space of the complex legs fields is that of Einstein-Schrödinger theory. In the case of four legs, Sciama proposed a geometrical theory of the electromagnetic field by using the complex four legs. He reached the interesting consequences that the charges of all bosons must be integer multiples of some basic charge, and that the direct interaction between a
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particle and the electromagnetic field must be invariant under space and time reflections. If one uses Sciama’s electromagnetic field, it is easily seen that one cannot obtain non-vanishing electromagnetic effects. To obtain the electromagnetic effect we must again introduce the electromagnetic potential artificially as in the case treated by Green. In § 5, it will be shown that the electromagnetic potential defined by Green is introduced through the consideration on the variational equation with respect to the complex six legs connection.

If we use Kramer’s representation of \( \gamma_{(a)} \), the spin transformation \( S_A \) in (4.1) is of the form
\[
\left( \begin{array}{ccc}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{array} \right)
\]
and \( S_B \)
\[
\left( \begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
\end{array} \right)
\]
formation in vector space can correspond to \( S_A \). In § 6, it will be shown that the generalized Pauli-Gürsey transformation corresponds to the spin transformation \( S_B \).

§ 2. Preliminaries: generalized Dirac equation in the general relativity

In this section we shall sketch the outline of the general relativistic Dirac equation which is invariant under the general coordinate transformations and spin transformations, following the line of Green’s formalism. Since Green’s formalism contains few unsatisfactory points, slight modifications will be made. Throughout this paper the space-time coordinates \( x^p(x^1, x^2, x^3, x^4=t) \) are used and the familiar summation-rule concerning the super- and sub-scripts in tensor calculus is applied to every kind of indices.

The \( \gamma^a \) in (1.1) are those four-four matrix functions which satisfy the relations
\[
\{ \gamma^a, \gamma^b \} = \gamma^a \gamma^b + \gamma^b \gamma^a = 2 g^{ab}
\]
where \( g^{ab} \) are defined by the relation \( g^{ab} g_{bc} = \delta^a_c \) in which \( g_{bc} \) are non-singular metric field. We start with the postulate that the derivatives of matrix \( \gamma^a = g_{bc} \gamma^b \) have a general form
\[
\partial_a \gamma^a = E_{aq} \gamma^q + E_{aq} \gamma^q + E_{ar} \gamma^r + E_{aq} \gamma^q
\]
where \( E_{aq} \)'s are tensors depending on the coordinates, with the exception of \( E_{aq} \) which is affinity, and
\[
\gamma_{pq} = \frac{1}{2} [\gamma_p, \gamma_q], \quad \gamma_{rs} = \gamma_r \gamma_s - \gamma_s \gamma_r, \quad \gamma_{rs} = \frac{1}{4!} \varepsilon_{pqrs} \gamma_p \gamma_q \gamma_r \gamma_s, \quad \varepsilon_{pqrs} = i E_{pqrs} / \sqrt{-g}
\]
in which \( g \) means the determinant of \( g_{pq} \), and \( E_{pqrs} \) is the Levi-Civita symbol whose non-vanishing components are those for which all four indices are different and are equal to 1 or -1 according as \( (p, q, r, s) \) is an even or an odd permutation of \( (1, 2, 3, 4) \). Here we use the different quantities from Green’s. The relation

\* For instance, the second equation (26) and that of (34) in his paper are incompatible with the third equation (27) and that of (33), respectively.
between our quantities and those of Green is as follows:

\[ 1 = \gamma^2 = \left( \frac{\gamma^2}{\gamma} \right) / g, \quad E_{pq}^5 = \sqrt{g} \quad E_{pq}^5, \quad E_{pq}^5 = \sqrt{g} \quad E_{pq}^5 \quad (2.3) \]

where " " means the quantities of Green. The other quantities are same as those of Green.

The coefficients \( E^a_{pq}, E^b_{pq}, E^c_{pq} \) and \( E^d_{pq} \) which appear in (2.1) cannot be assigned arbitrarily, but must satisfy a set of identities in order that (2.1) be integrable. These identities are

\[ \begin{align*}
\partial_{(q} E^a_{rp)} + E^a_{(r} E^b_{pq)} + E^b_{(r} E^c_{pq)} & = 0 \\
\partial_{(q} E^a_{rp)} + E^a_{(r} E^b_{pq)} + E^b_{(r} E^a_{pq)} & = 0 \\
\partial_{(q} E^a_{pq)} + E^a_{(p} E^b_{rq)} + E^b_{(p} E^a_{rq)} & = 0 \\
\partial_{(q} E^a_{pq)} + E^a_{(p} E^b_{rq)} & = 0
\end{align*} \quad (2.4) \]

where

\[ A_{\alpha \beta} = \frac{1}{2} [A_{\alpha \beta} B_{\gamma \delta} - A_{\alpha \gamma} B_{\beta \delta}], \quad E_{pq}^a = g_{pq} E^a_q, \quad E_{pq}^a = g_{pq} E^a_q, \quad E_{pq}^a = E^a_{pq}. \]

The spin affine connection \( \Gamma^i_p \) is determined with the aid of Eq. (2.1) and

\[ \partial_p \gamma_q = \Gamma^i_q \gamma_i + [\Gamma^i_p, \gamma_i] \]

in which

\[ \begin{align*}
\Gamma^i_q & = (1/2) g^{pq} (\partial_p g_{qr} + \partial_q g_{rp} - \partial_r g_{pq}), \\
\Gamma^i_p & = - (1/2) \{ (1/2) C_{rp}^q \gamma^r + E^a_{rp} \gamma^a + E^a_{rp} \gamma^a + E^a_{rp} \gamma^a \} \quad (2.5) \end{align*} \]

where

\[ C_{rp}^q = E_{rq}^p - \Gamma_{rp}^q, \quad C_{rp}^q g_{ts} = - C_{qp}^r g_{ts}. \quad (2.6) \]

Here we emphasize that spin affine connection \( \Gamma^i_p \) can be expressed without use of \( E^a_{rp} \), etc., in the following form,

\[ \Gamma^i_p = (1/4) g^{st} \Gamma^t_s \gamma_{st} - (1/16) \{ \gamma^a \partial_r \gamma^a + \gamma^a \partial_r \gamma^a - (1/2) \gamma^a \partial_r \gamma^a \} + (1/8) \Gamma^a_{rp} \gamma^a \gamma^a \gamma^a \quad (2.7) \]

by a straightforward, though tedious, calculation. The above \( \Gamma^i_p \) is the most general form of the usual traceless \( \Gamma^i_{p'} \)

\[ \Gamma^i_{p'} = - (1/8) (\partial_p \gamma_{rs} + \partial_r g_{sp} - \partial_s g_{rp}) \gamma^a. \quad (2.8) \]

If the spin affine connection (2.5) is substituted into the generalized Dirac equation

\[ \gamma^a (\partial_p - \Gamma^a_p) \phi = 0, \quad (2.9) \]

(2.9) reduces to

\[ \gamma^p \partial_p \phi - (1/2) \{ g^{pq} C_{pq}^r \gamma_r + g^{pq} E^a_{pq} \gamma_a + E_{pq}^a + E_{pq}^a \gamma^a + (1/2) E_{pq}^a \gamma^a \} \gamma_{rs} \]
Green interpreted that the imaginary part of $g^{pq}C_{pq}^r$ has the character of an electromagnetic field, that the imaginary part of $g^{pq}E_{pq}^r$ has the character of a pseudo-scalar meson field, and that the real part of $E_{pq}^r$ acts as a mass field.\(^*\)

Of course, there is no necessary reason to accept this interpretation. To ensure that the Lagrangian density be real, $(\partial^p - \Gamma^p_\mu)$ must satisfy the following equation, 

$$\partial^p \gamma + \gamma \Gamma^p_\mu + \Gamma^*\gamma = 0, \quad (2\cdot11)$$

where $\gamma$ is an hermitian matrix defining the adjoint spinor $\bar{\psi} = \phi^\dagger \gamma$ and satisfying $\gamma \bar{\gamma} + \bar{\gamma} \gamma = 0$.\(^*\) The equation $(2\cdot11)$ can also be written in the form

$$\partial^p \gamma = \gamma [ (1/2) I^p_r \gamma + I^*_{pq} \gamma + I^*_{pq} \gamma + I^*_{pq} \gamma ] \quad (2\cdot11')$$

in which $I^p_r$, $I^p_q$, $I^p$ and $I^p_i$ represent $i$ times the imaginary parts of $C_{rp}^r$, $E^{rp}_{pq}$, $E^r_p$ and $E^s_p$, respectively. Since the definition of the adjoint field depends on $I^p$, etc., it will be difficult to interpret that $g^{pq}I^p_r$ and $g^{pq}I^p_s$ have the usual properties of the electromagnetic field and pseudo-scalar meson field. Each term in $(2\cdot7)$ has not any physical meaning independently of each other and it will be difficult to separate the irreducible entities. In fact, we can show that the electromagnetic interaction term does not appear in the Lagrangian density for Dirac particle, though it appears in $(2\cdot10)$. (See the last equation of $(4\cdot1)$ below.) In order to introduce the electromagnetic effect we must then add the interaction term. On the contrary, as the real part of $E_{pq}^r$ does not enter into $(2\cdot11')$, we shall interpret it as a mass field.

Green has also constructed an explicit representation of the $\gamma_p$ by introducing the six legs and by using the 8–8 constant mutually anticommuting matrices. But, as we start with 4–4 Dirac matrix and Dirac equation without a constant mass term, it is natural to use the four-dimensional representation instead of eight dimensional one. In the next section, this problem will be discussed in detail by introducing "sedenion conjugate".

§ 3. Introduction of legs connection and four-dimensional representation of $\gamma_p$

We shall first construct a representation of the matrices whose derivatives are defined by $(2\cdot1)$.

If we introduce $\beta^{\pm R}$ such as

$$\beta^p = -i\beta^R = \gamma^0, \quad \beta^{pq} = -\beta^{pq} = -\beta^{pq} = \gamma^1$$

the spin affine connection $\Gamma_p$ given by $(2\cdot7)$ can be written in the form

\[\]
To construct the four-dimensional representation of $\gamma_{\mu}$, it is instructive to remember the two-component theory of the general relativistic Dirac equation. As will be easily seen, the usual spin affine connection defined by (2·8) is expressed in the two-component theory as

$$\Gamma'_{\mu} = -(1/8) \left( \partial_{\mu} \sigma_{(a)} + \partial_{\mu} \sigma_{(a)} + (1/32) L \right) \sigma^{(a)}$$

where

$$\sigma^{(a)} = (1/2) \left( \sigma_{(a)}^{(a)} - \sigma_{(a)}^{(b)} \right), \quad \sigma_{(a)} = e^{(a)(a)} \sigma_{(a)}^{(a)}$$

in which $\sigma_{(a)}^{(a)}$ means the "quaternion conjugate" of $\sigma_{(a)}$ i.e. $\sigma_{(a)}^{(a)} = -\sigma_{(a)} = -E_{2}$, $\sigma_{(a)}^{(a)} = -E_{2}$ for $\gamma = 1, 2, 3$. Here and in what follows we use the Greek letters from the latter part of the alphabet for the range 1, 2, 3, 4, 5, 6, $\alpha$ and $\beta$ for the range 1, 2, 3, 4, and $\gamma$ for the range 1, 2, 3. $\sigma_{(a)}$ and four legs $e_{(a)}^{(a)}$ satisfy the following relations,

$$e_{(a)}^{(a)(a)} e_{(a)}^{(b)} = 1.$$ 

Since we take $x^{4} = t$ and $\gamma_{(4)} = -1$, it should be noted that our $\sigma_{(4)}$ plays the role of $\sigma_{(0)}$ in the usual formalism.

To achieve the four-dimensional representation of Eq. (3·1) impels us to introduce "sedenion conjugate" defined by

$$\tilde{\gamma}_{(a)} = \tilde{\gamma}_{(a)} = -E_{2}, \quad \gamma_{(a)} = -\gamma_{(a)}$$

where

$$\gamma_{(a)}^{(a)} = \rho_{(a)} \sigma_{(a)}^{(a)}, \quad \gamma_{(a)}^{(b)} = \rho_{(a)} \sigma_{(a)}^{(b)}$$

in which $\rho_{(a)}$, $\sigma_{(a)}^{(a)}$ are Pauli matrices. These matrices therefore satisfy

$$\tilde{\gamma}_{(a)}^{(b)} \gamma_{(a)}^{(b)} + \gamma_{(a)}^{(b)} \tilde{\gamma}_{(a)}^{(b)} = -2\gamma_{(a)}^{(b)}$$

where

$$\gamma_{(1)} = \gamma_{(22)} = \gamma_{(33)} = -\gamma_{(44)} = -1.$$ 

The legs indices are lowered and raised by means of this $\gamma_{(a)}^{(b)}$ and $\gamma^{(a)}_{(b)}$ ($=\gamma^{(a)}_{(b)}$) respectively. Further, we put

$$\tilde{\gamma}_{(a)}^{(b)} = (1/2) [\tilde{\gamma}_{(a)}^{(b)} \gamma_{(a)}^{(b)} - \gamma_{(a)}^{(b)} \tilde{\gamma}_{(a)}^{(b)}].$$

One can verify the following relations which are useful for later considerations,

$$[\gamma_{(a)}^{(b)}, \gamma_{(c)}^{(d)}] = \gamma_{(a)}^{(b)} \gamma_{(c)}^{(d)} - \gamma_{(c)}^{(b)} \gamma_{(a)}^{(d)} + \gamma_{(a)}^{(d)} \gamma_{(c)}^{(b)} - \gamma_{(c)}^{(d)} \gamma_{(a)}^{(b)}$$

$$\{\gamma_{(a)}^{(b)}, \gamma_{(c)}^{(d)}\} = 2(\gamma_{(a)}^{(b)} \gamma_{(c)}^{(d)} - \gamma_{(c)}^{(b)} \gamma_{(a)}^{(d)} + \epsilon_{(a)(d)(b)(c)} \gamma_{(e)}^{(f)}).$$
where $\varepsilon_{(\mu\nu\rho\tau\sigma)}$ is the antisymmetrical tensor of the sixth rank and has the value $\varepsilon_{(123456)} = i$.

As mentioned in the last section, the appearance of complex coefficients $E^{\nu}_{\rho}$, etc., makes it complicated to construct $\gamma$ and to interpret their physical meanings. Therefore, we shall first start with the real six legs $h_{\alpha}^\rho (A=1, 2, 3, 4, 5, 6)$ defined by

$$h_{\alpha}^A h_{\rho}^A = g^{AB}, \quad h_{\alpha}^A h_{\beta}^A = \gamma_{(\alpha\beta)}$$  \hspace{1cm} (3·10)

where

$$g^{AB} = g^{\nu\rho} \text{ for } A, B = 1, 2, 3, 4$$

$$g_{\alpha\beta} = g^{\rho\nu} = 0, \quad g^{\alpha\beta} = g^{\rho\nu} = 1.$$

We shall now introduce the legs connection $u_{\rho}^{(\nu)}$. For this purpose we transfer in parallel the legs at one point $P$ to a neighbouring point $P'$ by means of affine connection $\Gamma_{\rho\nu}^\sigma$ of the Riemannian space. These transferred legs will in general differ infinitesimally from the local legs at $P'$. It will be a simple assumption about this difference that it consists of an infinitesimal rotation in the six-dimensional pseudo-Euclidean space

$$L^{(\nu)} = \partial^{(\nu)} + u_{\rho}^{(\nu)} dx^\rho$$

where $dx^\rho$ is the displacement $PP'$. The $u_{\rho}^{(\nu)}$ is the legs connection we are seeking for. Accordingly, one finds

$$\partial_\nu h_{\rho}^{(\nu)} + \Gamma_{\rho\nu}^\sigma h_{\sigma}^{(\nu)} + u_{\rho}^{(\nu)} h_{\sigma}^{(\nu)} = 0$$  \hspace{1cm} (3·11)

$$\partial_\nu h_{\alpha}^{(A)} + u_{\rho}^{(\nu)} h_{\alpha}^{(A)} = 0 \text{ for } A = 5, 6. \hspace{1cm} (3·12)$$

We can find a representation of $\gamma^\rho$ with the help of the real six legs $h_{\alpha}^\rho$ and $\gamma_{(\nu)}$. Let

$$\gamma^\rho = h_{\rho}^{(\nu)} h_{\nu}^{(\sigma)} \gamma_{(\nu\sigma)}.$$  \hspace{1cm} (3·13)

From (3·9) and (3·10) it follows that

$$\{\gamma^\rho, \gamma^\sigma\} = 2g^{\rho\sigma}$$  \hspace{1cm} (3·14)

and other required anticommutation relations.

Also, by virtue of (3·11) and (3·12), one obtains

$$\partial_\nu \gamma^\rho - \Gamma_{\rho\nu}^\sigma \gamma^\sigma + h_{\alpha}^{(A)} u_{\rho}^{(\nu)} h_{\nu}^{(\sigma)} \bar{\alpha}_{\alpha} \alpha_{\beta} = h_{\alpha}^{(A)} u_{\rho}^{(\nu)} h_{\nu}^{(\sigma)} \bar{\alpha}_{\beta} \alpha_{\alpha} = 0$$  \hspace{1cm} (3·15)

where

$$\bar{\alpha}^A = h_{\alpha}^{(A)} \bar{\gamma}^{(A)}.$$

Putting

$$C_{\rho\nu}^{\sigma} = - h_{\rho}^{(\nu)} h_{\sigma}(\nu), \quad E_{\rho}^{\nu} = - h_{\sigma}^{(\nu)} h_{\rho}(\nu), \quad E_{\rho}^{\nu} = - h_{\sigma}^{(\nu)} h_{\rho}(\nu).$$  \hspace{1cm} (3·16)
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\[
E_q^\alpha = E_q^\beta = -h^\alpha_{(\mu)} h^\beta_{(\nu)} \quad u_q^{(\alpha\nu)} = h^\alpha_{(\mu)} h_{\mu(\rho)} u_q^{(\alpha\rho)},
\]

we obtain (2.1), i.e.

\[
\partial_q \gamma_p = E_p^{\alpha} \gamma_\alpha + E_p^{\nu} \gamma_\nu + E_p^{\beta} \gamma_\beta.
\]

Furthermore, it is a matter of course that the spin affine connection \( \Gamma_p \) defined by (2.5) takes a form of

\[
\Gamma_p = (1/2) u_p^{(\alpha\nu)} \gamma_\alpha \gamma_\nu
\]

by virtue of (3.16), as it should be.

Finally, we shall show that our \( C_p^{q \nu}, E_p^{\alpha} \), etc., satisfy the integrability condition (2.4). The integrability condition of (3.11) and (3.12) is

\[
\partial_\mu u_q^{(\alpha\nu)} + \gamma^{(\rho\sigma)} u_\mu^{(\rho\sigma)} u_q^{(\alpha\nu)} = (1/2) R^{\kappa\lambda\mu\nu} h_\kappa^{(\lambda)} h_\mu^{(\nu)}
\]

where \( R^{\kappa\lambda\mu\nu} \) is the Riemann-Christoffel curvature tensor. Of course, in the case of the integrability condition of (3.12), the right-hand side of (3.18) must be put to equal to zero, because the condition has a meaning only when it operates on \( h^{(\lambda)} (A=5, 6) \). Multiplying \( h_{\lambda(\omega)} h_\mu^{(\nu)} \), and contracting with regard to \( \mu \) and \( \nu \), we obtain

\[
C_p^{q \nu} + C_p^{q \nu} + E_p^{\alpha} E_p^{\alpha} + E_p^{\alpha} = (1/2) R^{\kappa\lambda\mu\nu}
\]

after putting \( s' = s, t' = t \), where \( \gamma^{(\rho\sigma)} \) means the covariant derivatives with respect to the Christoffel affinity \( \Gamma_\gamma^{(\nu)} \).

Similarly,

\[
E_p^{\beta} + E_p^{\beta} = 0
\]

Eqs. (3.19) and (3.19') are nothing but Eq. (2.4).

§ 4. Field equation derived from the variation of legs connection

From the analysis in the last section, we see that the legs connection \( u_p^{(\alpha\nu)} \) plays an important role in the presence of matter field. We shall therefore take \( h^{(\lambda)} (A=5, 6) \) as independent field variables. Taking the variation with regard to \( h_\lambda^{(\omega)} \), we get the usual equation for the gravitational field. Then, we shall consider only the variation with respect to \( u_p^{(\alpha\nu)} \).

We start with the Lagrangian

\[
L = \int \sqrt{-g} R d\tau + \int \sqrt{-g} L_0 d\tau + \int \sqrt{-g} L_{int} d\tau
\]

where

\[
R = 2h^{\alpha}_{(\omega)} h^{\beta}_{(\nu)} \left\{ \partial_\mu u_q^{(\alpha\nu)} + \gamma^{(\rho\sigma)} u_\mu^{(\rho\sigma)} u_q^{(\alpha\nu)} \right\}
\]

\[
L_0 = \kappa \left\{ \mathcal{L} h^{(\lambda)} \mathcal{L} h^{(\nu)} \gamma^{(\rho\sigma)} \partial_\mu \mathcal{L} - \partial_\mu \left[ \mathcal{L} h^{(\lambda)} \mathcal{L} h^{(\nu)} \gamma^{(\rho\sigma)} \right] \right\}
\]

\[
(4.1)
\]
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\[ L_{\text{int}} = - \left( \frac{\kappa}{2} \right) \overline{\psi} \{ \gamma^{(\mu\nu)}, \gamma^{(\rho\sigma)} \} \psi \mathcal{H}^{(\mu)} \mathcal{H}^{(\nu)} u_{\mu(\nu)} \]

\[ = \kappa u_{\mu(\nu)} [ \bar{h}^{(\mu)} h^{(\nu)} - h^{(\mu)} \bar{h}^{(\nu)} ] \psi \phi - \left( \frac{\kappa}{2} \right) \mathcal{E}^{(\mu\nu\rho\sigma)} \bar{h}^{(\rho)} h^{(\sigma)} u_{\mu(\nu)} \psi \phi \]

In which \( \kappa \) is Einstein's gravitational constant and \( \psi = \phi^* \eta \) with \( \eta = i \gamma^{(\alpha)} \) which results from the assumption of the real property of \( \bar{h}^{(\mu)} \) and \( u_{\mu(\nu)} \).

The variation with regard to \( u_{\mu(\nu)} \) leads to the following relations:

\[ h^{(\mu)} \partial_{\mu} \bar{h}^{(\nu)} - h^{(\nu)} \partial_{\mu} h^{(\mu)} = h^{(\mu)} u_{\mu(\nu)} h^{(\nu)} - h^{(\nu)} u_{\mu(\nu)} h^{(\mu)} \]

\[ + \kappa (h^{(\mu)} h^{(\nu)} - h^{(\nu)} h^{(\mu)}) \overline{\psi} \phi - \left( \frac{\kappa}{2} \right) \mathcal{E}^{(\mu\nu\rho\sigma)} \bar{h}^{(\rho)} h^{(\sigma)} \psi \phi \]

(4.2)

On the other hand, by eliminating \( \Gamma^{(\mu)} \) from Eq. (3.11), we directly get

\[ h^{(\mu)} \partial_{\mu} \bar{h}^{(\nu)} - h^{(\nu)} \partial_{\mu} h^{(\mu)} = h^{(\mu)} u_{\mu(\nu)} h^{(\nu)} - h^{(\nu)} u_{\mu(\nu)} h^{(\mu)} \]

(4.3)

instead of (4.2). Thus, between (4.2) and (4.3), there appears a difference by the terms which contain spinor field. Accordingly, to remove this disaccordance one must modify Eq. (3.11) defining legs connection.

As a simple modification, we postulate that

\[ \partial_{\mu} h^{(\mu)} + \Gamma^{(\mu)}_{\nu \rho} h^{(\nu)} + u_{\mu(\nu)} h^{(\nu)} + o_{\mu(\nu)} h^{(\nu)} = 0 \]

(4.4)

with

\[ o_{\mu(\nu)} = - \kappa (h^{(\mu)} h^{(\nu)} - h^{(\nu)} h^{(\mu)}) \overline{\psi} \phi \]

\[ + \left( \frac{\kappa}{4} \right) \mathcal{E}^{(\mu\nu\rho\sigma)} \phi (\nu) \phi (\rho) \psi \phi \]

(4.5)

It is readily shown that if we eliminate \( \Gamma^{(\mu)} \) from (4.4) the relation (4.2) is derived. Similarly, eliminating \( u_{\mu(\nu)} \) and \( o_{\mu(\nu)} \) we obtain

\[ \partial_{\mu} \phi^{(\mu)} + \Gamma^{(\mu)}_{\nu \rho} \phi^{(\nu)} + \Gamma^{(\mu)}_{\rho \nu} \phi^{(\rho)} = 0. \]

As we now regard Eq. (4.4) as defining the legs connection, the integrability condition (3.18) must be modified. Since the new condition has a very complicated form, we do not give it here. Multiplying this condition by \( \bar{h}^{(\mu)} h^{(\nu)} \) and contracting with regard to \( \mu, \nu, \) and \( \mu, \nu, \) we get the following rather simple expression by a straightforward but tedious calculation,

\[ R = 2 \bar{h}^{(\mu)} h^{(\nu)} \{ \partial_{\nu} u_{\mu(\nu)} + \gamma_{(\nu), \rho, \mu, \sigma} u_{\mu(\rho)} u_{\mu(\sigma)} \}
\]

\[ - 12 \kappa \left( \frac{\kappa}{2} \right) \overline{\psi} \phi \cdot \overline{\psi} \phi + \left( \kappa^2 / 16 \right) \mathcal{E}^{(\rho\sigma\mu\nu)} \mathcal{E}^{(\rho\sigma\nu\mu)} \phi (\rho) \phi (\sigma) \psi \phi \]

\[ \times \gamma_{(\rho\sigma), \nu, \mu} \overline{h}^{(\rho)} h^{(\nu)} \overline{h}^{(\mu)} h^{(\sigma)} \overline{h}^{(\mu)} \overline{h}^{(\nu)} h^{(\rho)} h^{(\sigma)} \phi (\rho) \phi (\sigma) \psi \phi \]

(4.6)

Further, it should be noticed that the interaction term in (4.1) must also be modified due to the addition of \( o_{\mu(\nu)} \) to the legs connection \( u_{\mu(\nu)} \). This additional interaction term is

\[ L_{\text{int}}(\text{add}) = - 8 \kappa \left( \frac{\kappa}{2} \right) \overline{\psi} \phi \cdot \overline{\psi} \phi - \left( \frac{\kappa^2}{8} \right) \mathcal{E}^{(\rho\sigma\mu\nu)} \mathcal{E}^{(\rho\sigma\nu\mu)} \gamma_{(\rho\sigma), \nu, \mu} \times \overline{h}^{(\mu)} h_{\mu(\rho)} h^{(\nu)} h_{\nu(\sigma)} \overline{\psi} \phi \cdot \overline{\psi} \phi \]

\[ \times \overline{h}^{(\rho)} h_{\rho(\nu)} h^{(\sigma)} h_{\sigma(\mu)} \overline{\psi} \phi \cdot \overline{\psi} \phi \]

(4.7)
Thus, by taking into account Eqs. (4.6) and (4.7), our Lagrangian reduces to

\[ L = \int \sqrt{-g} R d\tau + \int \sqrt{-g} L_0 d\tau + \int \sqrt{-g} L_{int} d\tau + F \]  
(4.8)

with

\[ F = -20\kappa^2 \overline{\psi} \psi \cdot \overline{\psi} \psi + F' \]  
(4.9)

in which

\[ F' = -\frac{\kappa^2}{16} \varepsilon^{(p\sigma \rho \tau \xi)} \varepsilon^{(p' q \sigma \rho' \tau' \xi')} \eta_{(p\rho') \eta_{(q\rho') \eta_{(\rho \tau') \eta_{(\xi \tau')}}} h_{(\rho')} h_{(\xi') \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')}} \]  
(4.9')

\[ = -\frac{3}{4} \kappa^2 \{ \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \} . \]  
(4.9'')

\[ F \]  
(4.9'')

\[ F \]  
(4.9''')

\[ F \]  
(4.9'''')

In other words, it can be said that our result (4.9) is an expansion of \(-17\kappa^2 \overline{\psi} \psi \cdot \overline{\psi} \psi\) in terms of \(\overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')}\).

Recently, Kita has advocated that \(sp\Gamma_p\) can include the terms \(\overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')}\) from the viewpoint of the spinor unified theory. Our result shows that this proposition is a necessary conclusion based on the consideration that \(u_{(\rho'\sigma')}(\xi')\) should be independent field variables subject to the variational principle. Moreover, our result involves the most general type of bilinear forms of \(\overline{\psi}\) and \(\psi\).

So far we have considered only one sort of Dirac particle. In the case where two Dirac particles described by \(\psi\) and \(\varphi\) coexist, the following Fermi-Fermi interaction term \(F_{int}\) is added to (4.8),

\[ F_{int} = -20\kappa^2 \{ \overline{\psi} \psi \cdot \overline{\psi} \varphi \psi \} \left[ \frac{3}{4} \kappa^2 \bigl\{ \{ \overline{\psi} \eta_{(\xi')} \psi, \eta_{(\xi')} \psi \bigr\} + 2h^{(s)}_{(\xi')}h^{(s)}_{(\xi')} \{ \overline{\psi} \eta_{(\xi')} \psi, \eta_{(\xi')} \psi \} \bigr\} \right] . \]  
(4.10)

If we interpret the above result in the usual four legs formalism by regarding \(h^{(s)}_{(\xi)}\) and \(h^{(s)}_{(\xi)}\) as scalar quantities, the last term in (4.10) contains the parity non-conserving terms.

The coupling constant of the above Fermi-Fermi interaction is very small. In order to introduce the coupling constant larger than the gravitational constant, we shall try to introduce electromagnetic-type and ps-\(ps\) meson-type interaction term. As a simple interaction term, we take

\[ \overline{L}_{int} = i\kappa h_{(\xi')}^{(s)} \overline{\psi} \eta_{(\xi')} \overline{\psi} \eta_{(\xi')} \{ \eta_{(\xi')} \eta_{(\xi')} \psi, \eta_{(\xi')} \eta_{(\xi')} \psi \} . \]  
(4.10')

* In the four legs formalism, we obtain Weyl's result, \(F_w = -\frac{3}{2} \kappa^2 \overline{\psi} \eta^{(s)} \psi \cdot \overline{\psi} \eta^{(s)} \psi\).
where $\tilde{\kappa}$ is a new interaction constant which need not be equal to $\kappa$. Here, $i$ must be multiplied in order to guarantee the hermitian property of the interaction Lagrangian density, because we have used the real legs $h_{\mu}^{(p)}$ and real legs connection $u_{\mu}^{(p)}$. At first sight, the introduction of the large coupling constant $\tilde{\kappa}$ raises a question whether the new interaction term (4·11) gives a large gravitational effect between the Dirac particles. However, we can show that it gives the second order gravitational effect, as seen from the linearized theory of gravitation where the term containing $u_{\mu}^{(p)}$ gives the higher order effect.

Putting

$$L_{int} = -2i\tilde{\kappa}h_{\mu}^{(p)}\bar{\psi}\gamma_{(\mu)}\psi + h_{\mu}^{(p)}$$

we can express (4·11) as

$$\tilde{L}_{int} = 2i\tilde{\kappa}\bar{\psi}\gamma_{(p)}\psi + 2i\tilde{\kappa}\bar{\psi}\gamma_{(p)}\psi.$$  

(4·11')

It should be noted that our procedure of introduction of $V^p$ and $\phi$ is different from Green's.

The introduction of new interaction term (4·11) involves the modification of Eq. (4·4) defining $u_{\mu}^{(p)}$ by similar arguments. (4·11) should be modified as

$$\partial_{\mu} h_{\mu}^{(p)} + G_{\mu}^{(p)} h_{\mu}^{(p)} + \omega_{\mu}^{(p)} h_{\mu}^{(p)} + \bar{\omega}_{\mu}^{(p)} h_{\mu}^{(p)} = 0$$

(4·12)

with

$$\bar{\omega}_{\mu}^{(p)} = -i\tilde{\kappa}(h_{\mu}^{(p)} h_{\mu}^{(p)} \bar{\psi}^{(p)} \gamma_{(p)} \psi - h_{\mu}^{(p)} h_{\mu}^{(p)} \bar{\psi}^{(p)} \gamma_{(p)} \psi).$$

The additive interaction term $\widetilde{F}$ which corresponds to (4·8) can also be found as

$$\widetilde{F} = 4\tilde{\kappa} h_{\mu}^{(p)} h_{\mu}^{(p)} \bar{\psi}^{(p)} \gamma_{(p)} \psi h_{\mu}^{(p)} h_{\mu}^{(p)} \bar{\psi}^{(p)} \gamma_{(p)} \psi,$$

(4·13)

in the derivation of which we have taken into account the relation

$$\partial_{\mu}(\sqrt{-g} h_{\mu}^{(p)} h_{\mu}^{(p)} \bar{\psi}^{(p)} \gamma_{(p)} \psi) = 0.$$  

The above result (4·13) may be considered to suggest that the vector field $V^p$ and pseudoscalar field $\phi$ play a role of intermediate boson whose second order effects are usual weak interactions.

§ 5. Extension to the complex legs

We have seen that $E_{\mu}^{p}$, etc., are real quantities so far as real legs are adopted. In order to obtain the complex $E_{\mu}^{p}$ etc., we must introduce complex legs $h_{\mu}^{(p)}$ at each point of space. This implies that the metric tensor $g^{\mu}$ of the space is no longer real symmetric tensor of Riemannian geometry, but it becomes complex and hermitian.

On the analogy of the complex four legs formalism of Sciama, we shall
introduce the complex six legs $h_\alpha^\beta$, hermitian connection $\Gamma^\tau_\rho_\sigma$ and antihermitian legs connection $u_\rho^\beta_\sigma$. They are connected by the equations similar to (3.11) and (3.12):

\[
\begin{align*}
\partial_\nu h_\rho^\tau_\sigma + \Gamma^\nu_\rho_\sigma h_\nu^\tau_\sigma + u_\rho^\tau_\sigma h_\sigma^\nu_\tau = 0 \\
\partial_\nu h_\alpha^\beta + u_\alpha^\beta h_\nu^\tau_\nu = 0 \quad \Lambda = 5, 6.
\end{align*}
\]

The metric tensor is defined by

\[
\begin{align*}
g^{ij} &= \delta^{ij}, \quad g^{ab} = -\delta^{ab} = 1 \\
g^{\alpha\beta} &= h_\alpha^\beta h_\beta^\alpha = 0, \quad g^{ab} = h_a^c h_b^d h_c^a h_d^b = 0.
\end{align*}
\]

Using the antihermitian property of $u_\rho^\beta_\sigma$, we can eliminate it and obtain

\[
g^{ij}_{\alpha\beta} = \partial_k g^{ij} + g^{ik} \Gamma^k_\nu + g^{jk} \Gamma^j_\nu = 0
\]

which is the well-known equation in Einstein’s unified theory.\(^9\)

We can also eliminate $\Gamma^\nu_\rho_\sigma$ by using its hermitian property

\[
h_\rho^\tau_\sigma h_\alpha^\beta u_{\nu(\rho)} - h_\rho^\tau_\sigma h_\alpha^\beta u_{\nu(\rho)} = h_\alpha^\beta \partial_\nu h_\rho^\tau_\sigma - h_\rho^\tau_\sigma \partial_\nu h_\alpha^\beta.
\]

We shall now consider variational equation with respect to $u_\rho^\beta_\sigma$. As a Lagrangian for the gravitational field without Dirac particle, we take

\[
L = \sqrt{-g} \ h_\nu^\beta_\nu \ U_\nu^{\alpha(\nu)}(\partial_\nu u_\nu^{\tau(\nu)} + \eta_{(\nu)}^{\tau(\nu)} u_\nu^{\alpha(\nu)} u_\nu^{\tau(\nu)}) d\tau.
\]

The legs connection $u_\rho^\beta_\sigma$ can be decomposed into irreducible constituents

\[
u_\rho^\beta_\sigma = u_{\rho(\nu)}^{\alpha(\nu)} + u_{\rho(\nu)}^{(1)(\nu)} + u_{\rho(\nu)}^{(2)(\nu)}
\]

where

\[
u_{\rho(\nu)}^{(1)(\nu)} = (1/6) \eta_{(\nu)}^{(\nu)} u_{\rho(\nu)}^{(\nu)} \quad \text{and} \quad u_{\rho(\nu)}^{(2)(\nu)} = u_{\rho(\nu)}^{(\nu)} - u_{\rho(\nu)}^{(1)(\nu)}
\]

in which $u_{\rho(\nu)}^{(\nu)} = \nu_{(\nu)}^{(\nu)} u_{\rho(\nu)}^{(\nu)}$ and $\nu$ or $\sqrt{\nu}$ under a couple of indices means symmetric or antisymmetric part of that couple. For the sake of simplicity, we omit here the spurless part $u_{\rho(\nu)}^{(3)(\nu)}$. The variation with respect to $u_{\rho(\nu)}^{(\nu)}$ leads to

\[
\begin{align*}
&\left[ -h_\nu^\tau_\nu (\partial_\nu h_\nu^\tau_\nu + \Gamma_\nu^\nu h_\nu^\tau_\nu + u_{\nu(\nu)} h^{(\nu)} h_\nu^\tau_\nu + h_\nu^\tau_\nu (\partial_\nu h_\nu^\tau_\nu + \Gamma_\nu^\nu h_\nu^\tau_\nu + u_{\nu(\nu)} h^{(\nu)} h_\nu^\tau_\nu) \right] \left( \partial_\nu h_\nu^\tau_\nu + \Gamma_\nu^\nu h_\nu^\tau_\nu + u_{\nu(\nu)} h^{(\nu)} h_\nu^\tau_\nu \right) \\
&+ \left[ -h_\nu^\tau_\nu (\partial_\nu h_\nu^\tau_\nu + \Gamma_\nu^\nu h_\nu^\tau_\nu + u_{\nu(\nu)} h^{(\nu)} h_\nu^\tau_\nu + h_\nu^\tau_\nu (\partial_\nu h_\nu^\tau_\nu + \Gamma_\nu^\nu h_\nu^\tau_\nu + u_{\nu(\nu)} h^{(\nu)} h_\nu^\tau_\nu) \right] \left( \partial_\nu h_\nu^\tau_\nu + \Gamma_\nu^\nu h_\nu^\tau_\nu + u_{\nu(\nu)} h^{(\nu)} h_\nu^\tau_\nu \right)
\end{align*}
\]

\[
-\left[ \nu \text{ and } \mu \text{ interchanged} \right] = 0.
\]

If $\Gamma = \left( \Gamma^\nu_\nu h_\nu^\tau_\nu + \Gamma^\nu_\nu h_\nu^\tau_\nu + h_\nu^\tau_\nu \right) - \left( \nu \text{ and } \mu \text{ interchanged} \right)$ were absent, (5.8) would imply that (5.5) holds. Therefore it can be said that Eq. (5.1) is derived from the variation principle. However, this is not the case.

The simplest way to remove $\Gamma$ is to put $\Gamma^\nu_\nu = 0$ as was done by Einstein.\(^9\)

This implies also that $\partial_\nu (\nu \nu) = 0$, where $\nu \nu = \sqrt{-g} g^{\nu\nu}$, as easily seen from (5.4).
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\[ \partial_i (g^\mu_\nu) = 0 \]

is nothing but the equation derived from the Lagrangian (5·6) by variation with respect to \( u^{(sp)}_\mu \). Sciama interpreted that \( F_{\mu\nu} = \partial_\mu u^{(sp)}_\nu - \partial_\nu u^{(sp)}_\mu \) (see (5·10) below) represents the electromagnetic field. Then the vanishing of \( \Gamma^r_{\mu\nu} \) means that there are no electromagnetic fields in the Einstein theory. Accordingly, this procedure may be not appropriate.

Therefore, in order to obtain non-vanishing electromagnetic effect, we shall now eliminate the term \( \Gamma^r \) by adding

\[
L_{\text{add}} = -\int \sqrt{-g} \Gamma^r_{\nu} u^{(sp)}_\mu (h^a_\mu h^b_\nu + h^b_\mu h^a_\nu) d\tau
\]

(5·9)
to Lagrangian (5·6).

From the relation (5·1) we have

\[ \Gamma^r_{\nu} = - u^{(sp)}_\mu - i \operatorname{Im} \partial_\mu \log \varepsilon \]

(5·10)

where

\[ \partial_\mu \log \varepsilon = h^a_\mu \partial_\mu h^a_\nu. \]

If we add \( \frac{1}{2} \int \sqrt{-g} u^{(sp)}_\mu u^{(sp)}_\nu g^{\mu\nu} d\tau \) to the Lagrangian density, the variation with respect to \( u^{(sp)}_\mu \) yields

\[ (1/3) \partial_i (g^\mu_\nu) + \sqrt{-g} u^{(sp)}_\nu (h^a_\mu h^b_\nu + h^b_\mu h^a_\nu) = 0 \]

(5·11)

which implies

\[ \partial_i (1/2) \partial_\mu \log A^\mu = 0 \]

(5·12)

with

\[ A^\mu = (1/2) \operatorname{Im} (u^{(sp)}_\mu h^a_\nu h^a_\nu). \]

(5·12) is the Lorentz condition for the vector potential \( A^\mu \) in the curved space. This suggests that we can interpret \( A^\mu \) tentatively as the electromagnetic four-potential. As mentioned in § 2, Green has interpreted that

\[ \bar{A}^\mu = (1/2) \operatorname{Im} (g^{\mu\nu} C^\nu_\rho) \]

has the character of an electromagnetic four-potential. Using (3·16) and (5·3), we can easily show that \( A^\mu \) is nothing but \( \bar{A}^\mu \) artificially introduced by Green. It is interesting that our electromagnetic potential based on the complex legs is the same as Green’s.

There remains the problem of formulating the interaction of Dirac particle and complex legs connection in the hermitian and complex metric space. As seen from Eq. (2·11), in the case of complex legs the interaction term \( \bar{\psi} (\gamma^a \Gamma^a_\nu + \Gamma^a_\nu \gamma^a) \psi \) is not hermitian and \( \gamma \) has a complicated form. Then the discussion in the last section does not hold without any radical modification, because one must add a non-hermitian additive interaction Lagrangian to get rid of the discrepancy con-
cerning the variation equation and (5·1). Therefore for the Dirac particle we shall use the physical real legs $H'_{(\alpha)}$ defined by

$$h^{(\alpha)} h'^{\alpha} + h'^{\alpha} h^{(\alpha)} = 2 \Re g^{ij} = 2 H'_{(\alpha)} H'^{(\alpha)}.$$  \hspace{1cm} (5·13)

The electromagnetic interaction term is

$$I_{E.M.}^{\text{imp}} = (1/2) \bar{\psi} H^{(\alpha)} H'^{(\alpha)} \gamma_{(\alpha \beta)} \bar{\psi} \cdot \Im (u_{\mu}^{(\alpha \beta)} h'^{\mu} h^{\nu}) \gamma_{(\nu \sigma)}.$$ \hspace{1cm} (5·14)

The discussion concerning this problem will be made on another occasion.

§ 6. The relation between the spin transformation and Pauli transformation

To discuss Pauli-Gürsey type transformation, it may be appropriate to start with the Gürsey's representation of spinor equation. The use of Gürsey's representation is not essential but instructive. Gürsey introduced the $2 \times 2$ wave matrix $\mathcal{F}(\psi)$ for $\psi$

$$\mathcal{F} = \mathcal{F}(\psi) = (\psi^+, (\psi^+)\tau_3)$$ \hspace{1cm} (6·1)

where $\psi^+ = (1 + P)/2 \cdot \psi$ and $\psi^+$ is the charge conjugate of $\psi$ in the usual theory. From the above definition, it is easily seen that

$$\mathcal{F}(i\psi) = i\mathcal{F}(\psi) \tau_3,$$

$$\mathcal{F}(\gamma \psi) = \sigma(\gamma) \mathcal{F}(\psi) \tau_3,$$

$$\mathcal{F}(\rho_1 \psi) = \mathcal{F}(\psi) \tau_3,$$

$$\mathcal{F}(\rho_2 \psi) = i \mathcal{F}^* (\psi),$$

$$\mathcal{F}(\rho_3 \psi) = \mathcal{F}(\psi),$$

where $\tau_3$ is a new set of Pauli matrices and $\mathcal{F}^*$ means the quaternion conjugate of $\mathcal{F} = \sigma^{(4)} \psi_\alpha$, namely $\mathcal{F}^* = -\sigma^{(4)} \psi_\phi - \sigma^{(3)} \psi_\phi - \sigma^{(2)} \psi_\phi + \sigma^{(1)} \psi_\phi$ in which $\sigma^{(4)} = E_2$.* In what follows, $\sigma(\gamma)$ operates on $\mathcal{F}$ from the left while $\tau_3$ from the right.

We shall first express $\mathcal{F}(\Gamma_p \psi)$ in which $\Gamma_p$ is the spin affine connection defined by (3·17), i.e. $\Gamma_p = (1/2) u^{(\alpha \beta)} \gamma_{(\alpha \beta)}$. Using the relation (6·2), we get

$$\mathcal{F}(\Gamma_p \psi) = (1/2) u_\mu^{(\alpha \beta)} \mathcal{F}(u_\mu^{(\alpha \beta)} \gamma_{(\alpha \beta)} \mathcal{F} + u_\mu^{(\alpha \beta)} \gamma_{(\alpha \beta)} \mathcal{F} \tau_3$$

$$+ i u_\mu^{(\alpha \beta)} \mathcal{F}(u_\mu^{(\alpha \beta)} \gamma_{(\alpha \beta)} \mathcal{F} \tau_3 + i u_\mu^{(\alpha \beta)} \mathcal{F}(u_\mu^{(\alpha \beta)} \gamma_{(\alpha \beta)} \mathcal{F} \tau_3.$$ \hspace{1cm} (6·3)

On the right-hand side of (6·3) appear both $\mathcal{F}$ and $\mathcal{F}^*$. We shall then express the third and fourth terms in terms of $\mathcal{F}$. For this purpose it is convenient to introduce the fundamental basic vector defined by

$$\rho e^{(1)} = (1/2) s^{(\alpha \beta)} \mathcal{F}(u_\mu^{(\alpha \beta)} \gamma_{(\alpha \beta)} \mathcal{F} \tau_3 = (i/2) \left( \bar{\psi} \gamma_{(\alpha \beta)} \psi + \bar{\psi} \gamma_{(\alpha \beta)} \psi \right)$$

$$\rho e^{(2)} = (1/2) s^{(\alpha \beta)} \mathcal{F}(u_\mu^{(\alpha \beta)} \gamma_{(\alpha \beta)} \mathcal{F} \tau_3 = (1/2) \left( \bar{\psi} \gamma_{(\alpha \beta)} \psi - \bar{\psi} \gamma_{(\alpha \beta)} \psi \right)$$

$$\rho e^{(3)} = (1/2) s^{(\alpha \beta)} \mathcal{F}(u_\mu^{(\alpha \beta)} \gamma_{(\alpha \beta)} \mathcal{F} \tau_3 = i \bar{\psi} \gamma_{(\alpha \beta)} \psi.$$ \hspace{1cm} (6·4)

* In this section, we use the small Greek indices also to represent quantities with respect to the flat space-time coordinate system where $h^{(\alpha)} = \partial^{(\alpha)}$. See, for instance, Eq. (6·7) below.
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\[ \rho \epsilon_a^{(r)} = (1/2) sp (\mathcal{V}^+ \sigma_{(a)} \mathcal{V} \tau^4) = i\bar{\psi} \gamma_{(a)} \psi \]

where

\[ \mathcal{V} = \Omega + \bar{\Omega} \]

with

\[ \Omega = \frac{1}{2} sp \text{Re} (\bar{\mathcal{V}} \mathcal{V}) = \bar{\psi} \psi, \quad \bar{\Omega} = \frac{1}{2} sp \text{Im} (\bar{\mathcal{V}} \mathcal{V}) = i \bar{\psi} \gamma_{(a)} \psi. \]

\[ e_{(a)}^{(r)} \text{ satisfies the relations } \]

\[ e_{(a)}^{(r)} e_{a(\beta)} = \gamma_{(a\beta)}, \quad e_{a(\beta)} e_{a(\beta)} = \gamma_{a\alpha}. \] (6·5)

Using the relation for any 2\times2 matrix \( M \)

\[ M = - (1/2) sp (M \bar{\tau}_3) \tau^3 = -(1/2) sp (M \bar{\tau}_3) \varphi \]

we find

\[ \bar{\sigma}^{(a)} \bar{\mathcal{V}}^+ \bar{\tau}_3 = -(1/2) sp (\bar{\mathcal{V}} \sigma_{(a)} \bar{\mathcal{V}}^* \bar{\tau}_3) \bar{\tau}_3 = -(1/2) sp (\bar{\mathcal{V}}^* \sigma_{(a)} \mathcal{V} \tau^3) \mathcal{V}_3 \bar{\tau}_3 / (\bar{\mathcal{V}} \mathcal{V}) \]

\[ = - \frac{\Omega}{2 \Omega + i \bar{\Omega}} e_{(a)}^{(r)} \tau^3 \tau_3. \] (6·6)

In the Appendix, we interpret the meaning of the above equation by expressing it in terms of the usual spinor. Eq. (6·6) shows that we are inclined to consider the spin affine connection \( \sigma_{(a)}^{(a)} \gamma_{(a)} \) as a spin connection in the \( \tau \) space. We shall now show that such an interpretation is possible.

The Lagrangian density for Dirac particle without mass in Minkowski space

\[ L_E = (1/2) (\bar{\psi} \gamma_{(a)} \partial_\alpha \psi - \bar{\psi} \gamma_{(a)} \partial_\alpha \psi) \] (6·7)

can be expressed by Gürsey's wave matrix

\[ L_E = (1/2) sp (\mathcal{V}^+ \gamma_{(a)} \mathcal{V} \tau_3). \] (6·7')

The above density is invariant under the following rotation in \( \tau \)-space

\[ \mathcal{V}' = \mathcal{V} \exp \{(i \tau^a \theta_a) \tau^3 \} \] (6·8)

or

\[ \mathcal{V}' = \mathcal{V} \exp (i \tau^3 \bar{\theta}_a + \tau^2 \bar{\theta}_a + \tau^1 \bar{\theta}_a + i \tau^4 \bar{\theta}_a) \] (6·8')

where \( \theta_a, \bar{\theta}_a \) are real parameters. The transformation (6·8') can be decomposed into two transformations

\[ \mathcal{V}' = \mathcal{V} \exp (i \bar{\theta}_a) \] (6·9)

and

\[ \mathcal{V}' = \mathcal{V} \exp (i \tau^2 \bar{\theta}_a + \tau^3 \bar{\theta}_a + i \tau^4 \bar{\theta}_a). \] (6·10)

These equations are expressed in \( \phi \) representation as

\[ \psi' = \exp (- i \bar{\gamma}_{(a)} \bar{\phi}_a) \psi \] (6·9')

and

\[ \psi' = a \phi + b \phi_0, \quad \bar{\psi}' = a^* \bar{\phi} + b^* \bar{\phi}_0, \quad |a|^2 - |b|^2 = 1. \] (6·10')
The transformation (6·10) is isomorph to the rotation in the three-dimensional Lorentz space. If the spin affine connection is \( \Gamma^i_p = \frac{1}{2} \left( \frac{1}{3} C_{0_p i}^{\alpha} \gamma^p + E_{0_p i}^{\alpha} \gamma^5 \right) \) instead of (2·5), \( \gamma^p \) is expressed as

\[
\gamma^p = \sum_{\alpha=1}^{5} h^p_{(\alpha)} \gamma^{(\alpha)}
\]

linearly by five legs contrary to (3·13) where it is expressed quadratically by six legs. It seems interesting to us that the spin affine connection containing (6·6) brings about the non-linear character of representation and also introduces the rotation in the three-dimensional Lorentz space.

Let us replace the transformation (6·8) with the wider one derived from replacing the constant parameter \( \hat{\theta}_a \) by a set of arbitrary function of coordinates \( \hat{\theta}_a(x) \). By postulating the invariance of Lagrangian under this wider transformations, a new field \( E^\beta_p(x) \) is introduced through the expression\(^6\)

\[
\mathcal{F}_\mu \mathcal{T} = \partial_\mu \mathcal{T} - i \mathcal{T} \tau_\beta \gamma_\beta E^\beta_p(x) .
\]  

(6·11)

Thus we have seen that the legs connection \( \Gamma^{(\alpha)}_p \) is related to the above \( E^\beta_p(x) \) by

\[
\Gamma^{(\alpha)}_p = -\frac{\rho + i \hat{\omega}}{\rho} \mathbf{e}^{(\alpha)}_p E^\beta_p(x) .
\]  

(6·12)

in terms of \( \rho, \Omega, \hat{\Omega} \) and fundamental basic vectors \( \mathbf{e}^{(\alpha)}_p \).

Following the same procedure as above, we relate the legs connection \( \Gamma^{(\alpha)}_p \) appearing in the fourth term of (6·3) to \( F^\beta_p(x) \) by

\[
\Gamma^{(\alpha)}_p = -\frac{\rho + i \hat{\omega}}{\rho} \mathbf{e}^{(\alpha)}_p F^\beta_p(x) .
\]  

(6·13)

In this case, \( F^\beta_p(x) \) is introduced through the expression

\[
\mathcal{F}_\mu \mathcal{T} = \partial_\mu \mathcal{T} - i \mathcal{T} \tau_\beta F^\beta_p(x) .
\]  

(6·14)

by requiring that the Lagrangian density

\[
L_\mathcal{T} = \frac{1}{2} \left( \bar{\psi}^{(\alpha)}_T \partial \psi^{(\alpha)}_T + \bar{\psi}^{(\alpha)}_T \partial \psi^{(\alpha)}_T \right) = \frac{1}{2} \text{sp}(\mathcal{T}^\alpha \tau^{(\alpha)}_a \partial_a \mathcal{T})
\]  

(6·15)

is invariant under the transformation

\[
\mathcal{T}' = \mathcal{T} \exp (i \bar{\theta}_a(x)) .
\]  

(6·16)

When the parameters \( \theta_a \) do not depend on the coordinates, the transformation (6·16) is nothing but the Pauli transformation

\[
\psi' = \exp (-i \bar{\theta}_a(\psi)) \psi,
\]

\[
\phi' = e^{\psi} + d \bar{\psi} \psi, \quad \bar{\phi}' = e^{\psi} - d \bar{\psi} \psi, \quad |c|^2 + |d|^2 = 1.
\]  

(6·16')

In this case, the second transformation of (6·16') is isomorph to a rotation in the three-dimensional Euclidean space. Thus we have shown that each of spin affine connections \( \Gamma^{(\alpha)}_p \) and \( \Gamma^{(\alpha)}_p \) has an intimate relation with Pauli-like
transformation.

Unfortunately, there is no Lagrangian density left invariant under the combination of two transformations (6.8') and (6.16), though they preserve the field equation invariant. Therefore, in order to obtain the Lagrangian density which is invariant under the transformation corresponding to the spin affine connection $u^{(6)}_\mu \bar{\Gamma}^{(6)}_\mu + u^{(6)}_\mu \lambda (6_\mu \bar{\lambda})$, we are impelled to regard $\phi$ as an eight-component spinor and to generalize the Pauli transformation.

We introduce $8 \times 8$ matrices

$$
\Gamma^{(a)} = i\gamma^{(5)(a)}\omega_1, \quad \Gamma^{(b)} = \omega_2, \quad \Gamma^{(8)} = i\omega_3
$$

(6·17)

where $\omega_a$ is a set of Pauli matrices. The adjoint spinor $\bar{\varphi}$ and the charge conjugate $\varphi_c$ of $\varphi$ are defined by

$$
\bar{\varphi} = i\varphi^+ \Gamma^{(6)} \Gamma^{(5)} = -\varphi^+ \Gamma^{(6)} \Gamma^{(5)} \omega_3,
$$

(6·18)

$$
\varphi_c = C\times \omega_2 \bar{\varphi}^T.
$$

We shall take the Lagrangian density

$$
L = (1/2) \left[ \bar{\varphi} \Gamma^{(5)} \Gamma^{(6)} \omega_2 + \bar{\varphi} \Gamma^{(6)} \Gamma^{(5)} \omega_3 \right]
$$

(6·19)

as a generalization of (6·7) and (6·15). The transformations corresponding to $u^{(6)}_\mu \Gamma^{(6)} + u^{(6)}_\mu (6_\mu \bar{\lambda})$ are

$$
\varphi' = (A \omega_3 + B \omega_2) + (C \omega_2 + D \omega_3) \bar{\Gamma}^{(8)} \varphi_c
$$

$$
\bar{\varphi}' = \bar{\varphi} (A* \omega_3 - B* \omega_2) + \bar{\varphi} \Gamma^{(8)} (C* \omega_2 - D* \omega_3)
$$

(6·20)

with

$$
|A|^2 + |B|^2 + |C|^2 + |D|^2 = 1
$$

and

$$
\varphi' = \exp (i\gamma^{(6)} \theta) \varphi.
$$

(6·20')

It is evident that these transformations with eight parameters preserve the Lagrangian density (6·19) invariant. Our eight-dimensional representation is necessary only when it is regarded that the spin transformation corresponds to a product of the usual Lorentz transformation and the generalized Pauli transformation under which the Lagrangian density is invariant. Therefore, the introduction of an eight spinor does not contradict with the discussion in § 3 where we represent $\gamma^a$ by $4 \times 4$ matrices.

§ 7. Concluding remarks

Though we have restricted the problems to c-number theory, the discussion in § 6 will be applicable also to q-number theory. By using Gürsey representation, the transition from c-number theory to q-number one is performed by interchanging $\tau_3$ and $\tau_4$. For instance, while $i \bar{\psi} \gamma^{(a)} \psi$ is represented as $\frac{1}{2} sp (\gamma^{(a)} \sigma^{(a)} \gamma)$ in c-number theory, it is represented as $\frac{1}{2} sp (\gamma^{(a)} \sigma^{(a)} \sigma^{(a)} \gamma)$ in q-number theory. But, if we treat
q-number theory, there arises a very difficult problem concerning the quantization of general relativity. This is one of the reasons why we do not enter into q-number theory. However, our treatment, in which the legs connection \( u_p^{(a \omega)} \) is regarded as an independent field quantity, will be useful for the discussion of quantization. Arnowitt and Deser\(^4\) have shown that the Palatini formulation of the Einstein theory may circumvent the difficulty in quantizing the gravitational field. This technique, by making the metric tensor and the affinity independent field variables, will be useful only in the case where the Dirac particle is absent. From the Lagrangian density (4·1) we see that our field variables are suitable to quantize the interaction between the gravitational field and Dirac particle.

Finally, we must mention the question whether the requirement of the general covariance is necessary in the region of atomic scale or not. Even when taking into account the gravitational effect, many people have usually considered that it is sufficient to use Newtonian gravitational theory, because it represents a very good approximation for all gravitational phenomena inside the solar system. The most elementary effect which represents a crucial test of the general relativity is the gravitational shift of spectral lines, which is a direct consequence of the principle of equivalence. By using the Mössbauer effect of recoilless nuclear resonance absorption of gamma ray in Fe\(^{57}\), Cranshaw et al.\(^5\) have recently found the line shift for the height difference (12.5 meters) available to us inside the laboratory. If we want to obtain the value of Einstein's gravitational red shift by solving the Dirac equation in the linearized gravitational theory such as Birkoff's, some rather artificial technique is necessary as shown by Moshinsky.\(^6\) Therefore the recent experiment seems to imply that it may be most natural to require the general covariance of the theory also in the region of atomic scale.

**Appendix**

We shall derive the relation (6·6) in the usual \( \phi \) representation to clarify the meaning of (6·6).

First we show that \( \gamma_{(a)\phi} \) is uniquely expressed by a linear combination of \( \psi, \gamma_{(a)\psi}, \psi_e, \gamma_{(a)\psi_e} \) in the form

\[
\gamma_{(a)\phi} = w\psi + x\gamma_{(a)\psi} + y\psi_e + z\gamma_{(a)\psi_e}. \tag{A·1}
\]

In order that the above equation can be solved with respect to \( w, x, y, z \) the determinant of the matrix

\[
X = \begin{pmatrix}
\psi & (\gamma_{(a)\psi})_1 & (\psi_e)_1 & (\gamma_{(a)\psi_e})_1 \\
\psi_2 & (\gamma_{(a)\psi})_2 & (\psi_e)_2 & (\gamma_{(a)\psi_e})_2 \\
\psi_3 & (\gamma_{(a)\psi})_3 & (\psi_e)_3 & (\gamma_{(a)\psi_e})_3 \\
\psi_4 & (\gamma_{(a)\psi})_4 & (\psi_e)_4 & (\gamma_{(a)\psi_e})_4
\end{pmatrix}
\]

must not be equal to zero. The determinant of the matrix \( X \) can be calculated...
as follows. Using the charge conjugation operator $C$, we have

$$X^\tau C^{-1}X = \begin{pmatrix}
\psi^T C^{-1} \phi & \psi^T C^{-1} \gamma(5) \rho_1 & \psi^T C^{-1} \psi_e & \psi^T C^{-1} \gamma(5) \psi_e \\
\psi^T \gamma(5) C^{-1} \phi & \psi^T \gamma(5) C^{-1} \gamma(5) \rho_1 & \psi^T \gamma(5) \gamma(5) C^{-1} \psi_e & \psi^T \gamma(5) \gamma(5) C^{-1} \gamma(5) \psi_e \\
\psi^T C^{-1} \rho_1 & \psi^T C^{-1} \gamma(5) \psi_e & \psi^T C^{-1} \gamma(5) \psi_e & \psi^T C^{-1} \gamma(5) \gamma(5) \psi_e \\
\psi^T \gamma(5) C^{-1} \rho_1 & \psi^T \gamma(5) C^{-1} \gamma(5) \psi_e & \psi^T \gamma(5) \gamma(5) C^{-1} \gamma(5) \psi_e & \psi^T \gamma(5) \gamma(5) C^{-1} \gamma(5) \gamma(5) \psi_e
\end{pmatrix}.$$ 

Adopting the antisymmetric nature of $C^{-1}$ and $C^{-1} \gamma(5)$, and taking the determinant of the matrix $X^\tau C^{-1}X$, we get

$$|X^\tau C^{-1}X| = |C^{-1}| |X|^2 = \begin{vmatrix}
0 & 0 & \bar{\psi}_1 & \bar{\psi}_1 \\
0 & 0 & \bar{\psi}_1 \gamma(5) \rho_1 & \bar{\psi}_1 \\
-\bar{\psi}_1 & -\bar{\psi}_1 \gamma(5) \rho_1 & 0 & 0 \\
-\bar{\psi}_1 \gamma(5) \rho_1 & -\bar{\psi}_1 & 0 & 0
\end{vmatrix}$$

$$= -\rho^2 = -(\Omega^2 + \Omega^2) = -[(\bar{\psi}_1 \rho_1)^2 - (\bar{\psi}_1 \gamma(5) \rho_1)^2] = -\bar{\psi}_1 (1 - \gamma(5) \rho_1 \cdot \bar{\psi}_1) (1 + \gamma(5) \rho_1).$$

Since $|C^{-1}| \neq 0$, $X$ does not vanish provided that

$$\rho^2 = \Omega^2 + \Omega^2 \neq 0.$$  
(A.2)

Therefore, if $\rho^2 \neq 0$, we can solve (A.1) with respect to $w$, $x$, $y$, $z$. The result is as follows:

$$\gamma(5) \rho_1 \phi = (1/\rho) \left[ \{(1 - \Omega \rho_1 \gamma(5) \rho_1) + (\hat{\Omega} \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \} \phi \\
- (\Omega + i\Omega \gamma(5) \rho_1) (\rho_1 \gamma(5) \rho_1) \phi \right]$$ 
(A.3)

where $\rho_1 \gamma(5) \rho_1$ is given by (6.4).

Similarly,

$$\gamma(5) \rho_1 \phi_e = (1/\rho) \left[ \{- (1 - \Omega \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \phi_e + (\hat{\Omega} \rho_1 \gamma(5) \rho_1 + i\Omega \rho_1 \gamma(5) \rho_1) \rho_1 \gamma(5) \rho_1 \phi_e \\
+ (\rho_1 \gamma(5) \rho_1 + i\rho_1 \gamma(5) \rho_1) (\hat{\Omega} \rho_1 \gamma(5) \rho_1) \phi_e \right]$$ 
(A.4)

$$\gamma(5) \rho_1 \phi_\gamma = (1/\rho) \left[ \{(1 - \Omega \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \phi_\gamma - (\hat{\Omega} \rho_1 \gamma(5) \rho_1 + i\Omega \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \phi_\gamma \\
- (\rho_1 \gamma(5) \rho_1 + i\rho_1 \gamma(5) \rho_1) (\hat{\Omega} \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \phi_\gamma \right]$$ 
(A.5)

$$\gamma(5) \rho_1 \phi_\gamma = (1/\rho) \left[ \{(i\rho_1 \gamma(5) \rho_1 - \hat{\Omega} \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \phi_\gamma - (\hat{\Omega} \rho_1 \gamma(5) \rho_1 - i\Omega \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \phi_\gamma \\
+ (\rho_1 \gamma(5) \rho_1 + i\rho_1 \gamma(5) \rho_1) (\hat{\Omega} \rho_1 \gamma(5) \rho_1) \gamma(5) \rho_1 \phi_\gamma \right].$$ 
(A.6)

By combining Eqs. (A.3)–(A.6), new sets of Pauli matrices are introduced. As an example, putting

$$\phi_\gamma = \begin{pmatrix}
\psi \\
\bar{\psi}_1 \gamma(5) \rho_1 \phi_e 
\end{pmatrix}$$ 
(A.7)

we obtain from (A.3) and (A.4)

$$\gamma(5) \rho_1 \phi_\gamma = (1/\rho) \left[ \{(1 - \hat{\Omega} \gamma(5) \rho_1) \rho_1 \gamma(5) \rho_1 \phi_\gamma + (\hat{\Omega} \gamma(5) \rho_1 - i\Omega \gamma(5) \rho_1) \rho_1 \gamma(5) \rho_1 \phi_\gamma \right]$$ 
(A.8)
where \( \hat{\xi}_\gamma \) is a set of Pauli matrices. Therefore, \( \gamma_\gamma \) is isomorph to a rotation with the generators \( \hat{\xi}_\gamma \) and \( \gamma_\gamma \).

Further, if we put

\[
\psi_{11} = \begin{pmatrix}
\gamma_\gamma \phi_c \\
\phi \\
\phi_c \\
\gamma_\gamma \phi
\end{pmatrix}
\quad \text{and} \quad
\psi_{111} = \begin{pmatrix}
\gamma_\gamma \psi_c \\
-\psi \\
\psi_c \\
\gamma_\gamma \psi
\end{pmatrix},
\]

then

\[
\gamma_\gamma \gamma_\gamma \psi_{11} = (1/\rho) \hat{\omega}_2 \left[ -i (\Omega \omega_3 + \hat{\Omega} \omega_3) \phi_{a}^{(4)} + (\hat{\Omega} \omega_2 - \Omega \omega_2) \phi_{a}^{(5)} \hat{\xi}_\gamma \right] \psi_{11}
\]

and

\[
\gamma_\gamma \gamma_\gamma \psi_{111} = (1/\rho) \hat{\omega}_3 \left[ -\omega_3 (i \hat{\Omega} \phi_{a}^{(4)} + \Omega \phi_{a}^{(5)} \hat{\xi}_\gamma) + \Omega (\phi_{a}^{(1)} \hat{\xi}_\gamma + \phi_{a}^{(2)} \hat{\xi}_\gamma) \\
-\omega_2 (i \phi_{a}^{(4)} \hat{\xi}_\gamma - \hat{\Omega} \phi_{a}^{(5)} \hat{\xi}_\gamma) + \hat{\Omega} \omega_1 (\phi_{a}^{(1)} \hat{\xi}_\gamma - \phi_{a}^{(2)} \hat{\xi}_\gamma) \right] \psi_{111}
\]

where \( \omega_\gamma \) is a set of Pauli matrices.

If we express the transformation (6·20) in terms of 2-4 wave matrix, it has a similar form to (A·11).

References

2) W. Pauli, Nuovo Cimento 6 (1957), 204.
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5) H. S. Green, Nuclear Phys. 7 (1958), 373.
12) H. Jehle, Z. Physik 100 (1936), 702.