types arise from actual wear tests. The mechanisms responsible for one or the other type of change are of course vital for the failure parameter, and therefore must be represented in the model.

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APPENDIX A

Contact Solutions for Elastic Penetration of a Spherical Indenter Against a Cratered Half-space, in Axial Symmetry

Consider Fig. 7(c). The contact pressure was numerically solved by Engel [4]; an exact solution was obtained by Barber [13]:

$$\frac{\pi^2 q(r)}{2E_r} = 4\beta a\sqrt{1 - r^2/a^2} - \frac{2a}{R_1}\sqrt{\frac{1 - s^2/a^2}{1 - r^2/a^2}} + \frac{2s}{R_1}E(\mu', \tau') - \frac{s}{R_1}F(\mu', \tau'), \quad (a \geq s \geq r \geq 0) \tag{A1}$$

$$\frac{\pi^2 q(r)}{2E_r} = 4\beta a\sqrt{1 - r^2/a^2} - \frac{2a}{R_1}\sqrt{\frac{1 - r^2/a^2}{1 - s^2/a^2}} + \frac{2r}{R_1}E(\mu, \tau) - \frac{2r^2 - s^2}{R_1 r}F(\mu, \tau), \quad (a \geq r \geq s \geq 0) \tag{A2}$$

where we define some new variables

$$\mu = \sin^{-1}\sqrt{\frac{1 - r^2/a^2}{1 - s^2/a^2}}; \quad \tau = s/r$$
$$\mu' = \sin^{-1}\sqrt{\frac{1 - s^2/a^2}{1 - r^2/a^2}}; \quad \tau' = r/s \tag{A3}$$

and elliptic integrals:

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{A4}$$

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2\theta}\, d\theta \tag{A5}$$

At the middle of the crater $r = 0$, the pressure is

$$q(0) = \frac{E_r a}{4\pi^2 R_1}\left[1 + \rho\left(1 + \sqrt{1 - \xi^2} - \frac{1}{2}\xi \sin^{-1}\sqrt{1 - \xi^2}\right)\right] \tag{A6}$$

The elastic approach is

$$\alpha = \frac{s^2}{R_1}\frac{1 + \rho}{\xi^2}F_1(\rho, \xi) \tag{A7}$$

The contact force is

$$P = \frac{4E_r s^3}{3\pi R_1}\frac{1 + \rho}{\xi^3}F_2(\rho, \xi) \tag{A8}$$

where [4]:

$$F_1(\rho, \xi) = 1 - \rho(1 + \rho)^{-1}\sqrt{1 - \xi^2} \tag{A9}$$

$$F_2(\rho, \xi) = 1 - \rho(1 + \rho)^{-1}\sqrt{1 - \xi^4} \tag{A10}$$

In expressions (A7–A9), $\xi \equiv s/a$, $\rho \equiv R_1/R_2$. Equations (A7–A9) were verified by Barber [13], and (A10) was found extremely close: the term $\sqrt{1 - \xi^4}$ in equation (A10) is $\sqrt{1 - \xi^2}(1 + \xi^2/2)$ in the exact solution.

DISCUSSION

J. R. Barber[3]

If the law describing the local wear behavior of a system is known (e.g. wear rate proportional to local pressure and sliding speed) it is always at least theoretically possible to use it to determine the wearpath, without reference to the author's optimal principle. This principle must therefore be seen as an alternative statement of the laws of wear in the same way as optimal principles in mechanics can be used (for example) to replace some of the equations of elasticity or dynamics.

The application of the method depends crucially on our ability to select the correct failure parameter to optimise, since different parameters will predict different wearpaths. Indeed, this degree of generality must be present in the principle to enable it to accurately model the variety of wear laws which might be encountered.

Consider an example which is simple enough to be treated exactly by both methods. A solid consisting of a set of n parallel cylindrical pins of identical cross section but slightly differing lengths slides on a plane surface. The load is sufficient to ensure that all the pins make contact with the plane, but the longer pins will take the larger share of the load. If we assume that the wear rate is proportional to load, we find that differential wear at the longer pins causes the system to approach exponentially to a state of uniform loading. At any instant, the deviations of the individual loads P_i on the pins from the mean value $\overline{P}$ are linearly proportional to the initial values. i.e.

$$\frac{P_1 - \overline{P}}{P_{01} - \overline{P}_0} = \frac{P_2 - \overline{P}}{P_{02} - \overline{P}_0} = \ldots \frac{P_i - \overline{P}}{P_{0i} - \overline{P}_0} \text{ etc.} \tag{B1}$$

where P_{0i} is the load on the ith pin when sliding commences.

Equation (B1) therefore defines the wearpath in terms of the n parameters P_i and we can now use the author's principle in reverse to find what failure parameter we must optimize to obtain equation (B1) as a line of greatest slope.

The function

$$f(P_i) = \sum_{i=1}^{n} (P_i - \overline{P})^2 \tag{B2}$$

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satisfies this condition. It is worth noting that other superficially plausible failure parameters would give the wrong answer here. For example, taking the maximum pin load P_i (max) as a parameter, the optimal principle would predict that wear would occur initially *only* at the most heavily loaded pin.

The expression of the wearpath as the steepest descent of the function in equation (B2) has the advantage of emphasizing that the system tends to a steady-state of uniform pressure ($f(P_i) = 0$). This fact, which applies to any system of constant contact area, is often overlooked. For example, in the author's Fig. 1, we should not expect the hammer to approach complete conformity with the plane, since some convexity is needed to obtain a uniform pressure distribution.

It is tempting to extend the above argument to the cases of continuous contact area (A) by writing

$$f(\sigma) = \int \int_A (\sigma - \sigma_m)^2 dA \qquad \text{(B3)}$$

where σ is the local contact stress and σ_m the mean value over A. However, while the optimization of $f(\sigma)$ will give the correct steady-state solution ($\sigma = \sigma_m$ if A remains constant), the intervening wearpath will be a straight line and it is easily shown that this is inconsistent with a linearly pressure dependent wear rate unless the local elastic deformation is proportional to the local contact pressure alone.

The author may like to comment on the form of failure parameter which would give an exact solution in cases of constant but continuous contact areas. It may not be possible to give a form which is independent of the particular contact geometry. If so, the method loses most of its practical value in view of the impossibility of knowing in advance how much error is introduced by an incorrect choice of failure parameter.

Pradeep K. Gupta[4] and Bharat Bhushan[4]

The prediction of changes, due to wear, in the general geometry of interacting surfaces is indeed of great interest in numerous practical applications and the author should be commended for his continuous interest in this highly applied area.

With regard to the asperity models and the interaction of a pair of practical rough surfaces a few points need some further clarification simply due to the complexities associated with the problem.

(1) Authors' assumptions (a) to (c) are reasonable, but the assumption (d) is somewhat questionable [14].[5] Although the author is in general agreement with this fact it will be very valuable if the practical applications, where such an assumption could be valid, are mentioned.

(2) The assumption (g), which forms the basis of the analytical formulation, when combined with (d), creates some confusion pertaining to the general nature of interaction. It is quite reasonable to say that when the Hertz elastic solution provides the solution on a macroscopic scale, the deformation at the asperities within the contact zone are likely to be plastic, even at loads determined by author's assumption (g). This plastic deformation will certainly cause changes in the topography of the surfaces and hence assumption (c) will be violated.

(3) It may be suggested that the surface model based on assumptions (a) to (c) will be only relevant under light loads when the individual asperity interactions can be considered independently [15]. In other words the junctions formed by interacting asperities are far apart from each other so that the elasticity of the substrate plays no role in influencing the deformation at one junction due to a load applied at the other. Under such light loads, the author agrees with the fact that the Hertz solutions will not be valid on a macroscopic scale. At heavy loads, however, when the macroscopic problem may be Hertzian, the local asperities will perhaps coalesce and the surface model adopted by the author will not be valid. Under such conditions a more rigorous model for surface interaction, such as [16], must be considered. Authors' comments to this effect will be highly appreciated.

(4) The author's general comment that the wearpath principle postulated in the paper may be applicable to only "mild" or "gradual" wear processes where elastic contact stresses may be valid, once again needs support of some practical applications. It appears to the discussors that the asperity interactions problem, even under "mild" wear conditions will be plastic in nature.

Additional References

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Author's Closure

The author thanks his discussors for their interesting comments.

The foremost intent of this paper was to systematize predictive methods for the growth-history ("wear law") of wear scars; this is achieved in equation (8). In some simple wearing systems the statement of the wear mechanism completely determines a unique wearpath, and then no optimization is necessary. In the continuous wearing system of the spherical slider (Example of Section 2.4) more than one admissible wearpath exists; the one chosen by the process tends to induce uniform pressures, and is therefore "optimal" in a sense. The instantly conforming feature of many (sliding) contacts cannot however be taken as a rule, and the growth of the contact area does require a selective process. Certainly this is the case for the continuously growing contact of the hammer in Fig. 1.

We shall define a wearing system as determinate if all the geometric wear parameters (throughout the wear life) can be found by application of the statements of the wear mechanism, given the external load conditions. For such a system, a single wearpath exists, and since optimization is bypassed, no separate failure parameter is needed.

A constant-area wear system is determinate since the wear rate (i.e. the change of n geometric parameters x_i) may be expressed uniquely by a matrix-relation,- like the following one for linear wear mechanisms:

$$\{x_i\} = [K]\,\{\sigma_i\} = [K][F]\{x_i\} \qquad \text{(C1)}$$

where K is the wear-coefficient matrix (usually diagonal), and $[F]$ is the surface elasticity matrix-operator, relating $\{x_i\}$ to the tractions $\{\sigma_i\}$ over the surface elements.

Therefore no choice for wearpath exists in Barber's first example of a constant-area system of rods (Fig. 15(a)). If the compressed length of all rods is kept a constant L_f, then the force P_i in the rods diminishes as wear w_i takes place. The elastic deformation at any stage is

$$\alpha_i = L_i - L_f,$$

and the force, (further approximated for moderate deformation):

$$P_i = AE\,(1 - L_f/L_i) \approx AE\,((L_i/L_f) - 1)$$

The statement of the wear mechanism may be written with some generality, between two consecutive states:

$$w_i = L_{ie} - L_i = KP_{ie}^c$$

[4] Mechanical Technology Inc., Latham, N. Y.

[5] Numbers in brackets designate Additional References at end of discussion.

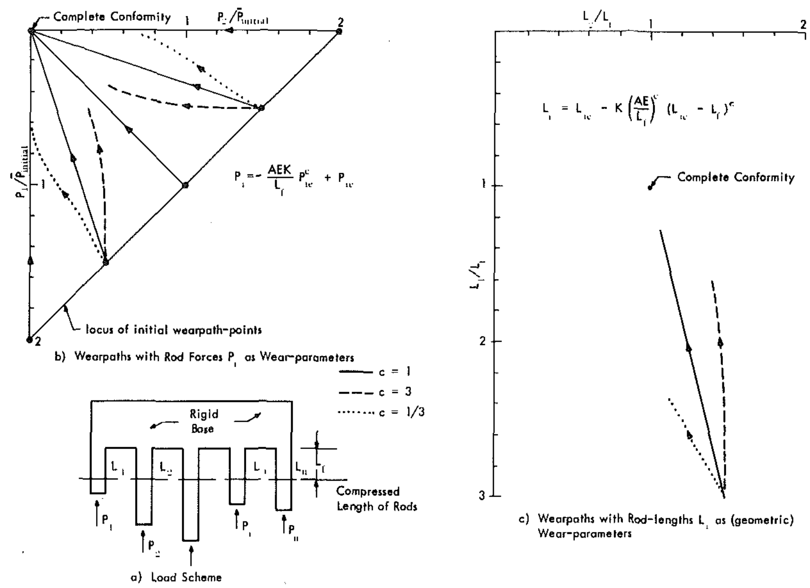

Fig. 15 Wearpaths for constant-area system of uniaxial rod assembly; $n = 2$. The wear mechanism statement, $W = KP^c$ uniquely determines the wearpath.

Selecting P_i as the wear parameters, the unique wearpath is then:

$$P_i = P_{ie} - (AEK/L_f)P_{ie}^c$$

which is shown in Fig. 15(*b*), for $c = 1/3$, 1 and 3. All the wearpaths (i.e., for any value of c) are directed toward the origin where conformity is complete, $L_i = L_f$. Regardless the initial set of rod-lengths L_i, the wearpath is straight for $c = 1$.

If the rodlengths L_i are chosen as (geometric) wear parameters, it can be similarly shown that the unique wearpath for any rod is, (Fig. 15(*c*)):

$$L_i = L_{ie} - K(AE/L_f)^c(L_{ie} - L_f)^c$$

It is remarked that a constant-force system (ΣP_i = const) has a similar unique wearpath.

For the continuous constant-area system (Barber's second example) the contact pressures may be written in complete generality (and therefore, "exactly") in the matrix form $\{\sigma_i\} = [F]\{x_i\}$. Now, however, $[F]$ is no longer diagonal as it was in the previous example for uniaxial rod elements. Thus a unique wearpath can be derived by the use of a known wear mechanism $[K]$, (as stated in equation (C1)), assuring the full analytical value of the procedure.

As opposed to a constant-area wear system, one with a (progressively) growing area cannot be written simply as equation (C1). This is because the expanding contact area affects and alters $[F]$ as well. Since the change of the parameters x_i is no longer uniquely determined by the wear mechanism, the optimal wearpath principle must be invoked to supply the wearpath.

Regarding the choice of failure parameter,- this may be suggested in a given case by experience. Several possible candidate-functions may be tried, and the results compared with experimental data. While this may appear as a bit of hindsight, most wear-work contain a measure of the latter, in the author's experience.

In response to Gupta and Bhushan's comments, the author can quote several examples where no substantial change in microtopography (roughness) occurs in a sustained wearing process. Fig. 6 of reference [1] shows consecutive Talysurf profiles of repetitively impacted spring steel, where the developing wear scar does not show essential change of roughness until past 13 million cycles. As a contrast, surface finish gets progressively rougher in ballistic impact wear tests of carbon steel projectiles, especially when a tangential speed component is imposed on the normal approach [5]; the surface finish of type is, on the other hand, generally reduced by impact wear, in the printing process.

The author agrees with the latter discussors on the importance of eventual plastic deformations occurring on the asperities. It is entirely possible of course to incorporate plastic deformations into both the stress analysis and/or the wear mechanism and failure parameter. On the other hand, Greenwood and Williamson [17] have shown through a simplified model, that, by virtue of the asperity height distribution, a wide region of the plasticity parameter will admit large pressures without plastic deformations taking place on the asperities. It is clear that models with a larger number of asperity parameters should also be investigated. The problem of the progressively wearing microtopography has been a neglected area so far in the literature; the author hopes that the procedures outlined in this paper will facilitate rational analytic approaches.

Additional Reference

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