If one quantizes with conventional methods the classical internal motion of the hyperspherical (Nakano) relativistic rotator, the relativistic Hamiltonian splits into the sum of two complex conjugate spherical complex three-dimensional rotators. The corresponding excited "levels" also separate into fine structure states, each "level" being naturally associated to a vector space irreducible under complex transformations isomorphic to the full Lorentz group. It finally turns out that every state is characterized by a set of quantum numbers with which we can classify typical families of rotator levels.

Introduction

In a recent paper in collaboration with T. Takabayasi we have developed the classical theory of relativistic rotators on the basis of Lagrangian and Hamiltonian formalism and stressed its connection with bilocal theory. This raises immediately a new problem: What happens when we substitute relativistic rotators for point particles as possible starting point of quantum mechanics? As we shall see it turns out that the exploration of this problem raises many difficult questions, but may also open far-reaching perspectives on the nature and physical interpretation of quantum mechanics itself.

However, in order not to confuse the picture, let us make a few preliminary remarks.

The first is that we consider this paper as a very simple first step towards the complete understanding of the preceding problem.

The second remark is that we shall now voluntarily and systematically leave aside all problems of physical interpretation (which shall be treated in subsequent papers*). The reader will find here only some mathematical consequences of the formal application of conventional methods of quantization. We only wish to say at this stage:

a. that we shall connect these rotators with real physical properties of the "subquantum level of matter" recently introduced as a necessary consequence of the causal interpretation of quantum mechanics.

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* To be submitted shortly to the Physical Review.
b. that we aim to justify later this conventional procedure on the basis of material properties of this subquantum level (such as continuity and pulse-like motions) so that its utilization is not necessarily connected with the usual difficulties of present field theories.

Now let us return to our problem. As one knows the passage from a line classical Hamiltonian \( H(\dot{q}_a, q_a^2) \) for the corresponding operator (acting on the wave function \( \phi \)) is realized by substituting operators \( -i\hbar \partial / \partial q_a^2 \) for the conjugate momenta \( \Pi_a^2 = (\partial / \partial q_a^2) L(q_a^2, \dot{q}_a^2) \) associated with the classical Lagrangian \( L(q_a^2, \dot{q}_a^2) \) or alternatively by replacing Poisson brackets \( [\Pi_a^2, q_a^2] = -i\hbar \).

One must remark here, as Möller and Synge\(^3\) have shown, that when one operates in Minkowski space a distinction must be made between the imaginary units. Indeed, when we introduce relativistic Euler angles the theory will be closely bound to the quaternion group (with complex coefficients) and we shall show that \( i \) and \( j \) can be identified with two of the three classical quaternionic units.

To illustrate this procedure we can apply this method of quantization to the classical point particle. As we have shown\(^3\) the line Lagrangian associated with its motion can be written:

\[
L = \frac{1}{2} m \dot{x}_a \dot{x}_a .
\]

If we therefore substitute \( -i\hbar \partial / \partial x_a \) for \( G_a = \partial L / \partial \dot{x}_a \) in the relativistic Hamiltonian,

\[
H = \frac{1}{2m} G_a G_a = -\frac{1}{2} mc^2,
\]

we get

\[
H = -\frac{\hbar^2}{2m} \Box,
\]

where \( \Box \) is the dalembertian and the corresponding wave equation just reduces to

\[
(\Box - \tau^2) \phi = 0
\]

with \( \tau^2 = (m^2 c^2 / \hbar^2) \), that is, the usual Klein-Gordon equation.

If we start now from a Nakano rotator\(^*\) one can show that its internal Lagrangian with the use of a moving tetrad \( \beta_a^2 \):

\*

Various types of Lagrangians implying a constant rest mass are possible. One of these proposed by one of us (J.P.V.) takes the simple form \( L_{\text{total}} = L_N + L \) with \( L_N = \alpha^2 \beta_a^2 \dot{x}_a^2 + (\lambda/2) (\dot{x}_a^2 + \vec{\alpha} + \beta) + \rho \lambda + \mu \delta \) where \( \alpha (r=1, 2, 3) \), \( \lambda, \rho, \mu \) (and \( \lambda, \rho \)) are Lagrange multipliers.
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\[ L = -\frac{1}{4}I\omega_{\alpha\beta}\omega_{\alpha\beta} + \lambda_{\mu}(b^{\mu}_\eta b^{\eta}_\xi - \delta_{\mu\xi}), \]

\[ (\omega_{\alpha\beta} = b^{\mu}_\alpha b^{\eta}_\beta, \ I: \text{moment of inertia}) \]

discussed in our first paper leads,\(^1\) when we introduce as new variables relativistic Euler angles to the Hamiltonian:

\[ H = \frac{1}{2I} (S_k^+ S_k^- + S_k^- S_k^+), \quad k = 1, 2, 3 \]

where the \( S_k^\pm \) are defined as projections of the angular momentum \( I\omega_{\alpha\beta} \) (expressed in terms of these new variables) on self-dual tensors associated by Einstein and Meyer to the moving tetrad. As we have seen, this Hamiltonian splits into two complex conjugate parts associated with complex conjugate three-dimensional complex rotations. According to our program, our present objective is just to analyze some mathematical aspects of the conventional quantization of this Hamiltonian using these new basic variables. This is an important point since the result of any quantization procedure may vary according to the system of variables chosen. In the point particle case we used \( x_\mu \) thus obtaining a wave in space-time. With relativistic Euler angles we quantize the tetrad orientation directly.* The quantum levels obtained in this way evidently correspond to what we can call “internal” states to distinguish them from the quantum “external” states associated with the motion of the tetrad’s origin \( x_\mu \). This latter question or more precisely the problem of the quantization of the motion of \( x_\mu \) associated with given “internal” states will be discussed in a subsequent paper.

This quantization will be done in three steps. In the first section we shall briefly discuss the new variables, the so-called relativistic Euler angles, which shall greatly simplify our quantization procedure and give the explicit form of the new Hamiltonian operator. As we shall see the theory will be invariant under the full group of complex conjugate three-dimensional rotations isomorphic to the full Lorentz group.

In § 2, we shall study the corresponding “eigenfunctions” and summarize certain mathematical results established by two of us (P. H. and J. P. V.).\(^4\)

In § 3, we shall study the connection between the new “internal” quantum rotator levels and irreducible representations of the complex threedimensional rotation group, connection which leads to a classification of these excited levels.

\[ \text{§ 1} \]

According to this program, let us first transform the Lagrangian with the help of the relativistic Euler angles:

* This quantization can only have a clear physical meaning if this orientation is quantized with respect to a kinematical frame \( a_\xi \) attached to the moving body. Such a frame can be determined for example by the principal axis of dilatation introduced by Fukutome. In the rest of this paper however we only assumed that the \( a_\xi \) correspond to a well defined frame in which \( L(\text{total}) \) is given.
The real quantities $\varphi_1^+, \theta_1^+, \phi_1^+$, represent the usual space Euler angles, while $\varphi_2^+, \varpi_2^+, \varphi_2^-$ represent hyperbolic angles (varying from $-\infty$ to $+\infty$) expressing pure Lorentz transformations. The physical interpretation of these six angles is clear. If one starts from a fixed tetrad $a_k^\pm$ we can determine the orientation of the moving tetrad $b_k^\pm$ which defines the relativistic rotator by a homogeneous Lorentz transform:

$$b_k^\pm = A_k^\pm a_k^\mp, \quad \mu, \xi \sim 1, 2, 3, 4.$$ 

where the $A_k^\pm$ are functions of $\omega^+$ and $\omega^-$ which we have given explicitly in a preceding paper.\(^5\) More precisely, if we associate to the tetrads $a_k^\pm$ and $b_k^\pm$ complex self-dual antisymmetrical tensors:

$$\Gamma_{\mu\nu}^\pm = \varepsilon^{\mu\nu\rho} b_\rho^+, b_\rho^\pm \pm (b_\rho^+ b_\rho^\pm - b_\rho^- b_\rho^\pm),$$

and

$$A_{\mu\nu}^\pm = \varepsilon^{\mu\nu\rho} a_\rho^+, a_\rho^\pm \pm (a_\rho^+ a_\rho^\pm - a_\rho^- a_\rho^\pm), \quad r, s, t \sim 1, 2, 3$$

we can show that their space-like parts:

$$B_k^\pm = b_k^+ b_k^\pm - b_k^- b_k^\pm \pm \varepsilon_{ijk} b_i^+ b_j^\pm,$$

$$A_k^\pm = a_k^+ a_k^\pm - a_k^- a_k^\pm \pm \varepsilon_{ijk} a_i^+ a_j^\pm \quad r \sim 1, 2, 3 \quad i, j, k \sim 1, 2, 3$$

behave exactly like three orthogonal unitary complex vectors $B^r$ and $A^r$ belonging to two complex conjugate three-dimensional spaces $E^\pm$.

Indeed, one can show\(^3\) that under a homogeneous Lorentz transform defined by the six Euler angles, the $A^\pm$ transform into the $B^\mp$ in $E^\mp$, according to the complex three-dimensional rotation defined by the Euler angles $\omega^+$; while $A^\mp$ transform into $B^\mp$ in $E^\mp$ under the rotation determined by $\omega^-$. We note here that in the non-relativistic limit $B_k^\pm \rightarrow b_k^\pm$ and $A_k^\pm \rightarrow a_k^\pm$ so that the classical limit of our Nakano rotator is just the classical spherical rigid body.

If we consider an observer in the frame $B_k^\pm$ or $b_k^\pm$ he will see $B_k^\mp = B_k^\mp \equiv b_k^\mp$, while an observer in the frames $A_k^\pm$ will see $A_k^\mp = A_k^\mp \equiv a_k^\mp$ so that the preceding formulas just correspond to the relativistic transformation of orthogonal three frames. Mathematically, this implies that if we define in ordinary three-dimensional space three fixed unitary vectors $a_k^\pm$ and three moving unitary and orthogonal vectors $b_k^\pm$ we can determine matrix elements $A_k^\pm$ as functions of the three real Euler angles $\omega = \{\theta, \varphi, \psi\}$ by the relation..
We are now in a position to quantize $H$. But before we do that, in order to clarify our procedure, let us briefly recall certain well-known results of the quantization of the three-dimensional rigid spherical rotator, or rather the quantization of its rotational motion, when we leave aside its radial behaviour.

If we characterize this rotator by a moving frame of three orthogonal unitary vectors $b^r_\ell$ and depict their orientation with respect to a fixed frame $a^r_\ell$ by three Euler angles $\theta, \varphi, \psi$, the classical rotation Lagrangian associated to the spherical case becomes ($I=$ moment of inertia)

$$L = \frac{1}{4} I \omega_{ij} \omega_{ij} + \lambda_{ij} (b^r_\ell b^r_\ell - \delta_{ij}),$$

where

$$\omega_{ij} = b^r_i b^r_j,$$

or

$$L = \frac{1}{2} \dot{b}^r_i \dot{b}^r_i + \lambda_{ij} (b^r_i b^r_j - \delta_{ij}),$$

$$L = \frac{I}{2} (\dot{\varphi}^2 + \dot{\psi}^2 + \dot{\varphi} \dot{\psi} + 2 \dot{\varphi} \dot{\psi} \cos \theta),$$

with $\dot{G} = dG/dt$, when expressed in terms of Euler angles. Introducing the canonical momenta $p_\theta = \partial L/\partial \dot{\theta}$, $p_\varphi = \partial L/\partial \dot{\varphi}$ and $p_\psi = \partial L/\partial \dot{\psi}$ we obtain for the projections of the angular momentum $S$ on both frames certain functions of $\theta, \varphi, \psi, p_\theta, p_\varphi, and p_\psi$. Let us call

$$S_k \quad \text{the projection of } S \text{ on } a^r_\ell,$$

$$S'_k \quad \text{the projections of } S \text{ on } b^r_\ell,$$

where

$$S_i = I \omega_i = \frac{I}{2} \epsilon_{ijk} \omega_{jk}.$$

We discover after a simple calculation that the Hamiltonian becomes

$$H = \frac{1}{2I} S_k S_k = \frac{1}{2I} S'_k S'_k.$$

The quantization is performed as usual. Replacing $p_\theta, p_\varphi, p_\psi$ by $-i\hbar \cdot \partial/\partial \theta$, $-i\hbar \cdot \partial/\partial \varphi$ and $-i\hbar \cdot \partial/\partial \psi$, we get for the projections of $S$ on $a^r_\ell$ the operators $J_i$ defined by the expressions
These operators are exactly proportional to the infinitesimal rotation operators associated with the directions $a_k$ and $b_k$. We see also that the quantized Hamiltonian becomes

$$H = \frac{1}{2I} (J)^2,$$

with

$$(J)^2 = J_k J_k = J'_k J'_k.$$

We know also that these operators satisfy the commutation relations of the three-dimensional infinitesimal rotation group:

$$[[J_i, J_j]] = -j\hbar J_k,$$

$$[[J'_i, J'_j]] = -j\hbar J'_k,$$

$$[[J_i, J'_j]] = 0,$$

besides

$$[H, J_k] = [H, J'_k] = 0,$$

and then we obtain the angular part of all eigenstates by the simultaneous eigenfunctions of

$$(J)^2, J_3 \text{ and } J'_3,$$

that is, the set $Y_{l,m,m'}(\omega)$ satisfying

$$J^2 Y_{l,m,m'}(\omega) = l(l+1) Y_{l,m,m'}(\omega),$$

$$J_3 Y_{l,m,m'}(\omega) = m Y_{l,m,m'}(\omega),$$

$$J'_3 Y_{l,m,m'}(\omega) = m' Y_{l,m,m'}(\omega),$$

where we put $\hbar = 1,$
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with \( l=0, 1/2, 1, 3/2, \ldots \); \( m, m'=-l, -l+1, \ldots, l-1, l \),

and

\[
Y^m_{l, m'}(\omega) = \left( \frac{\sin \frac{\theta}{2}}{d} \right)^{-m+m'} \left( \frac{\cos \frac{\theta}{2}}{d} \right)^{-m-m'} \frac{d^{l-m}}{d \left( \sin^{2} \frac{\theta}{2} \right)^{l-m}}
\]

\[
\times \left[ \left( \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right)^{l-m'} \left( \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right)^{l+m'} \right] e^{i\omega (m+m') \phi}, \tag{2}
\]

(corresponding to the well-known energy levels \( l(l+1) \)). We know moreover that when we fix \( l \) and \( m' \), the set \( Y^m_{l, m'}(\omega) \) determines a subspace of the classical Hilbert space which transforms into itself under an irreducible representation \( \mathcal{D}(l) \) of the three-dimensional rotation group.

We are now in a position to quantize \( H \). As we have seen in a preceding paper\(^1\) if we replace \( d/dt \) by \( d/d\tau \) the scalar relativistic Hamiltonian \( H \) splits into the sum of two complex conjugate Hamiltonians,

\[
H = \frac{1}{2l} (M_{\alpha\beta} M_{\alpha\beta} + M_{\bar{\alpha}\beta} M_{\bar{\alpha}\beta})
\]

\[
= \frac{1}{2l} (\dot{B}_k^+ \dot{B}_k^- + \dot{B}^- k_+ \dot{B}^- k_-)
\]

\[
= H^+ + H^-.
\]

That is exactly the sum of the Hamiltonians of two complex conjugate three-dimensional rigid rotators (with \( d/d\tau \) replacing \( d/dt \)). We have shown also that if we call \( S^\pm_k \) the projections of the total angular momentum \( M_{\alpha\beta} \) on \( A^\pm_\nu \), that is,

\[ S^\pm_k = M_{\alpha\beta} A^\pm_{\alpha\beta}, \]

and similarly the projections on the moving frame

\[ S'^\pm_k = M_{\alpha\beta} \Gamma^\pm_{\alpha\beta}, \]

we have

\[
H = \frac{1}{2l} (S^+_k S_k^+ + S^-_k S_k^-)
\]

\[
= \frac{1}{2l} (S'^+_k S'^-_k + S'^-_k S'^+_k),
\]

where \( S^\pm_k \) and \( S'^\pm_k \) are exactly given by the three-dimensional usual relations when we replace \( \omega \) by \( \omega^\pm \). This is remarkable since \( S^+_k \) and \( S'^+_k \) depend only on \( \omega^+ \), while \( S^-_k \) and \( S'^-_k \) are the same functions with \( \omega^- \) replacing \( \omega^+ \).

As stated before, we perform the quantization by substituting the operators \(-j\hbar \cdot \partial / \partial \omega^\pm \) for the quantities \( \rho_{\pm} = \partial L / \partial \omega^\pm \). This replaces the quantities \( S^\pm_k \) by the set of operators.
which are proportional to infinitesimal transformations conserving bivectors $A^\pm_{\nu}$ in space-time (or infinitesimal three-dimensional complex rotations around $A^\pm_\nu$ in $E^\pm$), while $S^\pm_\nu$ become

$$
J^\pm_\nu = -j\hbar \left( \sin \phi^\pm \partial / \partial \theta^\pm - \cos \phi^\pm \cot \theta^\pm \partial / \partial \theta^\pm + \frac{\cos \phi^\pm}{\sin \theta^\pm} \partial / \partial \phi^\pm \right),
$$

having the same signification with respect to $P^\pm_\nu$ and $B^\pm_\nu$. These operators satisfy the commutation relations of the three-dimensional infinitesimal rotation group, that is,

$$
[J^\pm_i, J^\pm_j] = -j\hbar J^\pm_k, \quad i, j, k = 1, 2, 3.
$$

This is not astonishing since they are associated with complex rotations in $E^\pm$ isomorphic to Lorentz transformations in Minkowski space-time. We find also

$$
[H, J^+_3] = [H, J^-_3] = [J^+_k, J^-_k] = 0.
$$

Besides

$$
[H, J^+_3] = [H, J^-_3] = 0,
$$

Introducing further the operators

$$
J^\pm = (J^\pm_1)^2 = (J^\pm_2)^2 = J^\pm_3 J^\pm_1 = J^\pm_2 J^\pm_2,
$$

satisfying the commutation relations

$$
[(J^\pm)^2, J^\pm_3] = [(J^\pm)^2, J^\pm_2] = 0,
$$

we see that the quantized Hamiltonian becomes

$$
H = H^+ + H^- = \frac{1}{2\hbar} (J^+_3 J^+_1 + J^-_3 J^-_1),
$$
$H^+$ commuting with $H^-$. Naturally this implies, if we generalize the usual three-dimensional procedure, that the “levels” of the hyperspherical rotator are products of the eigenfunctions $Y_{l^+}^{m^+, m_l^+}(\omega^+)$ of the three operators $J_{k}^{+}, J_{k}^{-}, J_{k}^{0}$, by the eigenfunctions $Y_{l^+}^{m^-, m_l^-}(\omega^-)$ of $J_{k}^{+}, J_{k}^{-}$ and $J_{k}^{0}$ with

$$
\begin{align*}
J_{k}^{+} J_{k}^{-} Y_{l^+}^{m^+, m_l^+}(\omega^+) &= [l^+ (l^+ + 1)] Y_{l^+}^{m_l^+, m_l^+}(\omega^+), \\
J_{k}^{0} Y_{l^+}^{m^+, m_l^+}(\omega^+) &= m^+_l Y_{l^+}^{m^+, m_l^+}(\omega^+), \\
J_{k}^{-} Y_{l^+}^{m^+, m_l^+}(\omega^+) &= m^-_l Y_{l^+}^{m^+, m_l^+}(\omega^+), \\
(\text{we put here } \hbar = 1)
\end{align*}
$$

$$
\begin{align*}
l^+, l^- &= 0, 1/2, 1, 3/2, \ldots , \\
m^+, m^- &= -l^+ + l^+ - 1, \ldots , l^+ - 1, l^+, \\
m^+_l, m^-_l &= -l^+, -l^+ + 1, \ldots , l^+, l^+.
\end{align*}
$$

However, if we impose on $H$ the physical condition that it is a scalar under transformations of the full complex rotation group (that is, including reflections through the origin of the moving frame) we are led to associate the “angular” part of the eigenfunctions with simultaneous eigenfunctions of the six commuting operators

$$
J_{k}^{+}, J_{k}^{0}, J_{k}^{-}, J_{k}^{0}, S_{k}^{+}, S_{k}^{-}, S_{k}^{0}
$$

with

$$
S_{k}^{+} = J_{k}^{+} + J_{k}^{0}.
$$

As we shall see in the next section these are linear products of the $Y_{l^+}^{m^+, m_l^+}(\omega^+)$ and $Y_{l^+}^{m^-, m_l^-}(\omega^-)$ multiplied by suitable Clebsh-Gordan coefficients. These eigenfunctions which we shall denote $Z_{l^+}^{m^+, m_l^+}(\omega^+, \omega^-)$ constitute the four-dimensional generalization of the levels of the three-dimensional rotator. They satisfy transformations of the full complex rotation group and will now enable us to study the internal level classification of our relativistic rotator.

§ 2

According to our program we shall now recall certain mathematical results concerning the eigenfunctions of the operators defined in the preceding section. No detailed mathematical proofs shall be given here since they have already been published in specialized papers which will be quoted as references and can be consulted if necessary. Such a summary is however useful as starting point for the quantum theory of relativistic rotators. The following theorems can be established:

**Theorem I:** If one considers the set of operators $J_{k}^{+}, J_{k}^{0}, J_{k}^{-}, J_{k}^{+}$, it is possible to find a denumerable set of eigenfunctions $Y_{l^+}^{m^+, m_l^+}(\omega^+)$ satisfying relations (5) where the eigenvalues $l^+$ may take all real integer or half-integer values: 0, 1/2, 1, 3/2, \ldots ; the associated values of $m^+$ and $m^+_l$ being distributed over the set:
The theorem states that Pauli first remarked there is no mathematical reason to eliminate these levels, with half integer levels necessarily come back to the original orientation after a rotation of a rigid rotator by three moving axis (with components $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$). In this case if we describe as before the orientation under an irreducible representation of the three-dimensional rotation group. To show this it is enough to recall that if we describe a classical rigid rotator by three real axis (with components $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$), the components of the angular momentum are just given by operators $J_x, J_y, J_z$ which form a vector space which transforms under an irreducible representation $\mathcal{D}(l^+, l^-)$ of the three-dimensional complex rotation group. This result is a straightforward generalization of the classical three-dimensional theory of the rotation group. To show this it is enough to recall that if we describe a classical rigid rotator by three real axis (with components $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$) and determine their orientation with respect to a fixed frame $a_k^x$ by three real Euler angles, the components of the angular momentum are just given by operators $J_x, J_y, J_z$ deduced from $J_3^\pm$ and $J_5^\pm$ by putting $i\phi_3 = i\theta_3 = i\psi_3 = 0$. The square of the angular momentum given by $J^2 = J_3^2 + J_5^2$ is proportional to the classical quantum Hamiltonian so that the "energy" levels are proportional to $l(l+1)$. The corresponding angular part of eigenfunctions can written $Y_{l, m}^\omega (\omega)$ (with $\omega$ representing the real Euler angles $\theta, \phi, \psi$) and each "level" when we fix $l$ and $m$, splits into "sublevels" $Y_{l, m}^\omega (\omega)$ with $m$ taking all values $-l, -l+1, \ldots, l-1, l$ which form a vector space transforming under an irreducible representation of the three-dimensional rotation group.

The third remark is the most important from the physical point of view. The theorem states that $l^+$ (or $l^-$) can take all possible integer and half-integer positive values so that $l^\pm = 0, 1/2, 1, 3/2, \ldots$. To make this point clear, let us go back to three-dimensional theory. In this case if we describe as before the orientation of a rigid rotator by three moving axis $b_k^x$, we usually eliminate all half integer levels $l=1/2, 3/2, \ldots$ with the argument that with a rigid body we necessarily come back to the original orientation after a $2\pi$ angular rotation. As Pauli first remarked there is no mathematical reason to eliminate these levels.
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Indeed, as Lochak\(^7\) has demonstrated these levels are even necessary if the state function is to be continuous on the sphere of Cayley-Klein parameters. Physically their existence is evidently connected with non-rigid rotators. In that case, materialized for example by a liquid droplet, there is no reason why the body should not recover its original distribution after a \(4\pi\) rotation only or any higher multiple of \(2\pi\). Naturally one would expect such highly excited states to be rather unstable, but they are evidently permissible. Indeed, interesting consequences can be deduced from their possible existence in the domain of atomic spectra.\(^7\) This reasoning can evidently be extended to the relativistic domain. First, we know that in that case it is impossible to generalize in a covariant manner the classical notion of rigid body. The rotation of our \(b^x_k\) frame is then evidently physically connected with a relativistic fluid droplet so that there is no reason to eliminate the half integral values of \(l^+\) and \(l^-\) if we connect the self-dual tensors \(B^+_k\) with internal physical structures. Moreover, any disparity between \(l^+\) and \(l^-\) must be interpreted as connected with an internal disparity or distortion of the internal matter. Any attempt to transfer without precaution into the relativistic domain results connected with the classical notion of rigidity might lead to contradictory results.

The second theorem we want to discuss here is related to the physical assumption that the Lagrangian is invariant under transformations of the full complex rotation group; that is, including reflections through the origin of the moving tetrad \(b^x_k\).

We know already that products of functions \(Y^{m^+,m^-}_r(l^+,l^-)\) transform under the \(\mathcal{D}(l^+,l^-)\) representation of the proper complex rotation group.

Simple considerations\(^5\) then show the validity of the following theorem:

**Theorem II:** The set of eigenfunctions \(Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-)\) of the set of commuting operators \(J^+_k J^-_k, J^+_k J^-_k, J^+_k J^-_k, J^+_k, S_k^+ S_k^+\) and \(S'_k\) (with \(S'_k = J^+_k + J^-_k\)), that is, the functions satisfying

\[
\begin{align*}
J^+_k J^-_k Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-) &= \pm(l^+ + 1) Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-), \\
J^-_k J^+_k Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-) &= \pm Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-), \\
S'_k S'_k Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-) &= s'(s' + 1) Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-), \\
S'_k Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-) &= m' Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-),
\end{align*}
\]

(6)

transform when we fix the values of \(l^+, l^-, s', m'\) according to the representation \(\mathcal{D}(l^+,l^-) \oplus \mathcal{D}(l^-,l^+)\) of the full complex rotation group. In other terms, the the set of functions \(Z^{m^+,m^-,m^-,m^+}_{r',r',s',s'}(l^+,l^-)\) with \(l^+, l^-, s', m'\) fixed constitute a set of independent orthogonal vectors of a finite dimensional vector space transforming into itself under the complex rotation group according to the corresponding
representation. Evidently such vector spaces are subspaces of the general numerably infinite dimensional Hilbert space containing all finite dimensional representations of the full complex rotation group.

Before we discuss this last statement, we recall\(^{6}\) that these \(Z_{l_1^+}^{m_1^+, m_1^-} (\omega^+, \omega^-)\) functions are given in terms of the preceding \(Y_{l_2^+}^{m_2^+, m_2^-} (\omega^+)\) and \(Y_{l_2^-}^{m_2^-, m_2^+} (\omega^-)\) functions by the expressions

\[
Z_{l_1^+}^{m_1^+, m_1^-} (\omega^+, \omega^-) = \sum_{-m_1^-' \leq m_1^- \leq m_1'^{+}} (l_1^+, l_1^-, -m_1^-' | l_1^+, l_1^-, s_1', -m_1')
\times Y_{l_1^+}^{m_1^+, m_1'^{+}} (\omega^+) Y_{l_1^-}^{m_1^-, m_1'^{-}} (\omega^-)
\]

where

\[
s_1' = l_1^+ + l_1^-, l_1^+ + l_1^- - 1, \ldots, |l_1^+ - l_1^-|
\]

\[m_1' = -s_1', -s_1' + 1, \ldots, s_1' - 1, s_1'.\]

and \((l_1^+, l_1^-, -m_1^{'}, l_1^-, s_1', -m_1')\) are the usual Clebsch-Gordan parameters.

Now the problem of defining an orthogonal basis within each subspace associated to an irreducible representation of the full complex rotation group is closely connected with the question of the metric associated with this group. In order to clarify this point we shall briefly recall the classical results of the three-dimensional real rotation group expressed in terms of real Euler angles.

Introducing as before the operators \(J_1, J_2, J_3\) given by relations (1a), we know that the eigenfunctions of the three commuting operators \(J_3, J_2', J_3\) and \(J^2 = (J_1J_2') = (J_1' J_2')\), namely \(Y_{l,m} (\omega)\) (with \(J_3 Y_{l,m} (\omega) = m Y_{l,m} (\omega)\), \(J_2' Y_{l,m} (\omega) = m' Y_{l,m} (\omega)\) and \(J^2 Y_{l,m} (\omega) = (l+1) Y_{l,m} (\omega)\) build, when we fix \(l\) and \(m'\), a finite dimensional subspace of a general Hilbert space, which transforms according to the irreducible representation \(\hat{\mathcal{D}} (l)\) of the three-dimensional rotation group.

The metric of that Hilbert space is well known. If we take two functions \(f(\omega)\) and \(g(\omega)\), and call \(f^*(\omega)\) the complex conjugate function of \(f(\omega)\) (corresponding to \(j \rightarrow -j\)) we define the scalar product of \(f\) and \(g\) by the relation

\[
\langle f | g \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^\pi f^*(\omega) g(\omega) \sin \theta d\theta.
\]

This definition implies that \(\langle f | f \rangle\) is always positive (which means that the space has a positive definite metric) and that the preceding operators \(J_3, J_2', J_3\) are Hermitian operators in the sense that they satisfy the general relation \(\langle f | Af \rangle = \langle Af | g \rangle\). We also know that the \(Y_{l,m} (\omega)\) vectors satisfy the general orthogonality conditions:

\[
\langle Y_{l,m}^{m'} (\omega) | Y_{l,m}^{m''} (\omega) \rangle = \delta_{mm'} \delta_{m'm''}.
\]

Now our problem is to generalize these results to the space built on the \(Z_{l_1^+}^{m_1^+, m_1^-} (\omega^+, \omega^-)\) eigenfunctions of the operators \(J_3^+, (J^+)^2, J_3^-, (J^-)^2, S_x^2, (S_y)^2\). This is evidently a rather complex mathematical problem since the new space
cannot be compact because \( i \psi, i \theta, i \phi \) vary from \(-\infty\) to \(+\infty\). It can however be solved, as two of us (P. H. and J.-P.V.)\(^9\) have shown, if one introduces an indefinite metric in the sense recently utilized by Heisenberg\(^9\) and other authors.\(^10\)

We shall just summarize here results (for detailed demonstration, see ref. 4). First let us recall two definitions. We have seen, following Møller, that we must distinguish between two types of complex quantities, the first with \( i \) resulting from Minkowski's space-time, the second with \( j \) introduced by the quantization procedure. This implies two operations corresponding to complex conjugation \( i \to -i \) and \( j \to -j \). The second type we shall denote by \( * \) so that

\[
 f^*(j, \omega^+, \omega^-) = f(-j, \omega^+, \omega^-). 
\]

We can now demonstrate the following theorem.

**Theorem III:** If we associate to the Hilbert space of functions \( f(\omega^+, \omega^-) \) the indefinite metric defined by the scalar product:

\[
\langle f(\omega^+, \omega^-) | g(\omega^+, \omega^-) \rangle = \lim_{n_1, n_2} A \int d\psi_2 \int d\psi_3 \int d\theta_3 \\
\times \left[ \left( f^*(\omega^+, \omega^-) g(\omega^+, \omega^-) \right) | 1 - \cosh^2 \theta_3 \cos^2 \theta_1 | d\theta_1 \right],
\]

where the \([ \ ]\) under the \( \{ \} \) indicates that it is first performed with respect to \( \theta_1, \psi_1, \phi_1 \), \( A \) is a weighing factor defined by

\[
 A = e^{-(k+1)(n_0+n_0')} (B_1+B_2') (B_2+B_2'),
\]

\( L \) being the maximum of the numbers \( l^+ \) and \( l^- \) of the representation on which we can project \( f(\omega^+, \omega^-) \) and \( g(\omega^+, \omega^-) \) and we can show that the operators \( J_z, J_x, S_z, (J^+)^2, (J^-)^2 \) and \( (S')^2 \) are pseudo-Hermitian operators, meaning that they satisfy \( \langle f | A g \rangle = \langle A f | g \rangle \). We can also show that if we define the average value of an operator \( A \) by

\[
\overline{A} = \frac{\langle f | A f \rangle}{\langle f | f \rangle},
\]

we have

\[
 J_z = m_z, \\
 S_z = m', \\
 (J^+)^2 = l^+ (l^+ + 1), \\
 (S')^2 = s' (s' + 1),
\]

the numbers \( m_+, m_-, m', l^+, l^-, s' \) being all real.

One can also show that we have
\[ \langle Z_{\zeta^+}^{m_1, m_2}, m_3' \mid Z_{\zeta^-}^{m_1, m_2}, m_3' \rangle = \varepsilon \delta_{\zeta^+} \delta_{\zeta^-} \delta_{m_1 \rightarrow m_1'} \delta_{m_2 \rightarrow m_2'} \delta_{m_3' \rightarrow m_3}, \]

(with \( \varepsilon = \pm 1 \)), a relation which shows that it is possible in the vector space constructed on the functions \( Z_{\zeta^+}^{m_1, m_2}, m_3' \) to define a scalar product (and therefore a metric) in such a way that these vectors \( Z_{\zeta^+}^{m_1, m_2}, m_3' \) constitute an orthogonal and unitary basis. Naturally in this generalization of the usual Hilbert space we must distinguish two types of vectors, for they split in space-like vectors (\( \langle Z|Z \rangle > 0 \)) and time-like (\( \langle Z|Z \rangle < 0 \)) since the metric is indefinite.

We must remark also that the \( Z_{\zeta^+}^{m_1, m_2}, m_3' \) functions are not bounded. This raises many new physically interesting points which we shall discuss later. Evidently the introduction of this new metric is a new qualitative factor introduced in the theory. This is not very surprising. One would hardly expect that the new “internal” variables should be governed by exactly the same mechanics and interpretation as the usual “external” variables, for, according to our views, they belong to a new “subquantum level” of matter defined by distances smaller than \( 10^{-13} \) cm. Moreover, the corresponding operators are pseudo-Hermitian, that is, Hermitian with respect to the above defined indefinite metric. One notices also that this new metric is connected with the fact that the “state vectors” \( Z_{\zeta^+}^{m_1, m_2}, m_3' \) must be invariant under transformations of the full complex rotation group. It might be necessary, however, as suggested by Heisenberg, that one must restrict oneself for physical reasons to certain regions of this new Hilbert space such as the space-like part. We shall conclude this mathematical section by a few results established by two of us (P. H. and J.P. V.) on the symmetry properties of the \( Z_{\zeta^+}^{m_1, m_2}, m_3' \) functions under certain transformations such as \( P(x \rightarrow -x) \) since they will turn out later to be useful for further steps in the theory of relativistic rotators. We know, moreover, that such symmetry properties play an essential part in the theory of interactions.

First let us study the parity operation \( P \). We have by definition

\[
P a_{\pm} = -a_{\mp}, \quad P a_{\pm} = a_{\mp}, \quad P A_{\pm} = A_{\mp}, \quad \text{so that} \quad P B_{\pm} = B_{\mp}.
\]

We deduce therefrom that \( P \) transforms the angles \( \omega^+ \) into \( \omega^- \) (and vice-versa), so that taking into account the symmetry properties of the Clebsch-Gordan coefficients we finally get

\[
 P Z_{\zeta^+}^{m_1, m_2}, m_3' \mid Z_{\zeta^-}^{m_1, m_2}, m_3' \rangle = (-1)^{r^+ + r^- + r'} Z_{\zeta^+}^{m_1, m_2}, m_3' \mid Z_{\zeta^-}^{m_1, m_2}, m_3' \rangle.
\]

By definition also the time reversal operation \( T \) implies the relations

\[
 T a_{\pm} = -a_{\mp}, \quad T b_{\pm} = -b_{\mp}.
\]
But these correspond to non-linear transformations which imply the following operation on the relativistic Euler angles:

\[ \omega^+ \rightarrow \omega^-, \quad \omega^- \rightarrow \omega^+ , \]
\[ \theta_1 \rightarrow \theta_1 + \pi, \quad \theta_2 \rightarrow \theta_2 + i\pi , \]

so that we finally get after a short calculation

\[ T \mathcal{Z}_{r^+, r^-, r^+}^{m^+ m^- m'}(\omega^+, \omega^-) = (-1)^d \mathcal{Z}_{r^+, r^-, r^+}^{m^+ m^- m'}(\omega^+, \omega^-) , \]

with

\[ d = |m' - m^+ - m^-| . \]

The preceding operations evidently imply the existence of a third operation \( C \) (defined by \( CPT = 1 \)) which we shall call the transition from “state” to “anti-state”.

We find after a short calculation

\[ C \mathcal{Z}_{r^+, r^-, r^+}^{m^+ m^- m'}(\omega^+, \omega^-) = (-1)^a \mathcal{Z}_{r^+, r^-, r^+}^{m^+ m^- m'}(\omega^+, \omega^-) , \]

with \( a = l^- + l^- + d - s' \). Evidently the so-called “anti-states” are just as real as the “states” themselves. They are just associated with a different irreducible representation of the complex rotation group since we pass from state \( \mathcal{Z}_{r^+, r^-, r^+}^{m^+ m^- m'}(\omega^+, \omega^-) \) to anti-state by the transformation:

\[ m^+ \rightarrow -m^+ , \quad m^- \rightarrow -m^- , \quad m' \rightarrow -m' . \]

We have only adopted this denomination because, as we shall see later, the spinors corresponding to particles and anti-particles are precisely obtained through the preceding transformation.

§ 3

We are now in a position to classify the “levels” associated to our new “internal” Hamiltonian \( H \).

This classification is evidently a straightforward generalization of the classification of the angular part of the energy level of the three-dimensional rotator expressed in terms of real Euler angles \( \theta, \varphi, \phi \). In order to clarify our procedure, we shall once more recall very briefly certain points of that well-known case, closely connected with the four-dimensional relativistic rotator.

As we have seen in the first section the angular parts of the state functions are just the eigenfunctions \( Y_{l^-, l^-, l^+}^{m, m'}(\theta, \varphi, \phi) \) corresponding to the well-known energy levels \( I(l + 1) \).

When we fix \( m' \), the energy levels split into sub-levels corresponding to different values of \( m \); the corresponding vector space transforming under the irreducible representation \( \mathcal{D}(l) \) of the three-dimensional rotation group.
The corresponding classification of levels is evident. For example:

I. for \( l=\frac{1}{2}, m'=\frac{1}{2} \) (states \( \frac{1}{2} \)), we have two sub-levels \( Y_{\frac{1}{2},\frac{1}{2}}(\theta, \varphi, \psi) \) and \( Y_{\frac{3}{2},\frac{1}{2}}(\theta, \varphi, \psi) \) represented by the table:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( m' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{1}{2})</td>
<td>(-\frac{1}{2})</td>
</tr>
<tr>
<td>(+\frac{1}{2})</td>
<td>(-\frac{1}{2})</td>
</tr>
</tbody>
</table>

The columns characterizing the values of \( m \) and \( m' \).

II. for \( l=\frac{1}{2}, m'=-\frac{1}{2} \) (anti-states \( \frac{1}{2} \)), we obtain two sub-levels described by the table:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( m' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>(-\frac{1}{2})</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

III. for \( l=1, m'=1 \) (states \( 1 \)), we get the three sub-levels corresponding to \( m=1, 0, -1 \) corresponding to the table:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( m' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( -1 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

and so on ad infinitum for all possible values of \( l \) and \( m' \).

We can now pass to four dimensions and Minkowski space.

The “levels” associated to \( H \) are evidently also determined by a set of state (or anti-state) functions \( Z_{l',-l'; m',-m'}(\omega^+, \omega^-) \), the eigenvalue associated to \( H \) being

\[ l^+ (l^+ + 1) + l^- (l^- + 1). \]

Each “level” evidently splits into “sub-levels” for if we fix \( l^+, l^-, s', m' \), we have various possibilities for \( m^+ \) and \( m^- \). According to the results of the preceding section we shall classify the levels into families, each family being associated with a specific irreducible representation of the full complex rotation group.

Every sub-level is associated to a function \( Z_{l^+, l^-, m^+, m^-}(\omega^+, \omega^-) \) and characterized in the classification by fixed values of \( l^+, l^-, s', m' \), that is to say, by table elements of the form:

<table>
<thead>
<tr>
<th>( m^+ )</th>
<th>( m^- )</th>
<th>( m' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m^+ )</td>
<td>( m^- )</td>
<td>( m' )</td>
</tr>
</tbody>
</table>
To every sub-level corresponds an anti-level with the same values of $l^+$, $l^-$, $s'$; but opposite signs of $m^+$, $m^-$, $m'$, that is, table elements:

<table>
<thead>
<tr>
<th>$Z_{l^+, l^-, s', m'}^{\omega^+, \omega^-}$</th>
<th>$-m^+$</th>
<th>$-m^-$</th>
<th>$-m'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega^+$, $\omega^-$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega^+$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega^+$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega^+$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Naturally there is an infinite number of possible levels corresponding to increasing values of $l^+$, $l^-$ and $s'$. One sees also that higher levels contain more and more sub-level families (each associated with an irreducible representation of the full complex rotation group) since $m^+$ and $m^-$ vary between $l^+$ and $-l^+$, $l^-$ and $-l^-$ while $s'$ and also $m'$ can take any value between $l^+ + l^-$ and $|l^+ - l^-|$. As an example we shall only give here the tables corresponding to the lowest levels with minimum values of $s'$, that is, $s' = |l^+ - l^-|$.  

**A. Representation $\mathcal{D} (1/2, 0) \oplus \mathcal{D} (0, 1/2)$**

For $l^+, l^- = \{1/2, 0\}$, $s' = 1/2$, $m' = -1/2$ we have a family associated with the $\mathcal{D} (1/2, 0) \oplus \mathcal{D} (0, 1/2)$ representation of the full complex group and the table:

<table>
<thead>
<tr>
<th>$m^+$</th>
<th>$m^-$</th>
<th>$m'$</th>
<th>$m^+ + m^- + m'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1/2$</td>
<td>0</td>
<td>$-1/2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$1/2$</td>
<td>0</td>
<td>$-1/2$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$-1/2$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The corresponding sub-space being four-dimensional. The anti-states corresponding to the same representation with $l^+, l^- = \{1/2, 0\}$, $s' = 1/2$, $m' = +1/2$. The introduction of the fourth column in which are indicated the values of $m^+ + m^- + m'$ of each sub-level $Z_{l^+, l^-, s', m'}^{\omega^+, \omega^-}$ will be explained later.

**B. Representation $\mathcal{D} (1, 0) \oplus \mathcal{D} (0, 1)$**

For $l^+, l^- = \{1, 0\}$, $s' = 1$, $m' = 0$ we have associated the $\mathcal{D} (1, 0)$ and $\mathcal{D} (0, 1)$ representations, that is,

<table>
<thead>
<tr>
<th>$m^+$</th>
<th>$m^-$</th>
<th>$m'$</th>
<th>$m^+ + m^- + m'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Corresponding to $\mathcal{D} (1, 0)$, and

<table>
<thead>
<tr>
<th>$m^+$</th>
<th>$m^-$</th>
<th>$m'$</th>
<th>$m^+ + m^- + m'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
corresponding to \( \mathcal{D}(0, 1) \), the sub-space of the general Hilbert space defined in the preceding section being six-dimensional.

C. Representation \( \mathcal{D}(1, 1) \):
   For \( \mathcal{D}(1, 1) \)  \( s' = 0, m' = 0 \) we shall not explicitly give the corresponding table.

D. Representation \( \mathcal{D}(1/2, 1/2) \):
   For \( \mathcal{D}(1/2, 1/2) \)  \( s' = 0, m' = 0 \) we have

<table>
<thead>
<tr>
<th>( m^+ )</th>
<th>( m^- )</th>
<th>( m' )</th>
<th>( m^+ + m^- + m' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1/2)</td>
<td>(-1/2)</td>
<td>0</td>
<td>(-1)</td>
</tr>
<tr>
<td>(1/2)</td>
<td>(-1/2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(-1/2)</td>
<td>(1/2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(1/2)</td>
<td>(1/2)</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

the last two sub-levels being the anti-levels of the two first.

E. Representation \( \mathcal{D}(1, 1/2) \oplus \mathcal{D}(1/2, 1) \):
   For \( \{l^+, l^-\} = \{1, 1/2\} \) with \( s' = 1/2, m' = 1/2 \) we get, corresponding to \( \mathcal{D}(1/2, 1) \),

<table>
<thead>
<tr>
<th>( m^+ )</th>
<th>( m^- )</th>
<th>( m' )</th>
<th>( m^+ + m^- + m' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1/2)</td>
<td>(-1)</td>
<td>(1/2)</td>
<td>0</td>
</tr>
<tr>
<td>(1/2)</td>
<td>0</td>
<td>(1/2)</td>
<td>1</td>
</tr>
<tr>
<td>(1/2)</td>
<td>1</td>
<td>(1/2)</td>
<td>2</td>
</tr>
<tr>
<td>(-1/2)</td>
<td>(-1)</td>
<td>(1/2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(-1/2)</td>
<td>0</td>
<td>(1/2)</td>
<td>0</td>
</tr>
<tr>
<td>(-1/2)</td>
<td>1</td>
<td>(1/2)</td>
<td>1</td>
</tr>
</tbody>
</table>

and corresponding to \( \mathcal{D}(1, 1/2) \),

<table>
<thead>
<tr>
<th>( m^+ )</th>
<th>( m^- )</th>
<th>( m' )</th>
<th>( m^+ + m^- + m' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(1/2)</td>
<td>(1/2)</td>
<td>0</td>
</tr>
<tr>
<td>(0)</td>
<td>(1/2)</td>
<td>(1/2)</td>
<td>1</td>
</tr>
<tr>
<td>(1)</td>
<td>(1/2)</td>
<td>(1/2)</td>
<td>2</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1/2)</td>
<td>(1/2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(0)</td>
<td>(-1/2)</td>
<td>(1/2)</td>
<td>0</td>
</tr>
<tr>
<td>(1)</td>
<td>(-1/2)</td>
<td>(1/2)</td>
<td>1</td>
</tr>
</tbody>
</table>

The anti-states are obtained with \( m' = -1/2 \). This classification has a series of remarkable characteristics.

We know that a Lorentz transform from an initial tetrad \( a^\xi_i \) to a final tetrad \( b^\xi_i \) does not define the latter completely since it is always possible to make afterwards an arbitrary space-like rotation in a space-like hyperplane. As a consequence the Lagrangian of the hyperspherical rotator is necessarily invariant under the transformation
Since $\varphi_1$ and $\psi_1$ precisely correspond to such rotations. Let us tentatively call such transformations gauge transformations. Under these transforms the $Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}$ functions become

$$e^{i\delta} Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-) = e^{i[(m^+ + m^-) \alpha + m' \beta]} Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-),$$

if we take into account a well-known property of the Clebsch-Gordan coefficients (they vanish if $m^+ + m^- \neq m'$), so that under the infinitesimal transform

$$\varphi_1 \rightarrow \varphi_1 + \varepsilon,$$

$$\psi_1 \rightarrow \psi_1 + \varepsilon,$$

we get

$$e^{i\delta} Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-) = Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-) + j(m^+ + m^- + m') \varepsilon Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-)$$

$$= Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-) + j \left[ -j \frac{\partial}{\partial \varphi_1} - j \frac{\partial}{\partial \psi_1} \right] \varepsilon Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-).$$

We deduce therefrom that the tetrad's transform

$$\varphi_1 \rightarrow \varphi_1 + \varepsilon,$$

$$\psi_1 \rightarrow \psi_1 + \varepsilon,$$

generates in the vector space of the $Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-)$ functions the operators

$$N = -j \frac{\partial}{\partial \varphi_1},$$

$$M = -j \frac{\partial}{\partial \psi_1},$$

$$Q = N + M = -j(\partial/\partial \varphi_1 + \partial/\partial \psi_1).$$

When applied to any state $Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-)$, they give

$$MZ^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-) = (m^+ + m^-) Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-),$$

$$NZ^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-) = m' Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-),$$

$$QZ^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-) = (m^+ + m^- + m') Z^{m^+, m^-, m'}_{\ell^+, \ell^-, \ell', s}(\omega^+, \omega^-),$$

with

$$\bar{M} = m^+ + m^-,$$

$$\bar{N} = n^+ + n^-,$$

$$\bar{Q} = m^+ + m^- + m'.$$

As $m^+ + m^-$ and $m'$ must be separately and simultaneously integer or half
integer numbers, $Q$ is always integer. Operator $Q$ can thus in a sense be compared to the usual charge operator with respect to our states, $N$ and $M$ being comparable to Fermionic and isotopic spin charge.

If we calculate matrix elements between states introducing the indefinite metric, we obtain certain rules comparable to selection laws.

For example, if we introduce the above definition of the metric (which we can call strong metric) we have shown elsewhere\(^5\) that the matrix elements

$$\langle Z_{\lambda_1, \lambda_2, \lambda_3}^{m_1, m_2, m_3} (\omega^+, \omega^-) | Z_{\lambda_2, \lambda_3, \lambda_4}^{m_2, m_3, m_4} (\omega^+, \omega^-) \rangle$$

are zero unless

$$\Delta m^+=0,$$
$$\Delta m^-=0,$$
$$\Delta m'=0.$$

This corresponds to strong coupling.

Another definition of the metric associated to representations $\mathcal{D}(l^+, 0) \oplus \mathcal{D}(0, l^+)$ called weak metric\(^6\) leads to the rules

$$\Delta (m^+ + m^-) = 0,$$
$$\Delta m' = 0.$$

All these results do not mean that one must assimilate elementary particles directly with these new rotator sub-levels; this would immediately lead to incorrect results.

As we shall see in the conclusion it seems more promising to compare them with particular sub-levels combinations. They do seem to indicate, however, that the basic physical idea from which we started (that the mutual transformations and interactions of elementary particles indicate they are excited internal states of basic physical structures in space-time) should be investigated seriously.

**Conclusion**

The preceding results suggest a certain number of line of research. The first is that an effort should be made to substitute systematically the relativistic rotator (equivalent as we have shown\(^5\) to a bilocal structure), to the point particle model as classical substratum of Quantum Theory. This proposal which we developed with de Broglie\(^11\) and Takabayasi\(^19\) implies new researches on the classical properties of relativistic rotators. Some results have already been obtained on the Hamiltonian formalism\(^8\) and the group theoretical properties of such models.

The second line is evidently connected with the classical and quantum interpretation of these results. We already know\(^16\) that classically the relativistic rotators can be interpreted as describing the average external and internal behaviour of relativistic fluid masses enclosed within time-like tubes. But this is not the
only possibility. Preliminary results obtained in Bristol indicate that this model or more exactly the hyperplanes $A_4^+ = B_4^+$ can be directly connected with collective material behaviour within small regions (or cells) of a possible sub-quantum level of matter. This would directly connect hyperplanes and bilocality with basic properties of a subquantum field.

The third line of research is more directly related with the interpretation of the results of § 3. At first sight, as two of us (P. H. and J.P. V.) had proposed, one is tempted to associate every elementary particle to a certain state $Z_{\ell', m', \omega'}(\omega', \omega^-)$ of the relativistic rotator.

We have now reasons to believe that this view is too simple and it is better to associate elementary particles with certain state vectors belonging to specific irreducible representations of the complex rotation group. This is strongly suggested by the non-relativistic limit of the theory. Indeed, if we return to the beginning of Section 3 and examine the solution associated with $l=1/2$ we find that for $m'=1/2$ the set of two wave functions is

$$\begin{align*}
\Phi &= (\Phi_1, \Phi_2) \\
\Phi_1 &= Y_1^{1/2,1/2}(\theta, \varphi, \psi) = \cos \frac{\theta}{2} e^{i(\psi+\phi)}, \\
\Phi_2 &= Y_1^{-1/2,1/2}(\theta, \varphi, \psi) = \sin \frac{\theta}{2} e^{-i(\psi+\phi)},
\end{align*}$$

while the set corresponding to $m'=-1/2$ can be written as

$$\begin{align*}
\Phi &= (\Phi_1, \Phi_2) \\
\Phi_1 &= Y_1^{-1/2,-1/2}(\theta, \varphi, \psi) = -\sin \frac{\theta}{2} e^{i(\psi+\phi)}, \\
\Phi_2 &= Y_1^{1/2,-1/2}(\theta, \varphi, \psi) = \cos \frac{\theta}{2} e^{-i(\psi+\phi)},
\end{align*}$$

This is very interesting since we recognize easily that for $l=1/2$, $m'=1/2$ the set of two eigenfunctions $\Phi_1$, $\Phi_2$ is identical with the classical expression of the two-component Pauli spinor expressed in terms of Euler angles by one of us (D. B.). One notices also that for $m'=-1/2$ the two eigenfunctions are the components of the so-called charge-conjugate spinor $\Phi'$ derived from $\Phi$ by the operation $C$ (see reference 8). This strongly suggests that we ought to associate particles (and antiparticles) not with specific sub-levels but with vectors (that is sub-level combinations) belonging to particular sub-spaces of our general Hilbert space, which transforms under irreducible representations of the complex rotation group.

Indeed, two of us (P. H. and J.P. V.) have already established elsewhere that one can discover in the sub-space transforming under $\mathfrak{D}(1/2, 0) \oplus \mathfrak{D}(0,1/2)$ four different types of vectors or Dirac spinors. These vectors can be associated to four different types of leptons (Fermions). This will be investigated in a further paper.
D. Bohm, P. Hillion and J.P. Vigier

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