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A STOCHASTIC MODEL DESCRIBING THE WATER MOTION IN A RIVER

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In this paper a mathematical model for the movement of water particles in a river is constructed.
Using the model it will be possible to find an analytic expression for the distribution function F(t) of the *transit time,* T, needed for a water particle to travel from one cross section of the river to another. This implies that we obtain a quantitative description of the longitudinal dispersion in the river.
Briefly the model can be described as follows. Firstly we divide the river into two layers, an upper layer and a lower layer. (The »border-line« between the two layers will depend on the mean velocity of the river and on the velocity profile across the river.) Secondly we assume that the longitudinal velocity is constant (and positive) in the upper layer and zero in the lower layer. Finally we assume that a water particle moves back and forth between the two layers in a random way. The probability law governing this movement will depend on a parameter characterizing the roughness of the bottom of the river.

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The purpose of this paper is to construct a mathematical model for the movement of water particles in a river.

Using the model we shall be able to calculate the distribution function F(t) of the *transit time,* T, needed for a water particle to travel from one cross section A_0 of the river to another cross section A_1. This implies that we obtain a quantitative description of the longitudinal dispersion in the river.

In order to apply the model the following conditions should be fulfilled: 1) the mean velocity between the cross sections A_0 and A_1 remains constant, 2) the shapes and the areas of the cross sections between A_0 and A_1 are (approximately) invariant, 3) the roughness of the bottom between A_0 and A_1 remains (approximately) constant.

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In the model there are three parameters depending on, respectively: the mean velocity, the velocity profile across the river and the roughness of the bottom of the river. The first two can be obtained rather easily from direct measurements, whereas the third parameter must be determined from experiments, for example dye-dispersion experiments.

The model is intended for problems dealing with transit times of particles *dissolved* in water. With some minor modifications it can also be applied to problems of particles *suspended* in water. Furthermore the technique used in this paper should prove effective for more complex water systems, e. g. lakes.

Hubbel & Sayre (1964) and Yang & Sayre (1971) have used the same technique as used in this paper, for transport problems of sand particles. However, in their papers they study the distance a particle has travelled during a fixed time interval. They do not calculate the distribution function for the *time* needed to travel a certain distance.

In order to understand the model some basic knowledge of probability theory is necessary. Moreover, in order to read Section 4 and the Appendix the reader must be familiar with the basics of stochastic processes.

The plan of the paper is as follows: In Section 2 we shall introduce some basic notations. In Section 3 we shall construct the model. In Section 4 we shall study the transit time T needed for a water particle to travel from one cross section of the river to another. In Section 5 we shall draw some conclusions and make some general remarks. (This last section can be understood without any knowledge of probability theory.) Finally, in the Appendix, we shall prove some mathematical theorems necessary for the proofs of Section 4.

The paper is not rooted in associated work. However, interesting papers dealing with the same or similar problems (besides the two mentioned above) are:

Danckwerts (1953), Andersson (1957), Thayer & Krutchkoff (1967), Fischer (1966, 1968), Howell, Kiser & Rumer (1970), and Eriksson (1971).

The model has not yet been tested by experiments.

2. BASIC NOTATIONS

Let R denote a river. Let R′ denote a volume of R, bounded by an upper cross section A_0 and a lower cross section A_1. We denote a point p in R by

$$p \equiv (x, y, z)$$

where x, y, z are, respectively, the longitudinal, lateral and vertical coordinates.

We put $x \equiv 0$ in A_0 and assume that x increases in the downstream direction.

Let the longitudinal distance between A_0 and A_1 be equal to d. Let us denote a cross section of R at x by A(x).

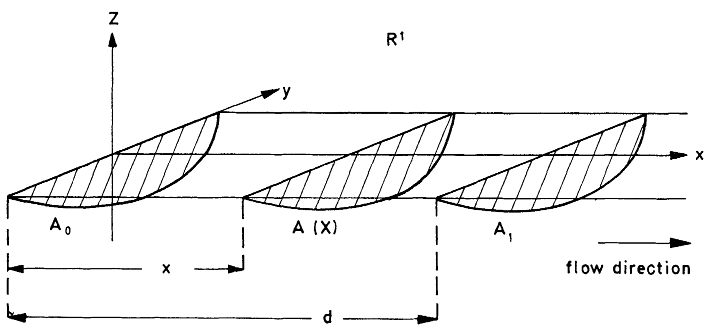

Fig. 1.

Let w denote a water particle in R. It is customary to regard the motion of w as a stochastic process and we shall do so. Let us define

T(x) ≡ the epoch*) at which a water particle w passes the cross section A(x), given that, at epoch 0, w was at A_0.

*) The word epoch will be used to denote a point on the time axis. (c.f. Feller 1967).

Define

$$F(x, t) \equiv \Pr [T(x) \leqq t].$$

Let us also define

$X(t) \equiv$ the longitudinal coordinate of a water particle w at epoch t given that at epoch 0 w was at A_0

and

$$G(x, t) \equiv \Pr [X(t) \leqq x].$$

The purpose of this paper is to construct a stochastic model for the movement of water particles in R′ from which we shall be able to calculate

$$E[T(x)], 0 \leqq x \leqq d$$

$$\mathrm{Var}[T(x)], 0 \leqq x \leqq d$$

$$E[(T(x) - E[T(x)])^3], 0 \leqq x \leqq d.$$

We shall also be able to find an analytic expression for $F(x, t)$, $0 \leqq x \leqq d$. Finally, we shall make some comments on the relationship between the distribution functions $F(x, t)$ and $G(x, t)$.

We shall make the following basic assumptions about R′:

A1. *The amount of water (per time unit) passing a cross section* A(x) *is independent of* x, $0 \leqq x \leqq d$, *and will be denoted by* Q_0.

A2. *The cross sections* A(x), $0 \leqq x \leqq d$, *are all equal (i. e. independent of* x). *Their area will be denoted by* S_0.

A3. *The roughness of the bottom is approximately constant all through* R′.

From A_1 and A_2 we find that the mean velocity u_0 of the river is equal to Q_0/S_0.

Finally, let $\bar{u}(p)$ denote the time average water velocity at a point p of the river. From the assumptions A1, A2 and A3 we see that it is reasonable to assume that $\bar{u}(p)$ ($= \bar{u}(x,y,z)$) is independent of x.

3. CONSTRUCTION OF THE MODEL

It is well known that the graph of the function $\bar{u}(p)$, when p moves along a vertical, is approximately as is Fig. 2. (See e. g. Bird et al. 1960, Chapter 5.)

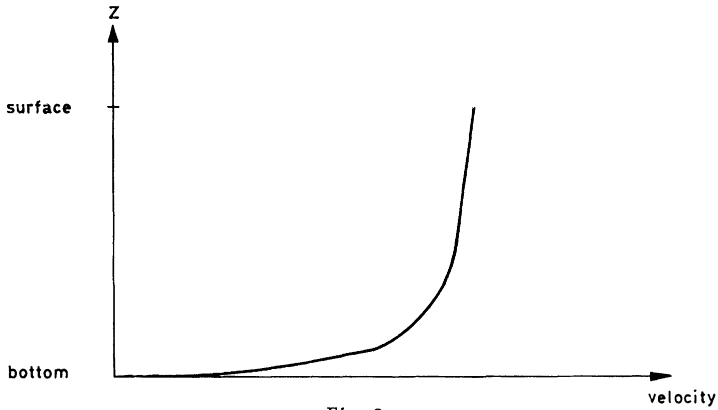

Fig. 2.

Let us define

$$B_1(x) \equiv \{\text{ those points in } A(x) \text{ for which } \bar{u}(p) \geqq u_0\} \tag{3.1}$$

and

$$B_2(x) \equiv \{\text{ those points in } A(x) \text{ for which } \bar{u}(p) < u_0\}. \tag{3.2}$$

Let us further define

$$L_1 = \{\text{ those points in } R' \text{ for which } \bar{u}(p) \geqq u_0\}$$

$$L_2 \equiv \{\text{ those points in } R' \text{ for which } \bar{u}(p) < u_0\}.$$

We shall call L_1 the upper, and L_2 the lower layer of R.
Let $S_{B_1(x)}$ denote the area of $B_{1(x)}$ and let $S_{B_2(x)}$ denote the area of $B_{2(x)}$. Since $\bar{u}(p)$ is independent of x we see that we can put

$$S_{B_1(x)} \equiv S_1, 0 \leqq x \leqq d$$

$$S_{B_2(x)} = S_2, 0 \leqq x \leqq d.$$

Define α_1 and α_2 by

$$\alpha_1 \equiv \frac{S_1}{S_0}, \alpha_2 \equiv \frac{S_2}{S_0}.$$

We are now ready for our first assumption about the water motion.

Assumption B1

If a water particle is located in L_1 at epoch $t \geqq 0$ then its longitudinal velocity is equal to u_0/α_1. If a water particle is located in L_2 at epoch $t \geqq 0$ then its longitudinal velocity is zero.

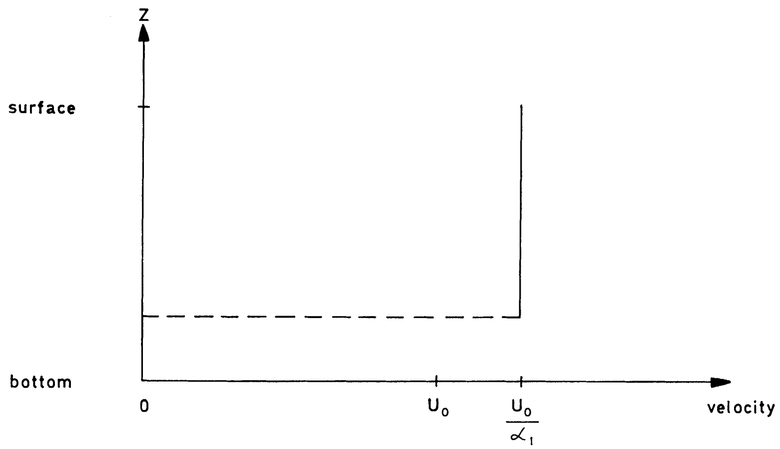

Fig. 3.

Remark 1. Observe that the constant u_0/α_1 is chosen such that the average velocity over a cross section is equal to the river velocity u_0.

Remark 2. Comparing the graph in Fig. 2 with the graph in Fig. 3 we observe that the second graph is a very rough approximation of the first. The reader might ask whether such a rough approximation can be of any use. If he desires a model which gives the most accurate possible description of the river *at any point,* then the answer is no. If, on the other hand, he is prepared to accept a model which does not give the best description of the river at every point, but which is useful for obtaining quantitative results for some characteristics of the river, then the technique in this paper should prove effective.

From assumption B_1 we see that, if we define

$V(t) \equiv$ the longitudinal velocity for a water particle w at epoch t, given that at epoch 0 w was at A_0

then

$$V(t) = \begin{cases} \dfrac{u_0}{\alpha_1} \text{ if w is in } L_1 \\ 0 \text{ if w is in } L_2. \end{cases} \tag{3.3}$$

So far we have made an assumption about the longitudinal movement of the water particles. We shall next make an assumption about the motion in the vertical and lateral directions.

Let w denote a water particle and let us define, for t, s $\geqq 0$, the following transition probabilities:

$P_{1,1}(t, s) \equiv$ Pr [w is in L_1 at epoch $t + s$, given that w was in L_1 at epoch t]

$P_{1,2}(t, s) \equiv$ Pr [w is in L_2 at epoch $t + s$, given that w was in L_1 at epoch t]

$P_{2,1}(t, s) \equiv$ Pr [w is in L_1 at epoch $t + s$, given that w was in L_2 at epoch t]

$P_{2,2}(t, s) \equiv$ Pr [w is in L_2 at epoch $t + s$, given that w was in L_2 at epoch t].

Assumption B2

During the time a water particle w *is in* R′ *the transition propabilities* $P_{i,j}(t, s)$ *are such that*

$$P_{i,j}(t, s) \equiv P_{i,j}(0, s) \equiv P_{i,j}(s), \ t, s \geqq 0, \ i,j = 1,2 \tag{3.4}$$

$$P_{1,1}(h) = 1 - \beta\alpha_2 + o(h), \ h \geqq 0 \tag{3.5}$$

$$P_{2,2}(h) = 1 - \beta\alpha_1 + o(h), \ h \geqq 0 \tag{3.6}$$

where o(h) *means that if we divide the term by* h *then its value tends to zero as* h *tends to zero.*

Remark 1. If we assume that a water particle w is located at A_0 at epoch 0 and define, for $t \geqq 0$

$$I(t) \equiv \begin{cases} 1 & \text{if w is in } L_1 \text{ at epoch t} \\ 2 & \text{if w is in } L_2 \text{ at epoch t} \end{cases} \tag{3.7}$$

then we see from Assumption B2 that I(t) can be regarded as a time continuous Markov chain with two possible states.

Remark 2. The coefficients $\beta\alpha_2$ and $\beta\alpha_1$ in (3.5) and (3.6) respectively are chosen such that

$$\Pr [I(t) = 1] \approx \alpha_1$$

$$\Pr [I(t) = 2] \approx \alpha_2$$

We shall prove this in the next section.

(Observe that $\frac{\alpha_1}{\alpha_2}$ can be regarded as the ratio between the volume of the upper and the lower layer of R′.)

Remark 3. We shall call the parameter β in (3.5) and (3.6) the *mixing coefficient* for R′.

4. CALCULATIONS OF E [T(x)], VAR [T(x)], E [(T(x)-E [T(x)]) ³] AND F (x, t)

Let us recall the definition of T(x):

$T(x) \equiv$ the epoch at which a water particle w passes the cross section A(x) given that w was located at A_0 at epoch 0.

Let us define

$T_1(x) \equiv$ the epoch at which a water particle w passes the cross section A(x) given that we was located *at $B_1(0)$* at epoch 0,

and

$T_2(x) \equiv$ the epoch at which a water particle w passes the cross section A(x) given that w was located *at $B_2(0)$* at epoch 0.

(For the definitions of $B_1(0)$ and $B_2(0)$ see (3.1) and (3.2).)

Let us also define

$F_1(x, t) \equiv \Pr[T_1(x) \leqq t]$

and

$F_2(x, t) \equiv \Pr[T_2(x) \leqq t]$.

Theorem 4.1

If we make the assumptions B_1 *and* B_2 *about the motion of a water particle* w *in* R′, *and assume that* $0 \leqq x \leqq d$ *then*

$$E[T_1(x)] = \frac{x}{u_0} \tag{4.1}$$

$$E[T_2(x)] = \frac{x}{u_0} + \frac{1}{\alpha_1\beta} \tag{4.1)'$$

$$Var[T_1(x)] \equiv \frac{x}{u_0} \cdot \frac{2\alpha_2}{\beta} \tag{4.2}$$

$$Var[T_2(x)] \equiv \frac{x}{u_0} \cdot \frac{2\alpha_2}{\beta} + \frac{1}{(\alpha_1\beta)^2} \tag{4.2)'$$

$$E[(T_1(x) - E[T_1(x)])^3] \equiv \frac{x}{u_0} \cdot \frac{6\,\alpha_2}{\alpha_1^2 \cdot \beta^2} \tag{4.3}$$

$$E[(T_2(x) - E[T_2(x)])^3] \equiv \frac{x}{u_0} \cdot \frac{6\,\alpha_2}{\alpha_1^2 \cdot \beta^2} + \frac{2}{(\alpha_1\beta)^3}. \tag{4.3)'$$

Furthermore

$$F_1(x,t) = \begin{cases} 0 \quad \text{if} \quad t < \dfrac{a_1 x}{u_0} \\ e^{-x_1} + e^{-x_1} \displaystyle\sum_{n=1}^{\infty} \Big(1 - e^{-t_1} \sum_{k=0}^{n-1} \frac{(t_1)^k}{k!}\Big) \cdot \frac{(x_1)^n}{n!} \quad \text{if } t_1 \geqq 0 \end{cases} \tag{4.4}$$

and

$$F_2(x,t) \equiv \begin{cases} 0 \quad \text{if} \quad t < \dfrac{a_1 x}{u_0} \\ e^{-x_1} \displaystyle\sum_{0}^{\infty} \Big(1 - e^{-t_1} \sum_{k=0}^{n} \frac{(t_1)^k}{k!}\Big) \cdot \frac{(x_1)^n}{n!} \quad \text{if } t_1 \geqq 0 \end{cases} \tag{4.4'}$$

where we have put

$$x_1 = \frac{a_1 \cdot a_2 \cdot \beta \cdot x}{u_0}$$

and

$$t_1 \equiv \beta \cdot a_1 \cdot \Big(t - \frac{a_1 x}{u_0}\Big).$$

Proof. Consider a water particle w starting in $B_1(0)$ (the upper layer) and oscillating between the two layers L_1 and L_2. As a function of time its movement is depicted in Fig. 4.

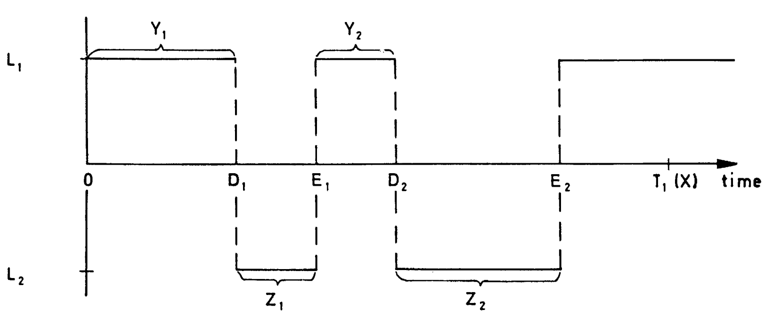

Fig. 4.

Let $D_1, D_2, \ldots, D_n$ denote the epochs at which w leaves the upper layer and $E_1, E_2, \ldots E_{n-1}$ those at which w enters it again. Define

$D_0 = E_0 = 0$

$N(x) \equiv$ the largest n for which D_n is less then the epoch at which the particle passes $A(x)$.

The movement and the epochs for a particle starting in $B_2(0)$ (the lower layer) is depicted in Fig. 5.

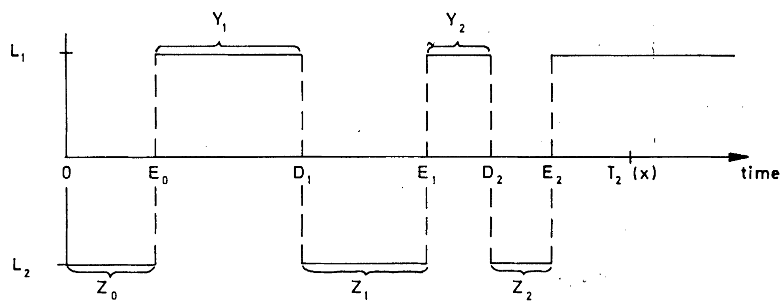

Fig. 5.

Here $D_0 = 0$ but $E_0 > 0$.

Name the time intervals that a particle spends in the two layers

$Y_0 = D_0 \ (= 0)$

$Y_n = D_n - E_{n-1} \qquad n = 1, 2, \ldots, N(x)$

$Z_0 = E_0$

$Z_n \equiv E_n - D_n \qquad n \equiv 1, 2, \ldots, N(x).$

(See also Fig. 4 and Fig. 5.)

Since, in the lower layer the longitudinal velocity is assumed to be zero, we conclude that all longitudinal movement takes place in the upper layer. Further, since the longitudinal velocity in the upper layer is assumed to be $\frac{u_0}{\alpha_1}$, we find that the time w spends in the upper layer before reaching $A(x)$ equals $\frac{x}{u_0/\alpha_1}$.

From the definition of Z_n and $N(x)$ we find that the time w spends in the lower layer equals $\sum_{0}^{N(x)} Z_n$.

(See also Fig. 4 and Fig. 5.)

Thus

$$T_i(x) = \frac{\alpha_1 x}{u_0} + \sum_{0}^{N(x)} Z_n, \quad i = 1, 2. \tag{4.5}$$

(Observe that if $i = 1$, then Z_0 is equal to zero, whereas if $i = 2$, then Z_0 is equal to the time spent before w leaves the lower layer for the first time.)

Further, let us observe that we can write

$$N(x) = \max \left\{ n \mid \frac{u_0}{\alpha_1} \sum_{0}^{n} Y_m < x \right\} \tag{4.6}$$

since

$\frac{u_0}{\alpha_1} \cdot \sum_{1}^{n} Y_m =$ the distance the particle w has travelled before it has left the upper layer for the n^{th} time.

Using the relations (3.5) and (3.6) of assumption B2, we obtain from the general theory about continuous Markov chains (see e.g. Karlin 1966, Chapter 7) that

$$Z_0, Y_1, Z_1, Y_2, \dots Y_{N(x)}, Z_{N(x)} \text{ are independent} \tag{4.7}$$

$$\Pr [Y_n \leqq t] = 1 - e^{-\alpha_2 \beta t},\ t \geqq 0,\ n = 1,2 \dots N(x) \tag{4.8}$$

$$\Pr [Z_n \leqq t] = 1 - e^{-\alpha_1 \beta t},\ t \geqq 0,\ n = 1,2 \dots N(x) \tag{4.9}$$

and if we assume that w starts in the lower layer then

$$\Pr [Z_0 \leqq t] = 1 - e^{-\alpha_1 \beta t},\ t \geqq 0. \tag{4.10}$$

(See also Theorem A3 of the Appendix.)

The relations (4.5) – (4.10) conform to the assumptions of Theorems A1 and A2 in the Appendix. Replacing the constants a, b, c and t_1 in the Appendix with $\alpha_2\beta$, $\alpha_1\beta$, u_0/α_1 and $t_1/\alpha_1\beta$ respectively, we obtain (4.1) – (4.4)′. This completes the proof.

Next let us consider the process $I(t)$, $t \geqq 0$, defined by (3.7). In the last section we stated that

$$\Pr [I(t) = 1] \approx \alpha_1$$

and

$$\Pr [I(t) = 2] \approx \alpha_2.$$

That these statements are correct is easily seen from Theorem A3 in the Appendix and the fact that

$$\frac{\alpha_1\beta}{\alpha_2\beta + \alpha_1\beta} = \alpha_1 \text{ and } \frac{\alpha_2\beta}{\alpha_2\beta + \alpha_1\beta} = \alpha_2$$

Moreover, if we assume that

$$\Pr [I(0)) = 1] = \alpha_1, \text{ and } \Pr [I(0) = 2] = \alpha_2,$$

then we see from Theorem A3 that

$$\Pr [I(t) = 1] = \alpha_1$$

$$\Pr [I(t) = 2] = \alpha_2$$

for all $t \geqq 0$ as long as the particle remains in R′.

We shall end this section making some remarks about the stochastic variable

$X(t)$ = the longitudinal position, at epoch t, of the water particle w, given that at epoch 0 w was at A_0.

Actually what we want to emphasize is the fact that,

$$\Pr [X(t) \leqq x] = \Pr [T(x) \geqq t]$$

since, if $T(x) \geqq t$, then $X(t) \leqq x$

and if $T(x) < t$, then $X(t) > x$.

Thus, if we denote

$$G(x, t) = \Pr [X(t) \leqq x]$$

and recall that

$$F(x, t) = \Pr [T(x) \leqq t]$$

then

$$G(x, t) = 1 - F(x, t). \tag{4.11}$$

Now, let us make the assumptions B1 and B2 about the motion of a water particle w in R′. Then using Theorem 4.1 we see that if $0 \leqq x \leqq d$, $F(x, t)$ can be determined, and consequently, we see from (4.11) that $G(x, t)$ can be determined for $0 \leqq x \leqq d$.

5. CONCLUSIONS AND SOME GENERAL REMARKS

In order to test the model and to determine the mixing coefficient β, some type of experiment is necessary, for example a dye-dispersion experiment. However, looking at the theoretical results obtained in the last section, the following observations make the model look promising:

1) The mean value of T(x) is at least approximately equal to the distance x divided by the mean velocity u_0. (See (4.1) and 4.1)′.)

2) The variance of T(x) increases with α_2 and decreases with β, i.e. the larger the proportion of slow moving water, the larger the variance; and, the larger the mixing coefficient, the smaller the variance. (See (4.2) and (4.2)′.)

3) The third central moment is positive, increasing with α_2 and decreasing with β. (See (4.3) and (4.3)′.)

We also want to emphasize the following properties of the model. Firstly, once the mixing coefficient β for a river section is determined the model is simple to handle, since the other parameters are easy to obtain. (It might even be possible to use the mixing coefficient obtained from one river as an approximate value for the mixing coefficient of some other river, if it is known that the roughness of the river bottoms is similar.)

Secondly, using the model, it is easy to obtain the distribution function for the transit time from one cross section of the river to another even when the roughness of the bottom is not invariant in between. Such a distribution function is obtained by a subdivision of the river into parts, each having constant roughness.

Thirdly, the general structure of the model can be used to treat transit time problems for substances other than water.

Finally, the technique of dividing a water system into different sections, and assuming a Poisson process for the water particle motion between the sections, can be used when treating transit time problems in lakes.

APPENDIX

In this appendix we shall state and prove those theorems to which we referred in Section 4.

Let Z_0, Y_1, Z_1, Y_2, $Z_2 \ldots$ be a sequence of independent, stochastic variables such that

$$\Pr [Y_n \leqq t] \equiv 1 - e^{-at}, \quad t \geqq 0, \quad n \equiv 1,2,3,\ldots \tag{1}$$

$$\Pr [Z_n \leqq t] \equiv 1 - e^{-bt}, \quad t \geqq 0, \quad n \equiv 1,2,3,\ldots \tag{2}$$

Let $Y_0 = 0$, let $c > 0$ and $x > 0$ be two constants, and define

$$N(x) \equiv \max \{ n \mid c \cdot \sum_{0}^{n} Y_m \leqq x \}. \quad (3)$$

Define

$$T(x) = \frac{x}{c} + \sum_{0}^{N(x)} Z_n \quad (4)$$

and

$$F(x, t) = \Pr [T(x) \leqq t]. \quad (5)$$

Theorem A1

Suppose that

$$Z_0 \equiv 0. \quad (6)$$

Then

$$E[T(x)] = \frac{x}{c} + \frac{x}{c} \cdot \frac{a}{b} = \frac{x}{c} \cdot (1 + \frac{a}{b}), \quad (7)$$

$$Var[T(x)] = \frac{x}{c} \cdot \frac{2a}{b^2}, \quad (8)$$

$$E[(T(x) - E[T(x)])^3] = \frac{x}{c} \cdot \frac{6a}{b^3}, \quad (9)$$

$$F(x, t) = \begin{cases} 0 \quad if \quad t < \frac{x}{c} \\ e^{-x_1} + e^{-x_1} \sum_{n=1}^{\infty} (1 - e^{-bt_1} \cdot \sum_{k=0}^{n-1} \frac{(bt_1)^k}{k!}) \cdot \frac{x_1^n}{n!}, \; if \; t_1 \geqq 0 \end{cases} \quad (10)$$

where

$$x_1 = \frac{a\,x}{c}$$

and

$$t_1 \equiv t - \frac{x}{c}.$$

Proof. From (4) and (6) we see that

$$E\,[T(x)] = \frac{x}{c} + E\,[\sum_{1}^{N(x)} Z_n]. \tag{11}$$

Now from the definition of N(x) (see (3)) we see that:

N(x) is independent of Z_n, $n = 1,2,3,\ldots$ (12)

Thus

$$E\,[\sum_{1}^{N(x)} Z_n] = E\,[N(x)] \cdot E\,[Z_1]. \tag{13}$$

(See e.g. Feller 1967, page 164.)

Next observing that $Y_1, Y_2, \ldots$ are exponentially distributed we find that

$$Pr\,[N(x) = n] = e^{-\frac{ax}{c}} \cdot \left(\frac{ax}{c}\right)^n \cdot \frac{1}{n!} = e^{-x_1} \cdot (x_1)^n \cdot \frac{1}{n!} \tag{14}$$

where we have put

$$\frac{ax}{c} = x_1$$

(N(x) is Poisson distributed.)

Thus

$$E\,[N(x)] = x_1. \tag{15}$$

From (2) we obtain

$$E\,[Z_1] = \frac{1}{b}. \tag{16}$$

Using (13), (15) and (16), we obtain

$$E\,[T(x)] = \frac{x}{c} + \frac{x_1}{b} = \frac{x}{c} + \frac{x}{c} \cdot \frac{a}{b} = \frac{x}{c} \cdot (1 + \frac{b}{a})$$

and thus (7) is proved.

Next let us define

$$\varphi\,(t) \equiv E\,[e^{\,it(T(x) - E\,[T(x)])}]. \tag{17}$$

From (4), (11) and (13) we find that

$$\varphi(t) \equiv E[e^{it(\sum_{1}^{N(x)} Z_n - E[N(x)] \cdot E[Z_1])}].$$

In order to simplify our notations we shall write

$N(x) \equiv N$, $Z_1 \equiv Z$, $E[N] = EN$, $E[Z] \equiv EZ$.

Thus

$$\varphi(t) \equiv E[e^{it(\sum_{1}^{N} Z_n - EN \cdot EZ)}].$$

From (15) and (16) we obtain

$$\varphi(t) = e^{-it\frac{x_1}{b}} \cdot E[e^{it\sum_{1}^{N} Z_n}], \tag{18}$$

Let us write

$$\varphi_1(t) = E[e^{it\sum_{1}^{N} Z_n}].$$

From (12) and the theory of random sums we find that

$$\varphi_1(t) \equiv e^{-x_1 + x_1 \cdot \varphi_2(t)}, \tag{19}$$

where

$\varphi_2(t) \equiv E[e^{it Z}]$

(See e.g. Feller 1967, page 478.)

We have

$$\varphi_2(t) \equiv \int_0^\infty e^{it\tau} b \cdot e^{-b\tau} d\tau \equiv \frac{b}{b-it} \equiv \frac{1}{1-i\frac{t}{b}}. \tag{20}$$

Combining (17), (18), (19) and (20) we obtain

$$\varphi(t) \equiv e^{it\frac{x_1}{b}} \cdot e^{-x_1 + x_1 \cdot \frac{b}{b-it}} \equiv e^{-\frac{itx_1}{b} + \frac{-bx_1 + itx_1 + bx_1}{b-it}} =$$

$$\equiv e^{itx \cdot (-\frac{1}{b} + \frac{1}{b-it})} \equiv e^{-\frac{t^2}{b \cdot (b-it)}}$$

Calculating the second and third derivatives of $\varphi(t)$ and putting $t = 0$ we find that

$$\varphi''(0) \equiv -\frac{2x_1}{b^2}$$

and

$$\varphi'''(0) \equiv -\frac{i\,6\,x_1}{b^3}:$$

Thus

$$\text{Var}\,[T(x)] \equiv \frac{\varphi''(0)}{(i)^2} = \frac{2x_1}{b^2}$$

and

$$E\,[(T(x) - E\,[T(x)])^3] \equiv \frac{\varphi'''(0)}{(i)^3} \equiv \frac{6x_1}{b^3}$$

proving (8) and (9).

Finally, since

$$\Pr\,[Z_1 + Z_2 + \ldots Z_n \leqq t] = 1 - e^{-bt}\,(1 + \frac{bt}{1!} + \ldots \frac{(bt)^{n-1}}{(n-1)!}) \tag{21}$$

(see e.g. Feller (1967), page 10,)

we obtain from (4) and (14) that

$$\Pr\,[T(x) - \frac{x}{c} \leqq t] \equiv e^{-x_1} + e^{-x_1} \sum_{n=1}^{\infty} (1 - e^{-bt} \sum_{k=0}^{n-1} \frac{(bt)^k}{k!}) \cdot \frac{(x_1)^n}{n!}.$$

From this (10) follows and the theorem is proved.

Theorem A2

If instead of (6) *we assume that*

$$Pr\,[Z_0 \leqq t] = 1 - e^{-bt},\ t \geqq 0 \tag{22}$$

then

$$E[T(x)] = \frac{x}{c} + \frac{x}{c} \cdot \frac{a}{b} + \frac{1}{b} \tag{23}$$

$$Var[T(x)] = \frac{x}{c} \cdot \frac{2a}{b^2} + \frac{1}{b^2} \tag{24}$$

$$E[(T(x) - E[T(x)])^3] = \frac{x}{c} \cdot \frac{6a}{b^3} + \frac{2}{b^3} \tag{25}$$

$$F(x,t) = \begin{cases} 0 \quad \text{if} \quad t \leqq \frac{x}{c} \\ e^{-x_1} \sum_{n=0}^{\infty} (1 - e^{-bt_1} \sum_{k=0}^{n} (bt_1)^k \cdot \frac{1}{k!}) \cdot \frac{(x_1)^n}{n!}, \text{ if } t_1 > 0 \end{cases} \tag{26}$$

where

$$x_1 = \frac{ax}{c}$$

and

$$t_1 \equiv t - \frac{x}{c}$$

Proof. From (4) we find that

$$E[T(x)] = \frac{x}{c} + E[\sum_{0}^{N(x)} Z_n]. \tag{27}$$

Since N(x) is independent of Z_n, $n \equiv 0, 1, 2, \ldots$ we see from Theorem A1 that

$$E[\sum_{0}^{N(x)} Z_n] = E[Z_0 + \sum_{1}^{N(x)} Z_n] = \frac{1}{\beta} + \frac{x}{c} \cdot \frac{a}{b}$$

which together with (27) proves (23). . .

Similarly we see, using Theorem A1 again, that

$$Var[T(x)] \equiv Var[Z_0 + \sum_{1}^{N(x)} Z_n] \equiv \frac{1}{b^2} + \frac{x}{c} \cdot \frac{2a}{b^2}$$

which proves (24). (Recall that the Z_n are independent.)

Furthermore,

$$E\,[(T(x)-E\,[T(x)])^3] = E\,[((\sum_{1}^{N(x)} Z_n - E\,[\sum_{1}^{N(x)} Z_n]) +$$

$$+ (Z_0 - E\,[Z_0]))^3] = E\,[(\sum_{1}^{N(x)} Z_n - E\,[\sum_{1}^{N(x)} Z_n])^3] +$$

$$+ E\,[(Z_0 - E\,[Z_0])^3] = \frac{x}{c} \cdot \frac{6a}{b^3} + \frac{2}{b^3},$$

where we once again have used Theorem A1. Thus (25) is proved.

Finally, using (2), (22), (14) and (21), we obtain

$$Pr\,[T(x) - \frac{x}{c} \leqq t] = e^{-x_1} \sum_{n=0}^{\infty} (1 - e^{-bt} \sum_{k=0}^{n} (bt)^k \cdot \frac{1}{k!}) \cdot \frac{(x_1)^n}{n!}$$

from which (26) follows and the theorem is proved.

Next let I(t), $t \geqq 0$, be a time continuous, time homogeneous, Markov chain with two possible states named 1 and 2 respectively.

Let us denote

$$Pr\,[I(s+t) = i \mid I(s) = j] = P_{i,j}(t),\ i,j = 1,2 \qquad t,s \geqq 0.$$

Let us further assume that I(t) is a Poisson process, i.e.

$$P_{1,1}(h) = 1 - \lambda h + o(h)$$
$$P_{2,2}(h) = 1 - \mu h + o(h) \qquad (28)$$

where $\lambda, \mu > 0$.

Let D_n denote that epoch at which I(t) leaves state 1 for the n^{th} time, $n = 1,2,\ldots$ Further, if, on the one hand, we assume that $I(0) = 1$ then we shall put $E_0 = 0$, and let E_n denote that epoch at which I(t) leaves state 2 for the n^{th} time, $n = 1,2,\ldots$ On the other hand, if we assume that $I(0) = 2$ then we shall denote by E_{n-1}, that epoch at which I(t) leaves state 2 for the n^{th} time, $n = 1,2,\ldots$ (See also Fig. 4 and Fig. 5.)

Next, let us define

$$\begin{aligned} &Y_0 = 0 \\ &Y_n = D_n - E_{n-1},\ n = 1,2,\dots \\ &Z_0 = E_0 \\ &Z_n = E_n - D_n,\ n = 1,2,\dots \end{aligned} \tag{29}$$

Theorem A3

Let I(t) *be a time continuous, time homogeneous, Markov chain fulfilling* (28). *Let* Y_n, $n = 0,1,2,\dots$ *and* Z_n, $n = 0,1,2$ *be defined by* (29). *Then*

$$\begin{aligned} P_{1,1}(t) &= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t},\ t \geq 0 \\ P_{1,2}(t) &= \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t},\ t \geq 0 \\ P_{2,1}(t) &= \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t},\ t \geq 0 \\ P_{2,2}(t) &= \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t},\ t \geq 0 \end{aligned} \tag{30}$$

$$\begin{aligned} \lim_{t\to\infty} P_{i,1}(t) &= \frac{\mu}{\lambda+\mu},\ i = 1,2 \\ \lim_{t\to\infty} P_{i,2}(t) &= \frac{\lambda}{\lambda+\mu},\ i = 1,2 \end{aligned} \tag{31}$$

and

$$\begin{aligned} \frac{\mu}{\lambda+\mu} P_{1,1}(t) + \frac{\lambda}{\lambda+\mu} P_{2,1}(t) &= \frac{\mu}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} P_{1,2}(t) + \frac{\lambda}{\lambda+\mu} P_{2,2}(t) &= \frac{\lambda}{\lambda+\mu} \end{aligned} \tag{32}$$

Furthermore

$$\begin{aligned} Pr\,[Y_n \leq t] &= 1 - e^{-\lambda t},\ t \geq 0,\ n = 1,2,\ .. \\ Pr\,[Z_n \leq t] &= 1 - e^{-\mu t},\ t \geq 0,\ n = 1,2,\ .. \end{aligned} \tag{33}$$

and, if I(o) ≡ 2 then

$$Pr\,[Z_0 \leqq t] = 1 - e^{-\mu t}\;,\; t \geqq 0. \tag{34}$$

Proof: Let us put

$$\mathbf{P}(t) \equiv \begin{pmatrix} P_{1,1}(t) & P_{1,2}(t) \\ P_{2,1}(t) & P_{2,2}(t) \end{pmatrix}$$

and

$$\mathbf{A} \equiv \begin{pmatrix} -q_{1,1} & q_{2,1} \\ q_{2,1} & -q_{2,2} \end{pmatrix}$$

where

$$q_{1,1} \equiv \lim_{h \to 0} \frac{1 - P_{1,1}(h)}{h}$$

$$q_{1,2} = \lim_{h \to 0} \frac{P_{1,2}(h)}{h}$$

$$q_{2,1} \equiv \lim_{h \to 0} \frac{P_{2,1}(h)}{h}$$

$$q_{2,2} = \lim_{h \to 0} \frac{1 - P_{2,2}(h)}{h}$$

From the general theory of time continuous, time homogeneous Markov chains with finite state space (see e.g. Karlin 1966, Chapter 7) we know that

$$\mathbf{P}(t) \equiv e^{\mathbf{A} \cdot t} \equiv I + \sum_{1}^{\infty} \frac{\mathbf{A}^n \cdot t^n}{n!}$$

From (28) we find that

$$\mathbf{A} \equiv \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

Representing $\mathbf{A}$ as

$$\mathbf{A} \equiv \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^{-1}$$

where

$$\mathbf{D} \equiv \begin{pmatrix} -(\lambda + \mu) & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathbf{Q} \equiv \frac{1}{\sqrt{\lambda + \mu}} \begin{pmatrix} \lambda & 1 \\ -\mu & 1 \end{pmatrix}$$

we find that

$$\mathbf{A}^n \equiv \mathbf{Q} \cdot \begin{pmatrix} (-(\lambda + \mu))^n & 0 \\ 0 & 0 \end{pmatrix} \cdot \mathbf{Q}^{-1} \quad n \equiv 1,2,\ldots$$

and thus

$$\mathbf{P}(t) \equiv \mathbf{Q} \cdot \left(\sum_{0}^{\infty} \frac{t^n}{n!} \begin{matrix} (-(\lambda + \mu))^n & 0 \\ 0 & 1 \end{matrix} \right) \mathbf{Q}^{-1} \equiv$$

$$\equiv \frac{1}{\lambda + \mu} \cdot \begin{pmatrix} \lambda & 1 \\ -\mu & 1 \end{pmatrix} \begin{pmatrix} e^{-t(\lambda + \mu)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ \mu & \lambda \end{pmatrix}$$

from which (30) follows.

Next we observe that (31) and (32) follow immediately from (30). Finally, (33) and (34) follow from the fact that the waiting times in a Poisson process are exponentially distributed (see e.g. Karlin 1966, Chapter 7) and thus the theorem is proved.

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