Non-Locality in the Second Order Potential

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The non-local properties of the second order two nucleon interaction are investigated using ps-ps meson theory and the Tamm-Dancoff approach. None of the customary transformations are applied to the potential or wave function, so the second order potential derived is both momentum dependent and energy dependent. The potential is hermitian, however, so that the two nucleon component of the wave function can be normalized in the usual manner, conserving probability. The size of the non-local corrections on scattering states is estimated as a function of energy and as a function of the internucleon separation distance. The Born approximation is used to show that the lowest order velocity correction accounts for almost all of the non-local corrections at 300 Mev laboratory energy where it decreases the static scattering amplitude by about 25 per cent. The energy dependence of the potential is instrumental for the Born approximation scattering amplitude being the same as given by lowest order S-matrix theory. The non-local corrections are estimated as a function of internucleon distance by defining an "effective" scattering potential which disguises most of the velocity dependences.

§ 1. Introduction

The recent trend in phenomenological studies of the two-nucleon problem has been to emphasize the velocity dependence of the two nucleon potential.¹,² The present situation may be summarized as follows. Using static potentials with widely varying radial shapes and a general spin angle dependence, the experimental data available for two-nucleon systems can not be adequately fit at all energies. This is true even though hard cores (or some other cutoff parameter) are introduced separately for even and odd states.³,⁴ Thus, if one wishes to retain the usual formulation of nuclear physics where only non-relativistic nucleons occur explicitly,

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one is forced to admit rather general interactions, i.e. certain velocity dependent (or non-local) potentials.

How one should proceed in introducing a velocity dependence into phenomenological investigations of the two-nucleon potential is really open to question. Present phenomenological investigations have taken the attitude that the familiar spin-orbit potential is the natural velocity dependence to consider since it is the only scalar invariant which depends linearly on the relative momentum. Although this idea is bolstered by the fact that phenomenological potentials with a spin-orbit dependence are considerably more successful than those without a spin-orbit potential, the special role of the spin-orbit potential is not so readily established from a theoretical point of view. In fact, other velocity dependences seem to be preferred. Furthermore, the spin-orbit potential may well arise as a correction not linear but quadratic in the relative momentum.

Most current theories of the two nucleon potential are based on the relativistic formulation of interacting fields, and extraction of accurate predictions from a relativistic theory of interacting fields corresponding to particles of finite mass has proven to be quite formidable. The basic difficulty is that there is no rapidly convergent method of calculation available. Consequently, the problem of finding the non-relativistic interaction of two nucleons at all separation distances in coordinate space (the operator which will yield the correct scattering phase shifts) is still not satisfactorily solved. If the separation distance of the two nucleons is kept large enough, however, the correctness of the potential can be insured.

In the present work, the Tamm-Dancoff method is used to investigate the velocity dependence of the two-nucleon potential. No innovation is introduced into the method of calculating the potential and, primarily for simplicity, the calculation is carried through only for the second order potential (the interaction arising from the exchange of a single meson). For a realistic example, the forms of non-locality that occur and their effective strength are investigated. Further, the velocity and energy dependent contributions are compared with potentials where these contributions have been transformed into local interactions.

§ 2. Second order potential in momentum space

The field equation for the strongly interacting nucleon and meson fields is

\[ H\psi = E\psi, \]

where

\[ H = H_N + H_M + H_I \]

is the system Hamiltonian and \( \psi \) is its state vector for the eigenvalue \( E \). \( H_N \) and \( H_M \) are the free nucleon and free meson field Hamiltonians, respectively, and \( H_I \) is the interaction energy between these fields. Explicitly,
where $G$ is the coupling constant between the nucleon and pion fields and $I$ is the operator coupling these fields:

\begin{align}
I = \phi(x) & \quad \text{scalar meson field,} \\
I = \gamma_5 \sum_a \tau_a \phi_a(x) & \quad \text{pseudoscalar meson field.}
\end{align}

(2a) \hspace{1cm} (2b)

The amplitude $\phi$ describes the meson field and $\tau$ is the nucleon isospin vector.

Really the derivation of the nuclear potential is only the process of finding the non-relativistic form of Eq. (1) for the case of two interacting nucleons. This non-relativistic equation will take the form of the familiar Schrödinger equation for two particles in which the two nucleon potential can be readily identified. We shall follow this procedure using the Tamm-Dancoff technique to account for the two-nucleon interaction only in the lowest approximation of single meson exchanges.\(^7\),\(^8\),\(^9\)

Using the center of mass system and the momentum representation, the second order relativistic equation of motion for the two-nucleon system is given by the following:

\[
(E - 2E(p)) F(p) = \frac{G^2 M^2}{4(2\pi)^3} \int \frac{dk}{\omega(k)} \left( \frac{E - E(p) - E(p-k) - \omega(k)}{E^2} \right) - 1 \\
\times \frac{E(p-k) + M}{E^2(p-k)} T(\sigma_1, \sigma_2, p, k) F(p-k). \tag{3}
\]

$F(p)$ is the two nucleon wave function (i.e. the projection of $\psi$ in (1) onto the state of two nucleons with no pairs or mesons), $E(p) = \sqrt{p^2 + M^2}$, $\omega(k) = \sqrt{k^2 + \mu^2}$, and $\sigma_1, \sigma_2$ are the Pauli spin operators for nucleons 1 and 2, respectively. In the scalar meson theory (Eq. (2a))

\[
T = \left[ 1 - \frac{p^2 - (p \cdot \sigma_1)(k \cdot \sigma_1)}{(E(p) + M)(E(p-k) + M)} \right] \left[ 1 - \frac{p^2 - (p \cdot \sigma_2)(k \cdot \sigma_2)}{(E(p) + M)(E(p-k) + M)} \right], \tag{4a}
\]

whereas in the pseudoscalar meson theory (Eq. (2b))

\[
T = \tau_1 \cdot \tau_2 \left[ \frac{(p \cdot \sigma_1)(p \cdot \sigma_2)}{(E(p) + M)^2} + \frac{(p - k) \cdot \sigma_1 ((p - k) \cdot \sigma_2)}{(E(p-k) + M)^2} - \frac{(p \cdot \sigma_1)(p - k) \cdot \sigma_2}{(E(p) + M)(E(p-k) + M)} \right], \tag{4b}
\]

In each case, the function $T$ contains all the interesting spin dependence in the potential. Equation (3) is well known and has been derived by several authors.\(^{20}\)

While only one meson is exchanged between the nucleons, the separate nucleons, their interaction with the meson field and the meson are treated exactly.
Thus $G, M,$ and $\mu$ are the renormalized quantities. Unfortunately we do not know how to conveniently describe the exact dependence of the pion-nuclear interaction on, say, the momentum transferred to the pion. This momentum dependence of the pion-nucleon interaction as compared with the unrenormalized interaction arises from two sources: renormalization of the interaction involving one nucleon, and the presence of the pion field associated with the second nucleon. The unrenormalized form of the pion-nucleon interaction is used here, considering this as an example of what the actual momentum dependence might be.

§ 3. Non-relativistic transformation to coordinate space

In order to obtain the two nucleon potential, Eq. (3) is reduced to its non-relativistic value. The Fourier transform of the resulting equation will have the form of the usual non-relativistic Schrödinger equation, and the second order potential can be readily identified.

In taking the non-relativistic limit, $p$ and $k$ (and therefore $p-k$) are treated on the same footing and $(p/M)^2$ is used as the expansion parameter. The non-relativistic energy (kinetic plus potential) is

$$\varepsilon = E - 2M.$$  

(5)

The essential energy denominator is

$$(E - E(p) - E(p-k) - \omega(k))^{-1} \approx \frac{-1}{\omega(k)} \left(1 - \frac{1}{2\omega(k)} \right)$$

$$\times \left(\frac{p^2}{M} + \frac{(p-k)^2}{M} + \frac{\varepsilon}{\omega(k)}\right).$$  

(6)

In the usual "static" theories, the right side of Eq. (6) is replaced by $-1/\omega(k)$. In calculating the Born approximation scattering amplitude (see § 5 below), Eq. (6) reduces exactly to the "static" result of $-1/\omega(k)$; this is perhaps an indication that the expansion in (6) is quite accurate.

The non-relativistic energy, $\varepsilon$, is treated as being of order $p^2/M$. (For large separation distances this is clearly correct, and for small separation distances the approximation depends on the sign of potential. In the region where the potential is accurately predicted ($r \geq 2f$), the approximation applies.)

In the non-relativistic (but non-static) limit, Eq. (3) has the form

$$(\varepsilon - p^2/M) F(p) = \int dk K\varepsilon(p, k) F(p-k).$$

Transforming to coordinate space, this equation becomes

$$(\varepsilon + F^2/M) \psi(r) = \int dr' U\varepsilon(r, r') \psi(r'),$$  

(7)

where
The integration over $p$ may be formally carried out as follows. First move $K_k$ to the left in the integrand in (8) and replace its dependence on $p$ with the operator $iF$. The remaining integration over $p$ is then simply a delta function and Eq. (7) becomes

$$ (\varepsilon + F^2/M)\psi(r) = V_c(r, -iF)\psi(r), $$

where

$$ V_c(r, -iF) = \int dk K_k(iF, k)e^{-ik \cdot r} $$

and

$$ V_c(r, -iF)\psi(r) = \int dr' U_c(r, r')\psi(r'). $$

From Eq. (7) it is clear that the scalar and pseudoscalar meson theories predict a non-local, energy dependent potential which, from Eq. (11), is equivalent to an ordinary multiplicative type potential with explicit energy and velocity dependence. The restriction only to lowest order velocity corrections (to order $(p/M)^2$), however, is a short range limitation on the amount of non-locality. As long as one uses the usual non-relativistic kinetic energy operator, i.e. $p^2/M$, this limitation must be imposed for overall consistency. Complete non-locality corresponds to a relativistic interaction which accounts for all orders of $(p/M)^2$; in this case, the kinetic energy operator is $E(p) = \sqrt{p^2 + M^2}$.

§ 4. Second order scalar potential

Following the procedure outlined above, the non-relativistic second order scalar potential is given by

$$ V_s = -\frac{G^2}{4\pi} \left[ \frac{\mu e^{-x}}{x} \left( 1 + \frac{3}{4} \left( \frac{\mu}{M} \right)^2 \right) + \frac{2}{\pi} \left( \frac{\mu}{M} \right)^2 \right] K_0(x) $$

$$ - \frac{M}{\pi} \frac{\mu^2}{M} K_1(x) - \frac{2}{M^2} \frac{\mu}{x} \left( \frac{\mu}{M} \right) \psi \cdot L \cdot \sigma, $$

where $x = \mu r$, $L$ is the total orbital angular momentum relative to the center of mass, and $\sigma = \sigma_1 + \sigma_2$ is twice the total spin operator of the two-nucleon system. $K_0$ and $K_1$ are the Bessel functions of imaginary argument of zero and first order, respectively.
In the limit $s/M \to 0$ and $p/M \to 0$, the scalar potential in (12) reduces to the familiar Yukawa potential.

It is of particular interest to note the presence of the spin-orbit term in the scalar potential. Explicitly,

$$V_L = \frac{G^2}{4\pi} \left( \frac{\mu}{M} \right)^2 \frac{\mu}{4x} \left( \frac{e^{-\rho}}{x} \right) \cdot L \cdot \sigma.$$ 

This potential arises from the spin, angle dependence of the nucleon spinors, a fact revealed by the $1/M^3$ dependence on the nucleon mass. $V_L$ is written with an explicit differentiation to be performed in order to emphasize that this potential is really of order $p^2$ in the nucleon momentum. The spin-orbit potential is characterized by one derivative acting on the wave function and the other only on the radial shape of the potential. The easiest way to keep account of the order of a given term with respect to the nucleon momentum is by the power of the dependence on $1/M$.

§ 5. Second order pseudoscalar potential

The non-relativistic second order pseudoscalar potential is given by

$$V_c = V_0 + \frac{\epsilon}{M} K - \left[ p^2/4M^3, 4V_0 + 2K \right], + \left[ p^2/4M^2, V_0 - A \right],$$

where $[A, B]_\pm = AB \pm BA$,

$$V_0 = \frac{\tau_1 \cdot \tau_3}{4\pi} \frac{G^2}{2M} \left( \frac{\mu}{2M} \right)^2 \frac{e^{-\rho}}{x} \left[ \sigma_1 \cdot \sigma_z + S \left( \frac{3}{x^2} + \frac{3}{x} + 1 \right) \right],$$

$$K = \frac{\tau_1 \cdot \tau_3}{4\pi} \frac{G^2}{2M} \left( \frac{\mu}{2M} \right)^2 \frac{2M}{3\pi} \left[ \sigma_1 \cdot \sigma_z \left( K_0(x) - \frac{K_1(x)}{x} \right) + S \left( K_0(x) + 2 \frac{K_1(x)}{x} \right) \right],$$

$$A = \frac{\tau_1 \cdot \tau_3}{4\pi} \frac{G^2}{2M} \frac{1}{x} \left[ (p \cdot \sigma_1) (r \cdot \sigma_2) + (p \cdot \sigma_2) (r \cdot \sigma_1) \right] \frac{\mu}{x} \left( \frac{e^{-\rho}}{x} \right),$$

$$S = 3(\sigma_1 \cdot r) (\sigma_2 \cdot r) / r^2 - \sigma_1 \cdot \sigma_2.$$

Equation (13) is a convenient way to write the second order potential because the commutator, which contains the most complicated spin, angle dependence of the potential, vanishes in the Born approximation. From this viewpoint, an even more useful form to write the second order potential is given below in Eq. (17).

The familiar static form of the potential is obtained by taking the simultaneous limits $\epsilon/M \to 0$, $p/M \to 0$; hence the static potential is given by (14).

The operator $A$ introduces a new spin, angle dependence into the nuclear potential, but it does not involve any spin-orbit coupling. (All other terms of the potential are clearly of the central force or tensor force variety.) To show this, only diagonal matrix elements of $V_c$ in the quantum numbers $J$, $L$, and $S$ need to be considered, i.e.
The Born approximation scattering amplitude giving a simple way of seeing exactly how much the non-locality contributes to the scattering at various energies. For a lab. energy of 300 Mev, the series in (16) becomes

\[ 1 - 0.320 + 0.063 = 1 - 0.320 + 0.077 - \ldots \]

The left side of this equation is the numerical value of \((M/E(p))^4\) at 300 Mev written to easily compare it with the term by term evaluation of the series in (16). Thus the non-locality of the interaction decreases the static Born approximation result by about 25 per cent at 300 Mev, and the non-locality is quite accurately represented by the lowest order velocity terms.*

The energy dependence of \(V_z\) plays an interesting role in calculating the Born approximation scattering amplitude. For in the Born approximation, \(\bar{e} = p^2/M\) and the operator \(\hat{p}^2\) becomes a numerical factor; thus the terms involving \(K\) in Eq. (13) cancel identically! This cancellation is instrumental in obtaining agreement with

\* Signell also pointed out that the corrections due to the long range component of the lowest-order velocity-dependent potential effectively weaken the second order potential. See P. Signell, Prog. Theor. Phys. 22 (1959), 492.
the lowest order S-matrix scattering amplitude and emphasizes the necessity of retaining an explicit energy dependence in the potential.

Writing Eq. (13) as

\[ V_\varepsilon = V_0 \left( 1 - 2 \frac{\varepsilon}{M} \right) + \left[ \frac{\varepsilon}{M} - \frac{p^2}{M^2}, U_1 \right] + \left[ \frac{p^2}{M^2}, U_2 \right], \tag{17} \]

where \( V_0 \) is given in (14) and

\[ U_1 = V_0 + \frac{1}{2} K, \]
\[ U_2 = \frac{1}{4} (V_0 - A), \]

only the first term of (17) contributes to the Born approximation scattering amplitude.

\section{7. Hermitian property}

It is easy to see that the condition for the potential to be hermitian is satisfied in both the scalar and pseudoscalar theories. The hermitian conjugate potential is

\[ V_\varepsilon^\dagger(r, p) = \int d\hbar e^{ik\cdot r} K_\varepsilon^\dagger(i\hbar, k) = \int d\hbar K_\varepsilon^\dagger(i\hbar + k, k)e^{ik\cdot r}. \]

Simultaneously reversing the sign of \( k \) and the limits of integration gives

\[ V_\varepsilon^\dagger(r, p) = \int d\hbar K_\varepsilon^\dagger(i\hbar - k, -k)e^{-ik\cdot r}. \tag{18} \]

The dependence of \( K_\varepsilon \) on spin and charge operators has been suppressed since these operators are hermitian and essentially play no role other than as specified by the superscript "T". The superscript "T" stands for transposing the order of these spin and charge operators whenever they appear in non-commuting combinations, and accounting for anything else that would change under transposition (as cross products). Comparing Eqs. (10) and (18) it follows that the potential is hermitian in case

\[ K_\varepsilon^\dagger(p - k, -k) = K_\varepsilon(p, k) \]

or that \( K_\varepsilon \) is invariant under the simultaneous operations (along with any necessary transposition):

\[ k \rightarrow -k, \]
\[ p \rightarrow p - k. \]

The kernel in Eq. (3) satisfies this condition using either (4a) or (4b) at all energies.

One can also see directly that the potential is hermitian. The scalar potential
in (12) is obviously hermitian, and the pseudoscalar potential in (17) is readily shown to be hermitian using the relation \( U_x^2 = -U_x \).

**§ 8. Estimate of velocity corrections**

In this section a crude estimation of the size of the velocity corrections in the two-nucleon interaction as a function of the separation distance of the nucleons is presented.

The non-relativistic scattering amplitude is proportional to

\[
(e_{12} | V_{\text{sb}}(r) | 21) = (-P^2/M + V_0)\psi(r) = \varepsilon \phi(r),
\]

where \( V_0 \) is the static potential, the "effective" second order potential is

\[
V_{\text{eff}} = (1 - 2(p/M)^2) V_0 + (U_1 + U_2) V_0/M.
\]

It is emphasized that this effective potential is defined only within the matrix element (19). The approximation in (20) amounts to removing the velocity dependence in the second order potential by a single iteration of the Schrödinger equation, retaining only the leading term in powers of \( 1/M \). Thus the second term in (21), which is the correction to the Born approximation result, naturally manifests itself as a potential of fourth order range. Since it is assumed that the non-local effects have a perturbation effect on the scattering, it is clear that the effective potential becomes completely meaningless at small distances where the fourth order term of (21) begins to dominate over the second order term. One can then make a meaningful quantitative statement only from a detailed treatment of the original Schrödinger equation. A detailed treatment suitable for comparing various potentials could take the form of integrating from large distances into some small distance.

In diagonal states, the effective potential may now be given by

\[
V_{\text{eff}} = V_0 + S V_r.
\]

The appearance of a spin-orbit term in (22) is regarded as spurious, and has been dropped, since the exact second order potential does not have such a term. To facilitate matters, the approximation

\[
r \cdot P \approx \frac{1}{2} \left( r \frac{\partial}{\partial r} - 3 \right)
\]

is used. This approximation is understood to be made within the matrix element (19) between diagonal states; it is exact in the Born approximation.

In terms of spin and parity eigenstates, the six potentials defined by (22)
have been numerically evaluated at 0 and 300 Mev lab. energies and compared with the respective static potentials. Table I is based on these calculations. Triplet states are affected more by the velocity dependence than are singlet states, and even parity triplet states are affected earlier (i.e. at larger ranges) than odd parity triplet states. Singlet even and singlet odd states are roughly affected to the same degree. At larger distances the change in any of the potentials from the static case is greater with increasing energy. At smaller distances, the velocity corrections (which are cast into a static mold in the second term of (21)) are predominant.

Table I. Ratio of effective to static potentials in each spin and parity state. Values are tabulated at various separation distances for lab. energies of 0 and 300 Mev. The separation distance is expressed in units of \( x = \mu r \); thus \( x = 1 \) corresponds to a separation distance of 1.414 f.

<table>
<thead>
<tr>
<th></th>
<th>Triplet Even States</th>
<th>Triplet Odd States</th>
<th>Singlet States</th>
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<tbody>
<tr>
<td></td>
<td>tensor</td>
<td>central</td>
<td>tensor</td>
</tr>
<tr>
<td></td>
<td>0 Mev</td>
<td>300 Mev</td>
<td>0 Mev</td>
</tr>
<tr>
<td>0.7</td>
<td>4.75</td>
<td>.43</td>
<td>-2.51</td>
</tr>
<tr>
<td>1.0</td>
<td>2.34</td>
<td>.202</td>
<td>.554</td>
</tr>
<tr>
<td>1.2</td>
<td>1.77</td>
<td>.45</td>
<td>.744</td>
</tr>
<tr>
<td>1.4</td>
<td>1.47</td>
<td>.15</td>
<td>-9.23</td>
</tr>
<tr>
<td>1.6</td>
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<td>1.20</td>
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<td>-2.64</td>
</tr>
<tr>
<td>2.0</td>
<td>1.13</td>
<td>.814</td>
<td>-1.34</td>
</tr>
<tr>
<td>2.5</td>
<td>1.06</td>
<td>.735</td>
<td>.117</td>
</tr>
<tr>
<td>3.5</td>
<td>1.01</td>
<td>.692</td>
<td>.825</td>
</tr>
<tr>
<td>( \infty )</td>
<td>1.00</td>
<td>.680</td>
<td>1.00</td>
</tr>
</tbody>
</table>

In the triplet even state, the effective tensor force is more attractive than the static result (it is less attractive at 300 Mev at large distances); in the triplet odd state, the effective tensor force is less repulsive than the static result. For \( r \leq 1.4f \), the velocity corrections begin to predominate and, therefore, the approximations begin to fail. In the triplet odd state, the effective tensor force actually "changes" the sign of the static result, becoming strongly attractive near the origin. As mentioned before, this change in sign is an indication of a complete breakdown of the approximation (20) and at best indicates the static potential is severely weakened.

In triplet states, the effective central potentials are drastically different from the static potentials. With the approximations made, the effective triplet even central potential "changes sign" to become repulsive at about 3f! The effective
triplet odd central potential is always repulsive but more than doubles the static result at even 2.5 f.

In singlet states, the approximations seem to apply quite well since velocity corrections are not very large even at r = 1.0 f. In the singlet odd state, the effective potential is slightly less repulsive than the static potential. In the singlet even state, the effective potential is slightly less attractive. So in both singlet states, the velocity corrections tend to slightly weaken the static potential.

(Comparisons of the effective potential with the work of others are discussed below.)

§ 9. Conclusions and discussion

The second order scalar and pseudoscalar potentials have been calculated taking account of first order recoil effects that the nucleons undergo in the process of interaction. Because of recoil, the two-nucleon interaction is of course non-local. Defining the two-nucleon probability amplitude by the Tamm-Dancoff method, the non-locality of the interaction is expressed by a "potential" which is both velocity dependent and energy dependent. The energy dependence in the potential is perhaps undesirable from the standpoint of simplicity, but if the Tamm-Dancoff technique is regarded as exact for exchanges of a given number of mesons, then the energy occurs in a natural application of the theory.

With local potentials, the stationary state continuity equation is expressed by

\[ \mathbf{P} \cdot \mathbf{S} = 0, \quad (23) \]

where \( \mathbf{S} \) is the familiar probability flux density. Non-local potentials, however, provide a source of probability at a point \( \mathbf{r} \) from all other points \( \mathbf{r}' \). Thus, for non-local potentials, the stationary state continuity equation is

\[ \mathbf{P} \cdot \mathbf{S} = -i(\phi^* V \phi - \phi (V \phi)^*). \quad (24) \]

The same flux density, \( \mathbf{S} \), is used in (23) and (24). (The right side of (24) of course vanishes for local potentials.) As long as the non-local potential in (24) is hermitian, the total probability will be conserved. This follows by integrating (24) over all space and using the hermitian property of the potential. Thus, for hermitian potentials, the stationary state wave function can be normalized in the usual way, conserving the total probability.

The energy dependence of the second order potential implies that non-degenerate eigenfunctions of the non-relativistic Hamiltonian are not orthogonal. The energy dependence in the two-nucleon interaction arises from the fact that the interaction Hamiltonian, \( H_I \), is non-diagonal in the particle number representation. If \( H_I \) were diagonal in the particle number representation, so that it could not cause transitions between states with different numbers of particles, then each of the Tamm-Dancoff amplitudes would satisfy an independent integral equation instead of coupled equations with the same energy. That the non-relativistic energy, \( \varepsilon \), is real is consis-
tent with the hermitian property of the Hamiltonian but actually follows from the relativistic formulation of the theory. (In the proof that the second order interaction is hermitian (§ 7), it is assumed that \( \varepsilon \) is real.) The important completeness property of the two nucleon wave functions also follows from the original formulation of the theory. For an arbitrary state of two nucleons is merely the projection of \( \psi \) in (1), which must be complete, onto the subspace of two nucleons and no anti-nucleons or pions.

Since the ideal purpose of calculations as in the present work is to arrive at the non-relativistic equation of motion for two nucleons starting from a postulated relativistic formulation involving any number of particles, and since this purpose is only partially accomplished, a consistent attitude would be to regard the non-relativistic energy as a given number and the non-relativistic Schrödinger equation as an equation for the wave function for each given eigenvalue (i.e. not as an eigenvalue equation). Since it is known empirically that none of the two nucleon stationary states is degenerate, the complication of finding independent degenerate states from an approximate Schrödinger equation is not present.

It is interesting to note that the essential difference among the static potentials calculated by other investigators arises from the way the energy dependence of the two-nucleon interaction is handled. Brueckner and Watson (BW), for example, simply set the energy equal to zero to calculate the static potential.\(^{14}\) Thus, their potentials give the local approximation to the two-nucleon potential. Others have chosen to remove the energy dependence according to some prescription. Feldman does this in a general way using canonical transformations (each written as an infinite series of operators), but his theory is not exact in the sense that the Tamm-Dancoff procedure is exact.\(^{15}\) Fukuda, Sawada, and Taketani (FST) regard the energy dependence as an inconsistency in the method of deriving the two-nucleon potential.\(^{16}\) They interpret the energy dependence of the interaction as expressing a possible absorption between, say, the state of two nucleons and the state of two nucleons and one meson.\(^{17}\) They also remove the energy dependence by employing canonical transformations, again in approximate form, and redefining the two-nucleon wave function. As a result, the fourth order static potentials of both Feldman and FST differ from those of BW; this is because Feldman and FST have partially accounted for the non-locality of the two-nucleon interaction and have introduced a fourth order potential very much as was done in the "effective" potential considerations of the present work (§ 8).

In the present work, no transformation is applied to the momentum and energy dependence of the potential in order to remove or partially remove these dependences as by FST, Feldman, and more recently by Sugawara and Okubo (SO).\(^{18}\) Although energy dependent, our potential is hermitian and it is of interest to contrast the present approach with others where various transformations have been applied. We are not sure these transformations are necessary and they raise additional doubts as to convergence. As a consequence of not making any transfor-
motions, the Schrödinger equation describes the two physical nucleon component of the full state vector. This component fully describes the asymptotic wave function and, therefore, all the scattering. At small distances it will differ from the wave functions of other theories. There is considerable arbitrariness as to the interpretation of a phenomenological wave function relative to these theories.

The role of the non-locality of the second order two-nucleon interaction in scattering states was estimated in two ways. As a function of energy, it was shown that at 300 Mev the static Born approximation scattering amplitude is reduced about 25 per cent by the velocity corrections. As a function of separation distance, an "effective" potential was defined. Here much of the velocity dependence of the interaction was "transformed" into a static fourth order potential. Because of the large magnitude of the corrections to the static potential, the approximations made to define the effective potential break down at rather large distances in triplet states (1.5~3.0 $\text{fm}$) and somewhat smaller distances in singlet states (1 $\text{fm}$). The large corrections in triplet states are expected when noting that the highly singular tensor part of the second order static potential contributes to the effective potential as a multiplicative factor in both the central and tensor parts of the fourth order term of (21). Although they are of fourth order range, the effective velocity corrections to the second order static potential are so large in triplet states that they exhibit themselves, from a practical point of view, as potentials of second order range. Accordingly, in triplet states, the fourth order static potentials (purely local), which are the BW fourth order potentials, are quite negligible at distances where the second order velocity corrections change the second order static potential, say, 50 per cent. In singlet states, the fourth order BW potentials are as large or larger than the effective second order velocity corrections. It is clear, however, that the large changes in triplet states cannot be regarded as spurious and that corrections to the static potential are likely to be important even at large separation distances.

It is interesting to compare the effective potential in the present work with the potentials of FST$^{(16)}$ and SO.$^{(18)}$ The difference of the fourth order potentials of FST and BW corresponds to non-local corrections to the static second order potential in the present work (especially at large distances). In the region $1 \leq x \leq 1.5$, the non-local corrections of the effective potential at zero energy are found to be generally larger than the difference of the FST and BW potentials, sometimes by as much as an order of magnitude. The signs of the non-local corrections agree except for the triplet odd tensor force where the effective potential gives a negative correction.

The SO potentials at zero energy differ only slightly from the second order static potentials (14). Thus in triplet states the effective potential gives much larger corrections than SO, and only in the singlet even state is the SO correction larger. In all states, the signs of the non-local corrections agree.

All the BW and FST potentials converge to the second order static potential,
Physically plausible result. The SO potentials also have a long range velocity
is the same as found in the Born approximation scattering amplitude, Eq. (16), a
physically plausible result. The SO potentials also have a long range velocity
correction and converge to \( V_0(1 - (p/M)^2) \). This discrepancy is related to the
manner in which the energy dependence has been removed from the SO potentials.

Phenomenological work with the two-nucleon problem projects the spin-orbit
coupling into a favored position over the long range velocity components of the
second order potential. This fact results from the phase shift fit to the 310 Mev
proton-proton scattering data that is available.\(^{10,20,21}\) In order to achieve the
indicated \(^3\)P phase shift splitting, a potential must have a spin-orbit component
satisfying a particular strength relation with the tensor force. Since the usual
phenomenological tensor force is quite strong and long ranged, the phenomeno-
logical spin-orbit force is required to be quite strong—much stronger than expected
from meson theory.

The velocity and energy dependence of the meson theoretic potential offers a
possible explanation of this situation. The reason is that in the rough considera-
tions made with the effective potential above, it was observed that the velocity cor-
rections in the triplet odd states weakened the tensor force, particularly as the
energy is increased. If this weakening were taken into account in phenomeno-
logical work, the strength of the appropriate spin-orbit force may fall into line
with theoretical expectations.

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References

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5) A summary of all the possible velocity dependences permitted by invariance principles is
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which is universally of good repute. A review of the discrepancies among the various
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No. 3 (1956).
10) For example: M. M. Levy, Phys. Rev. 88 (1952), 725; K. A. Brueckner and K. M. Watson,
12) This result for $V_{\alpha\beta}$ has also been obtained recently by S. N. Gupta, Phys. Rev. Letters 2 (1959), 124.
13) This was first pointed out by M. Sugawara.
17) This interpretation is, of course, inconsistent with the present work where it is shown that the energy dependent second order two-nucleon interaction is hermitian and, therefore, probability is conserved.