Author's Closure

The author thanks Professor Durvasula for his kind remarks and agrees with his constructive criticism. The additional results which Professor Durvasula has brought to the author's attention are also appreciated. The number of terms used by the author to obtain the published solutions was 25, and the author agrees that this information should have been supplied. With respect to symmetry groups, the author did not choose in the present work to split the equations into skew symmetric and skew antisymmetric sets, but rather elected to generate a single (larger) set and determine the lowest eigenvalue from the total set. Although this may seem computationally inefficient, it is necessary for more general application of the Ritz method, and the paper under consideration is but a special case of a much more general analysis capability contained within a single digital computer program (see, for example, references [5–7]). The method used by the author to obtain the lowest eigenvalue of the single combined set of equations was to iterate the eigenvector [8] which will converge on the lowest eigenvalue, symmetric or antisymmetric as the case may be. However, after reviewing the data presented by Professor Durvasula, the published result for pure shear with $\psi = 60$ deg seems suspect, and a more detailed investigation of this case has been conducted, this time using the approach suggested in reference [9] to obtain the eigenvalues.

Table 3 Buckling coefficients under the in-plane force N_{xy} alone

$a/b = 1 \quad \psi = 60$ deg

$M = N$	$\bar{R}_{xycr}$			$\bar{R}_{xycr}$ (Durvasula)		
5	−47.78	−49.02	+92.49			
6	−46.59	−48.35	+69.86	−46.58	−48.34	+69.86
7	−46.13	−48.11	+62.08			
8	−45.88	−47.98	+59.44			

Table 3 presents the results of this study. The lowest two negative buckling loads and the lowest positive buckling load are presented (in the discusser's notation) for 25, 36, 49, and 64 terms in the deflection series ($M = N$ in all cases). As indicated in the table, Professor Durvasula's suggestion that the published result for pure shear for a plate with $\psi = 60$ deg comes from the symmetric set of equations while a slightly lower result is obtained from the antisymmetric set is indeed found to be correct. Unfortunately, the author cannot determine the exact initial vector used in obtaining the original solution (obtained in Aug. 1967), but it is probable that the apparent convergence to the symmetric buckling load was caused by a "prejudiced" initial eigenvector. The very slow convergence of the iterative approach to obtaining an eigenvalue when two eigenvalues are very close is well known [8], and appears to have caused the error. The author believes the other published solutions to be correct within the accuracy cited by Professor Durvasula. Further investigation of the 60-deg skew plate using an increasing number of terms indicates, for this fairly extreme skewing angle, that 25 (or 36) terms is sufficiently accurate for the less buckling resistant direction of shear (negative in Table 3), but that the result presented by Professor Durvasula for positive shear is relatively inaccurate due to slow convergence.

Additional References

5 Ashton, J. E., and Waddoups, M. E., "Analysis of Anisotropic Plates," *Journal of Composite Materials*, Vol. 3, 1969, p. 148.
6 Ashton, J. E., "Analysis of Anisotropic Plates II," *Journal of Composite Materials*, Vol. 3, 1969, p. 470.
7 Ashton, J. E., "Anisotropic Plate Analysis," General Dynamics Research and Engineering Report, FZM-4899, October 12, 1967.
8 Faddeeva, V. N., *Computational Methods of Linear Algebra*, translated from the Russian by C. D. Benster, Dover, 1959.
9 Francis, J. G. F., "The QR Transformation," *The Computer Journal*, Oct. 1961, p. 265 and Jan. 1962, p. 332.

Plastic Failure of Fiber-Reinforced Materials[1]

TRYFAN G. ROGERS.[2] In his short paper, the author has presented a number of interesting results in the theory of plastic failure, under plane-stress conditions, of a sheet reinforced by unidirectional fibers, and of a two-ply laminate under simple tension.

The first problem, in which the fibers in the sheet are initially straight and parallel and remain straight and parallel during the deformation, has also been treated in [1].[3] There, as in the present paper, the continuum theory incorporates the kinematical constraints of incompressibility and inextensibility in the fiber direction. The two theories differ in the yield criterion governing the plastic behavior of the composite. Professor Prager assumes the extra stresses (i.e., the total stress less hydrostatic stress and longitudinal stress in the fiber direction) to satisfy von Mises' quadratic yield condition. In [1] the composite is modeled by a special type of transversely isotropic rigid/plastic solid with the preferred direction being in the fiber direction, convected with the material as it deforms; this is a consequence of assuming the yield function to depend on the extra stresses *and* the fiber direction. The problems of plane strain treated in [1] are analyzed using the most general form of yield criterion consistent with this model.

For the plane-stress problem directly pertinent to this discussion, the quadratic form of the yield criterion was adopted, in this case reducing to the condition

$$\frac{\sigma_{22}^2}{4k_2^2} + \frac{\sigma_{12}^2}{k_1^2} = 1.$$

Then, for example, the applied tension S at failure is related to the fiber angle α through

$$S = \frac{4k_1k_2}{\sin 2\alpha(4k_2^2 + k_1^2 \tan^2 \alpha)^{1/2}}.$$

The von Mises' yield condition is equivalent to $k_1 = k_2 = k$.

The constants k_1 and k_2 may be interpreted as the yield stresses in shear along, and in shear normal to, the fiber direction. Experimental data by Cooper [2] for a composite consisting of copper matrix and tungsten fibers, and by Jackson and Cratchley [3] for a composite consisting of an aluminum matrix and silica fibers, indicate that for these materials k_1/k_2 is about 1.6.

References

1 Mulhern, J. F., Rogers, T. G., and Spencer, A. J. M., "A Continuum Model for Fiber-Reinforced Plastic Materials," *Proceedings, Royal Society*, London, Vol. 301, Series A, 1967, pp. 473–492.
2 Cooper, G. A., *Journal of Mechanics and Physics of Solids*, Vol. 14, 1966, p. 103.
3 Jackson, P. W., and Cratchley, D., *Journal of Mechanics and Physics of Solids*, Vol. 14, 1966, p. 49.

Author's Closure

The author thanks Professor Rogers for his illuminating discussion.

Whereas reference [1] of the discussion treats the composite sheet as a *transversely isotropic* rigid, plastic solid, the paper under

[1] By W. Prager, published in the September, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 770–773.

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[3] Numbers in brackets designate References at end of Discussion.

discussion viewed the sheet as consisting of an *isotropic* rigid, plastic matrix that is constrained by inextensible fibers. The latter concept appears more natural to the author because, for most composites, the matrix material is isotropic. On the other hand, the equations expressing the former concept contain an additional constant, which is useful in fitting theory to experimental data.

Optimization of a Viscoelastic Structure: The Seat-Belt Problem[1]

RAYMOND R. McHENRY.[2] This paper presents an interesting analytical exercise. However, one of the major shortcomings of optimization studies of this type for the automobile restraint system problem (e.g., a similar simplified study is presented in reference [1][3]) is a tendency toward oversimplification. In the present case, the effects of the deceleration time history of the vehicle, the jackknifing motion of the occupant, belt angularity (side view) and slack, seat cushion deflection and friction, etc., are neglected.

There are in existence several relatively complete nonlinear mathematical models of the automobile crash victim in a longitudinal collision [2, 3] which could be applied in optimization studies with techniques that operate externally of the mathematical model, on the inputs and outputs (e.g., the method described in reference [4]). However, limitations on practical belt material properties and on achievable vehicle deceleration waveforms make the benefits of such a theoretical optimization rather doubtful.

It should be noted that an "optimum" restraint must be applied to wide ranges of occupant sizes and weights, that the interior dimensions of automobiles include wide variations, and that the different obstacles encountered in automobile collisions produce different deceleration waveforms for the vehicle. Also, the criterion for optimum performance must include a weighted sum of several individual responses (i.e., the peak belt load on the pelvic region is not as important a source of injury as are head and thorax impacts on the vehicle interior).

The general effect that is sought with viscoelastic materials in the present paper is the same as that which can readily be achieved with an inertia reel on a conventional belt.

The "severity index" is calculated for the belt loading, whereas the cited "critical value" of 1000 is related to head impact on the vehicle interior.

The cited 8 mph impact velocity threshold for windshield penetration was presented in the source document as a speed below which the head of an *unrestrained* occupant does not hit the windshield. With a belted occupant, the head impact velocity on the vehicle interior is substantially greater than the pelvic velocity (i.e., jackknifing motion occurs). Therefore, the use of an 8 mph limitation on pelvic velocity, at a specified forward displacement, is difficult to justify.

The numerical examples indicate an occupant mass corresponding to a weight of 180 lb. It is well established [5] that only approximately 60 percent of the mass of an occupant is decelerated by a lap belt.

Finally, the synopsis tends to be confusing because of the fact that constraints (*a*) and (*b*) are not compatible with each other, whereas each is applied in combination with constraint (*c*).

It is unfortunate that the cited difficulties, related to oversimplification of the particular application selected by the authors, detract from the excellent quality of both the analytical study and the technical presentation.

References

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2 McHenry, R. R., and Naab, K. N., "Computer Simulation of the Crash Victim—A Validation Study," Tenth Stapp Car Crash Conference, November 8–9, 1966.

3 Chaffin, D. B., "A Computerized Biomechanical Model—Development and Use in Studying Gross Body Motions," Technical Conference on Biomechanics, University of Michigan, June 12–13, 1969.

4 Fogerty, L. E., and Howe, R. M., "Trajectory Optimization by a Direct Descent Process," NASA CR-1070, June 1968.

5 Ryan, J. J., "Reduction in Crash Forces," *Proceedings of the Fifth Stapp Conference*, University of Minnesota, 1962.

L. W. Morland.[4] This paper makes a preliminary investigation of viscoelastic creep (relaxation) as a feature of seat-belt design. The model adopted is a point mass attached to the midpoint of a tension supporting belt fixed at both ends, and assumes small displacement and infinitesimal strain in the belt during the motion caused by a suddenly imparted velocity (change) to the mass. Two element (Maxwell and Voigt) models are considered to describe the viscoelastic properties, and quasi-static motion is assumed; for the given geometry wave travel times over the belt length are a factor 10^{-2} of the motion time. The initial velocity is maximized subject to restrictions on the final displacement and velocity, and maximum force exerted on the mass during the motion. While the displacements considered are not compatible with the linearity assumption in the geometry and the strain, so that the solution detail is not useful, the qualitative effect of viscoelasticity in comparison with purely elastic results indicates the value of further investigation. More realistic viscoelastic behavior and description of the body motion should be feasible, perhaps incorporating a double-belt arrangement for which displacements of the "main body" can be kept small.

Authors' Closure

The authors are grateful to Professor Morland and to Mr. McHenry for their comments on our paper. We recognize the complexity of the actual problem of design of automobile passenger restraint systems, but the paper was definitely not intended to be a study of such a system, as indicated by our choice of words: ". . . .highly idealized problem. . . . ." Rather, it is to show that viscoelastic properties of the belt material can be considered as additional parameters in the optimization, and that significant improvement in belt performance can be obtained by such consideration. Other optimization studies that we have seen have not included viscoelastic material parameters in the analysis. Since this is an initial study, a much-simplified geometry and material description was chosen to make the central point, and data, for the numerical examples in Table 1, then had to be chosen to be representative of the numbers that we found in the references rather than to be precise. For example, a point mass, which is the model chosen in the paper, cannot represent simultaneously the head and the pelvis, and the effect of jackknifing motion cannot properly be taken into account in the numerical example.

With regard to the choice of the 8 mph impact velocity threshold for windshield penetration, on which McHenry commented, we have the following response. It follows from the cited reference [6] that, for sled velocities below 8 mph, the head of an

[1] By W. Nachbar and J. B. Schipmölder, published in the September, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 565–572.

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[3] Numbers in brackets designate References at end of Discussion.

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