Some Characteristics of a Propagating Brittle Tensile Crack

Merle E. Hanson and Allan R. Sanford

(Received 1971 June 10)

Summary

A numerical technique was used to formulate the two-dimensional equations of motion for an elastic continuum. A brittle tensile crack was simulated to form and propagate in the continuum. The stress field in front of the fracture tip was found to become increasingly hydrostatic with increasing fracture velocity. A fracture criterion in terms of the values of the principal stresses near the fracture tip indicated a terminal velocity for a straight running fracture of approximately 0.39 of the dilatational wave speed. Part of the elastic energy residing initially in the continuum accumulated at the fracture tip. A quantitative fit of the excess elastic energy as a function of crack half length and velocity showed that excess energy at the fracture tip increases as the square of fracture length and decreases with fracture velocity.

Introduction

This paper discusses some of the dynamic characteristics of a bilateral brittle tensile fracture in an elastic continuum. A tensile crack was numerically simulated to form and propagate in an elastic medium having two-dimensional plane strain characteristics. The technique of calculation was to use a two-dimensional Lagrangian finite-difference formulation of the elastic equations of motion (Petschek & Hanson 1968). A uniform uniaxial tensile stress field was imposed on the elastic continuum for all time t. Several fracture velocities were simulated and in addition, a failure criterion was applied that resulted in a calculated terminal velocity similar to that given by Dulaney & Brace (1960) and Wells & Post (1957).

Studies of the effect of a propagating fracture on strain energy were made. Some of the initial strain energy was converted to kinetic energy, some was taken up by the creation of the fracture surfaces, and the remainder accumulated at the fracture tip. The excess strain energy in a region around the fracture tip was found to increase with the square of the fracture length. The accumulation of the strain energy decreased with fracture velocity.

Many investigators have studied the dynamic behaviour of a brittle tensile crack. For example, Yoffe (1951) analysed a constant-length fracture moving with constant velocity through a brittle elastic material. Her solution showed that, for large fracture velocities, the stress on a radius about the crack tip was maximum at some angle to the fracture axis. Thus, at some critical fracture velocity, the crack should either branch or curve. However, the solution is physically unrealistic because the...
length of a natural tensile fracture is not constant. Craggs (1960) analysed a two-dimensional brittle fracture extending unilaterally in an infinite elastic medium. He concluded that the force required to maintain a steady rate of extension of the crack decreases as the crack velocity increases. Baker (1962) analysed the case for a semi-infinite crack extending at constant velocity in a stretched elastic body. Baker contended that the stress field at the fracture tip is independent of fracture length.

Formulation of the problem and numerical technique of solution

The boundary conditions and fracture simulation were similar to those described by Hanson & Sanford (1970). The block was initially subjected to a uniaxial tensile stress ($\sigma_y$). A fracture nucleus from $z = -\infty$ to $+\infty$ was formed at the centre of the block ($x = 0$, $y = 0$) at $t = 0$ and was extended dynamically in both directions (bilaterally) along the x-axis. Fracture velocities were specified with the crack velocity starting and remaining at a constant value until the calculation was terminated. The fracture velocities used in the calculations were $0.1C_1$, $0.2C_1$, $0.3C_1$, $0.39C_1$, $0.45C_1$, and $0.5C_1$, where $C_1$ is the dilatational wave speed in plane strain.

Because of the symmetry of a bilateral tensile fracture, the problem could be solved by carrying out calculations in one quadrant only. In the calculation, the energy required to create the fracture surface was taken as approximately $5 \times 10^6$ erg cm$^{-2}$ which is in agreement with the values given by Dulaney & Brace (1960). The calculational grid size was chosen so that reflections from boundaries would not reach the areas of interest in the grid. The initial stress field in the elastic medium was uniaxial tension of magnitude $10^9$ dyne cm$^{-2}$.

The operation of the numerical Lagrangian two-dimensional computer code and the formation of cracks has been described by Hanson & Sanford (1970). A complete discussion of the difference forms used is given in Petschek & Hanson (1968). A tensor damping form was included in these calculations to reduce the numerically created high frequency oscillations resulting from the crack opening in a discretized mass network. The damping tensor is obtained from the time rate of change of the strain tensor.

The forms of damping used include both linear damping and quadratic damping. The quadratic damping is defined by

\begin{align}
q_{xx} &= K_1 (\dot{\epsilon}_{xx})^2 \\
q_{yy} &= K_1 (\dot{\epsilon}_{yy})^2 \\
q_{xy} &= K_1 (\dot{\epsilon}_{xy})^2
\end{align}

and the linear damping by

\begin{align}
q_{xx} &= K_2 \dot{\epsilon}_{xx} \\
q_{yy} &= K_2 \dot{\epsilon}_{yy} \\
q_{xy} &= K_2 \dot{\epsilon}_{xy}
\end{align}

where

\begin{align}
K_1 &= C_q^2 \rho_0 V_{R}^{n+\frac{1}{2}} A^{n+\frac{1}{2}} \\
K_2 &= C_L \sqrt[3]{[A^{n+\frac{1}{2}} \rho_0 V_{R}^{n+\frac{1}{2}} (2\mu + \lambda)]}
\end{align}

$C_q$ and $C_L$ are constants, $\lambda$ and $\mu$ are the Lamé constants for the material and $\rho_0$ is the initial density. $\epsilon_{ij}$ is the strain, $A^{(n+\frac{1}{2})}$ is the area, and $V_{R}^{n+\frac{1}{2}}$ (equal to $[V_0/V^{(n+\frac{1}{2})}]$) is the ratio of specific volumes of the Lagrangian zone defined as the average between the $n$ and $n+1$ time steps.
Application of the damping is accomplished by simply adding similar terms of the damping and stress tensors before differentiation in the momentum equations. In addition, conservation of energy is assured in the calculation if the sum of the stress and damping tensors is also applied in the energy equation.

The properties of these damping forms has been worked out by Petschek (1970 personal communication). If the effect of heating is ignored, a one-dimensional form of the elastic wave equation, with linear and quadratic damping, is

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} + K_2' \frac{\partial^2 u}{\partial x \partial t} + K_1' \frac{\partial}{\partial x} (\frac{\partial u}{\partial x} \frac{\partial u}{\partial t})^2$$  \hspace{1cm} (4)

where $E$ is Young's modulus, $\rho$ is density, and $K_2'$ and $K_1'$ are the coefficients of the linear and quadratic damping for the one-dimensional case. Multiplication of this equation by $\frac{\partial u}{\partial t}$, and integration by parts over the length of the rod, with the boundary condition that $\frac{\partial u}{\partial x} = 0$ at the ends of the rod, leads to

$$\frac{\partial}{\partial t} \int \left[ \frac{1}{2} (\frac{\partial u}{\partial t})^2 + \frac{1}{2} \frac{E}{\rho} (\frac{\partial u}{\partial x})^2 \right] dx = -K_2' \int (\frac{\partial^2 u}{\partial x \partial t})^2 dx - K_1' \int (\frac{\partial^2 u}{\partial x \partial t})^3 dx. \hspace{1cm} (5)$$

The left side of the preceding equation is the time derivative of the energy per unit mass of the system. If the amplitude of the oscillation is $A$, the frequency $\omega$, and the length $L$, then the integral is $A^2 \omega^2 L$, except for a constant. The term $\frac{\partial^2 u}{\partial x \partial t}$ is proportional to $A \omega^2 / V$, where $V$ is the wave speed. Hence the equation can be written as

$$\frac{\partial}{\partial t} A^2 \omega^2 L = -K_2'' A^2 \omega^4 L/V^2 - K_1''' A^3 \omega^6 L/V^3$$  \hspace{1cm} (6)

where $K_2''$ and $K_1'''$ are new constants. The equation is finally

$$\frac{\partial A}{\partial t} = -K_2'' A \omega^2/2V^2 - K_1''' \omega^4 A^2/2V^3. \hspace{1cm} (7)$$

If the second term on the right is ignored, the linear damping gives exponential decay. If the first term on the right is ignored, the quadratic damping gives

$$\frac{1}{A} = \frac{1}{A_0} + K_1''' t, \hspace{1cm} (8)$$

or $A \sim 1/t$, an effective damping proportional to the amplitude. In addition the damping increases rapidly with frequency.

In the preceding discussion, the oscillations are assumed to be small. The damping is used primarily to reduce the numerically created oscillations resulting from the technique used to simulate the fracture. Fig. 1 shows the displacement of a point as the fracture passes beneath and beyond the point and finally stops. Displacements parallel and perpendicular to the fracture are shown on this figure. The point is referenced to a Cartesian reference frame placed at the centre of the fracture. In non-dimensional co-ordinates, the position of the point with respect to the centre of the fracture is given by,

$$\xi = x/a \quad \text{and} \quad \eta = y/a \hspace{1cm} (9)$$

where $a$ is the half crack length. The displacement of a point from its initial strained position is given by the non-dimensional displacement forms

$$\frac{\delta \xi}{\varepsilon_p} \quad \text{and} \quad \frac{\delta \eta}{\varepsilon_p} \hspace{1cm} (10)$$

where $\varepsilon_p$ is the strain in the direction of the initial uniaxial stress. The non-dimensional time $\tau$ is defined by

$$\tau = tC_1/a \hspace{1cm} (11)$$

where $t$ is the problem time and $C_1$ is the P-wave speed.
Fig. 1 shows three traces for the displacements in the $x$ and $y$ directions. The fine solid lines are the calculated responses when no damping was used. The dotted lines are filtered versions of the undamped response. The heavy solid lines are the calculated responses when large amounts of damping were applied. The filter used for the dotted curves is similar to the one discussed in Hanson & Sanford (1970).

The numerically created high frequency oscillations are clearly evident in the trace of the $y$ displacement. The damping effectively attenuates this high frequency noise without significantly altering the response of the point. On the other hand, there is little difference between the undamped, filtered, and damped traces of the displacement in the $x$ direction.

**Characteristics of the stress field**

As expected, the initial uniaxial stress field of the medium was modified by the introduction of a moving fracture, particularly in the vicinity of the crack tip. An important change in the stress field was the creation just ahead of the fracture tip of tensile stress parallel to the fracture axis. As the fracture length increased this tensile stress, as well as that perpendicular to the fracture axis, increased. However, with increasing fracture velocity, the parallel stress increased faster than the perpendicular stress. In other words, the stress field in front of the tip moved toward a condition of hydrostatic tension. Fig. 2 shows how the ratio of $\sigma_1/\sigma_2$ ($\sigma_1$ is the principal stress parallel to the crack and $\sigma_2$ is the principal stress perpendicular to the crack) changed with fracture length and velocity.
Fig. 2. The ratio of the principal stresses $\frac{\sigma_1}{\sigma_2}$ zone ahead of the fracture tip as a function of crack half-length (in centimetres) for several fracture velocities.

For these computations, the Lagrangian zone was one centimeter square. Because the computational technique implies that the stresses are an average over the Lagrangian zone, interpretations of the stresses closer than a half zone to the singularity at the crack tip are not possible. Therefore, the stresses can only be compared a half zone ahead of the crack tip.

Possible experimental confirmation of a dilated region in front of the crack tip comes from the work of Wells & Post (1957). For a tensile crack propagating in photoelastic material, they found that a dimple-like distortion in the surface near the tip of the crack caused extinction of light in this region. The dimpling effect could be the result of the trend toward a hydrostatic tensile field found in our computations.

Fig. 3 is a plot of the maximum principal axis of stress (tensile) over the region above the right crack tip. If branch fractures are to form they should be oriented perpendicular to these principal stresses. A branch fracture that originates in the region just ahead of fracture tip will tend toward the original fracture axis. A branch fracture that originates just behind the fracture tip will tend to diverge from the original fracture axis. Fig. 3 is in agreement with the experimental results of Manogg (1966).

Figs 4 and 5 depict the rotation of the principal axes of stress as a moving fracture tip passes beneath the point. Again, Manogg (1966) showed comparable results from an experiment. Examination of these figures shows that the amount and rate of rotation of the principal axis increase with fracture length. In addition, the rotation rate and magnitude increase with fracture velocity. In the times considered on these plots, the principal axis angle is negative at the start. The negative angle results from the flow of material toward the fracture tip. The rotation indicates the changing shear stress in the region.
Fig. 3. Orientations of the maximum principal axis of stress in the vicinity of the fracture. The fracture extends from $X = 0, Y = 0$, to $X = 11, Y = 0$ centimetres with the fracture tip shown by the arrow. The orientations are not shown on the fracture axis, but are vertical. Fracture velocity was $0.3C_1$.

Fig. 6 is a contour plot of $2\tau = \sigma_2 - \sigma_1$ where $\sigma_2$ and $\sigma_1$ are the principal stresses and $\tau$ is the maximum shear stress. The stress differences shown in this figure include the initial applied stress as well as the stresses arising from the propagation of the tensile crack. A contour value of 8 is about the magnitude of $2\tau$ for the initial stress field alone. Thus, contours having values greater than 8 indicate regions where the shear stress has increased as a result of fracture. Notice that the lobe of maximum shear stress is at an angle greater than $45^\circ$ to the fracture axis. On the other hand, there is a lobe of low shear stress directly in front of the fracture tip. This lobe is the result of a decrease in distortion due to increasing hydrostatic tension in the region. Other characteristics of the distortional stress field are (1) the pattern increases in size with fracture length, and, (2) the relaxation zone behind the fracture tip expands at $P$-wave velocity.

Contours of constant dilation are shown in Fig. 7. The dilation contours in this figure include the dilation of the initial stress field in addition to the dilation arising from the stresses created by the moving crack. A contour value of four is about the magnitude of the dilation arising from the initial stress field alone. Thus, contour values greater than four indicate regions where dilation has increased as a result of fracture and vice versa. The maximum is directly in front of the fracture tip and the minimum occurs at the crack face in the expansion region behind the tip.
Characteristics of a propagating brittle tensile crack

Fig. 4. Rotation of the principal axis of stress at points 4, 7 and 10 cm from the centre of the crack as the fracture tip passes 1 zone beneath the points. The reference time is in microseconds with the zero time chosen so that the point observed relative to the crack tip is identical for the three cases. Fracture velocity was $0.3C_1$.

Fig. 5. Comparison of the rotation of principal axis for four fracture velocities at a distance of 5 cm from the centre of the crack. The reference time is in microseconds with the zero time chosen so that the point observed relative to the crack tip is identical for the cases.
Terminal velocity of a straight-running fracture

The terminal velocity of a straight-running fracture is defined as the maximum velocity at which a crack will grow without branching. If the terminal velocity of a straight-running fracture is a function of the dynamic stress field at the tip of the fracture, then it should be possible to establish a criterion for a straight-running fracture in terms of the values of principal stresses at the fracture tip.

Let the principal stress difference on the fracture axis a short distance ahead of the fracture be defined as \((\sigma_2 - \sigma_1)_a\). In addition, let the propagating fracture tip be enclosed by a small circle. In this small circle determine the largest principal difference off the fracture axis and define this stress difference as \((\sigma_2 - \sigma_1)_b\). A measure of the relative elongation between the principal axis is \((e_2 - e_1)\). For the linear elastic material, the strain difference differs from the stress difference by only a constant, \((\sigma_2 - \sigma_1) = 2\mu(e_2 - e_1)\), and therefore, the strain ellipse has the same shape as the stress ellipse.

Fig. 8 shows the ratio of the stress difference at \(b\) to the stress difference at \(a\), \((\sigma_2 - \sigma_1)_b/(\sigma_2 - \sigma_1)_a\), for the fracture velocities 0.2\(C_1\), 0.3\(C_1\), 0.39\(C_1\), 0.45\(C_1\), and 0.5\(C_1\) as a function of half crack length. As the damping is increased, the value of the ratio is reduced. For the fracture velocity 0.45\(C_1\), the ratio exceeds 1 after the
crack has run a short distance and continues to steadily increase as the fracture grows. Hence the eccentricity of the strain ellipse becomes greater at b than at a for velocities equal to or greater than 0.45C.

For a fracture to propagate in a straight line in homogeneous material the stress difference at a should be greater than at b, in other words, the ratio should be less than one. Hence it should be possible to find a fracture velocity where the ratio is a constant, less than one, for increasing fracture half-length. In equation form it can be stated that the terminal velocity will occur if

\[
\frac{dR}{dx} = 0 \quad \text{for} \quad R < 1,
\]

where \( R = (\sigma_2 - \sigma_1)_b/(\sigma_2 - \sigma_1)_a \) and \( x \) is the crack co-ordinate. Extrapolation between the velocities 0.39C1 and 0.45C1 in Fig. 8 shows that the velocity satisfying the conditions above will be closer to 0.39C1 than 0.45C1, probably about 0.4C1.
FIG. 8. The ratio of the principal stress difference in a small circle enclosing the fracture tip. The ratio is defined as the largest stress difference off the fracture axis, \( b \), to the stress difference on the fracture axis, \( a \), in the circular domain. The ratio is given as a function of fracture half-length in centimeters.

The criteria that the ratio of the stress difference had to be less than 1 was applied to the code. In equation form the fracture criteria applied to the code was

\[
(\sigma_2 - \sigma_1)_b \geq (1/R) (\sigma_2 - \sigma_1)_b.
\]

where \( R \) is some nominal value of \( R \). Terminal velocities of 0.39\( C_1 \) were predicted by the code. Variation of the parameter \( 1/R \) from 1.1 to 1.3 did not affect the terminal velocity. The terminal velocity predicted is close to the value of 0.38\( C_1 \) given by Wells \& Post (1957).

**Strain energies**

The excess elastic strain energies in a region about the crack tip were found to increase with the square of the fracture length. Excess total, distortional and dilational strain energies were determined by numerical integration over a region about the fracture tip. The integration was performed only where their value was greater than the initial value for the given fracture length. In equation form, the excess elastic energies are

\[
E_a' = E_a^{(e)} - E_a^{(0)}
\]

where \( E_a^{(e)} \) is the total, distortional, or dilational strain energy at time \( t \) and \( E_a^{(0)} \) is the initial value of the strain energy component.

The integrations were performed only over the regions where the elastic energies were larger than their initial values because these are the areas that can affect the propagate fracture. For example, if the total elastic energy decreased, the fracture could be expected to slow up or stop. On the other hand, excessive concentration of energy off the fracture axis could be expected to result in a branch or change in...
direction of the fracture. Because only the energies larger than the initial energies were included in the integration, the sum of the dilatational and distortional parts does not necessarily have to equal the total elastic energy.

Figs 9, 10 and 11 are examples of the regions over which integrations were performed, i.e. regions which had dilatational, distortional, or total strain energies in excess of the initial values. The plots shown are for a fracture velocity of $0.3C_1$. Ahead of the crack tip dilatational strain energy increases, whereas the distortional and total strain energies decrease. A similarity exists in the shape of the regions where the distortional and total strain energies are larger than the initial elastic energies. However, the region of excess distortional strain energy is somewhat larger.

A functional form found to express the change in the elastic strain energies with fracture length for a constant fracture velocity is

$$E'_a = a'_a(\lambda)X^2$$

where $X$ is the instantaneous fracture length and $a'_a(\lambda)$ is a constant for a given fracture velocity. $\lambda$ is the ratio of the fracture velocity to the $P$-wave speed. $a'_a(\lambda)$ can be scaled by defining

$$a_a(\lambda) = a'_a(\lambda)/\sigma_y$$

where $\sigma_y$ is the magnitude of the initial uniaxial stress. $a_a(\lambda)$ has the units of length per unit depth with the two-dimensional calculation. The magnitude of $a_a(\lambda)$ was found to decrease non-linearly with increasing velocity up to the terminal tensile fracture velocity of $\lambda = 0.39$. Fig. 12 shows $a_a$ as a function of fracture velocity to $\lambda = 0.39$ for the excess total, distortional, and dilatational strain energies.

![Fig. 9. An example of the integration domain for the excess dilatational energy for a fracture half-length of 5 cm and a fracture velocity of $0.3C_1$. Both co-ordinates $X$ and $Y$ are in centimetres. The half fracture length is shown by the solid line. The gradient of the function decreases to the right.](https://academic.oup.com/gji/article-abstract/24/3/231/629600)
Fig. 10. An example of the integration domain for excess distortional energy for a fracture half-length of 5 cm and a fracture velocity of 0.3C₁. Both co-ordinates X and Y are in centimetres. The half fracture length is shown by the solid line. The function decreases away from the tip and is small to the right.

Fig. 11. An example of the integration domain for excess total strain energy for a fracture half-length of 5 cm and a fracture velocity of 0.3C₁. Both co-ordinates X and Y are in centimetres. The half fracture length is shown by the solid line. The gradient of the function decreases away from the crack tip.
For a given length of fracture, the elastic energies near the crack tip decrease with fracture velocity. However, the decrease in elastic energy up to the terminal velocity does not result in the elastic energy less than the initial value. The rate of dilatational energy decrease with fracture velocity is less than the rate of decrease of the total or distortional energies.

An additional characteristic of the strain energy fields not shown in the figures is that the size of the regions having strain energies greater than the initial value increases with fracture length. The functional forms for the energies in terms of fracture length are for perfectly brittle elastic materials with energy dissipation resulting primarily from the creation of the fracture surfaces. A result of the analysis shows that a fracture can decrease the rate of energy accumulation at the tip by accelerating to a higher velocity.

The brittle dynamic fracture of these calculations differs from a stable Griffith-type fracture in that energy in excess of that required to form the fracture surface accumulates near the fracture tip. The physical effect of this excess energy may be to increase fracture roughness. An increase of fracture roughness with fracture velocity has been reported by Cotterell (1965, 1968), Creggs (1960), and Bieniawski (1968). In addition, excess energy must be available if branching is to occur.
Summary

The numerical simulation of a tensile fracture showed that a tensile stress parallel to the fracture axis is created in front of the moving crack tip. Because this tensile stress increases with crack length and/or velocity, the stress at the tip tends to become more hydrostatic (tensile) as crack length and/or velocity is increased.

A terminal velocity for a straight running fracture of $0.39C_1$ was obtained from the calculation. The fracture criterion used was that the principal stress difference be greater in front of the crack tip than at an angle to the fracture axis.

The elastic energy at the fracture tip was found to increase with (1) increase in fracture length and (2) decrease in fracture velocity. The total strain energy and hence the crack extension force continually increased with crack length, a result which is in agreement with work by Cotterell (1964). Dilatational energy in excess of the initial value was observed in a region directly in front of the propagating fracture tip. The lobes of highest distortional stress were at an angle of greater than 45° to the projected axis of the fracture. The high distortional stress lobes extend forward from the fracture tip. A lobe having a distortional stress less than the initial value extends in front of the fracture tip.

Acknowledgements

The authors wish to express their appreciation to Dr A. G. Petschek for his many helpful suggestions. Gary Mosley and Thomas Schellhase assisted in developing computer plot routines. This research was supported by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Office of Scientific Research under Contract Number F44620-70-C-0055.

New Mexico Institute of Mining and Technology
Campus Station
Socorro, New Mexico 87801.

References