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CROSS-SPECTRUM ANALYSIS OF GROUNDWATER LEVELS IN AN ESKER

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A study is made on 30-year records of precipitation, river water stage, and groundwater levels in the esker area south of Uppsala, Sweden, using cross-spectrum analysis. The results are summarized as coherence graphs for frequencies up to 18 cycles/year and tables of phase shifts for selected frequencies. The coherence between precipitation and the other variables is extremely weak or absent. The river water level and groundwater levels show appreciable coherence relatively independent of frequency, suggesting that variation in groundwater levels runs parallel to variations in river water stage. The coherence between groundwater levels, however, is such that interaction is suggested. The phase shift between river water level and groundwater level is approximately 10–30 days with the groundwater levels lagging. The magnitude of phase shift seems to depend on the distance between ground surface and groundwater level.

In a recent paper (Eriksson 1970) an analysis was made of time series on groundwater levels in an esker, a glaciofluvial deposit, south of the city of Uppsala in central-eastern Sweden. It was shown that the spectrum densities could be explained by a simple second-order autoregressive scheme which, in turn, could be explained by the general physical properties of a groundwater reservoir. In fact, the physical interpretation in this very simple one dimensional form offered possibilities of estimating means of yearly recharge of the reservoir.

In the present paper the analysis is extended to cross-spectrum analysis for

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studying the so-called coherence and phase lag spectra, i. e. the interrelation of groundwater levels between different points in a groundwater reservoir using the same set of 30-year records arranged with 10-days intervals. Also interrelations between other relevant factors – precipitation and the water levels of the adjacent River Fyri – are studied with the same technique. The physical interpretation of the results of such analysis are, naturally, of great interest.

COHERENCE SPECTRA, GENERAL REMARKS

The data were pretreated in the same way as in the earlier paper, i. e. means, trends, and seasonal mean fluctuations were removed before computing the spectrum, co-spectrum, and quadrature spectrum densities, the latter being computed by the formulas given by Panofsky & Brier (1958). Periodicity in the standard deviation was not removed since this does not necessarily make the covariances stationary.

As earlier, we used $S\ (K)$ for the spectral density estimates with $S_1\ (K)$ and $S_2(K)$ being the densities estimated from series 1 and 2. The co-spectrum density $C(K)$ and quadrature spectrum density, $Q(K)$, are, of course, computed from co-variances of the two series with the same number of lags as when computing the autocovariances of each series.

The coherence, $CO(K)$, is now defined by

$$CO(K) \equiv \frac{[C(K)]^2 + [Q(K)]^2}{S_1(K) \cdot S_2(K)} \tag{1}$$

and varies, as a correlation coefficient, between 0 and 1 (it corresponds rather to a squared multiple correlation coefficient). A coherence of 1 means that the variation in one series follows exactly the pattern of variation in the other series. There may or may not be a time lag between the series, phase shift, but this will not effect the coherence. In fact the phase shift can be computed from $C(K)$ and $Q(K)$ from

$$\Theta(K) \equiv \operatorname{arctg}\left(\frac{Q(K)}{C(K)}\right)$$

The coherence spectra thus show how the interrelation between the variables depends on the frequency. For some simple autoregressive processes the coherence is easily inferred. If, for instance,

$$x_t \equiv r_x x_{t-1} + \mu_t \tag{2a}$$

$$y_t \equiv r_y y_{t-1} + \eta_t \tag{2b}$$

and μ and η are correlated, the coherence between the two series (x_t) and (y_t) will be independent of frequency and equal to the square of the correlation coefficient between μ_t and η_t. But for a process like

$$x_t = a_1 x_{t-1} + b_1 y_{t-1} + \mu_t \tag{3a}$$

$$y_t = a_2 x_{t-1} + b_2 y_{t-1} + \eta_t \tag{3b}$$

the coherence will most unlikely be independent of frequency unless b_1 and a_2 are zero.

There is a significance test available for the coherence although it does not appear to be very strong (see Panofsky & Brier, 1958). The interpretation of this significance test is again the same as that for spectral densities, namely a test of the probability that two coherence spectra from different part of stationary infinite time series will show similar results. In the present case the 90 per cent confidence interval is at a coherence of 0.5; above this limit coherences have a probability of less than 10 per cent of being random concurrent fluctations.

RESULTS

The positions of the groundwater observation sites relative to the esker are shown in Fig. 1. The parts of the esker which reach the surface are enclosed by dashed lines. Topography is indicated by contour lines with figures which give the height about sea-level in meters. The slanted, larger number indicate the observation wells, referred to in the Figures and below. The water levels of the River Fyris were measured at the point marked "Pegel Fyrisån".

Precipitation – River Fyris water level

Although this relation is not of direct concern to the present subject, it may be of interest to study it. The coherence at different frequencies is shown in Fig. 2, upper graph. There is a very weak relation except at low frequencies. This is perhaps not surprising since the storage in the river basin will smooth shorter fluctuations in precipitation. There is, of course, some interrelation but this is most likely bound to the seasonal variation of the two which is eliminated in the present context.

Precipitation – groundwater levels

The coherence between precipitation and groundwater levels is shown in Figs. 2, 3, 4, and 5. There is very little coherence shown for frequencies higher than

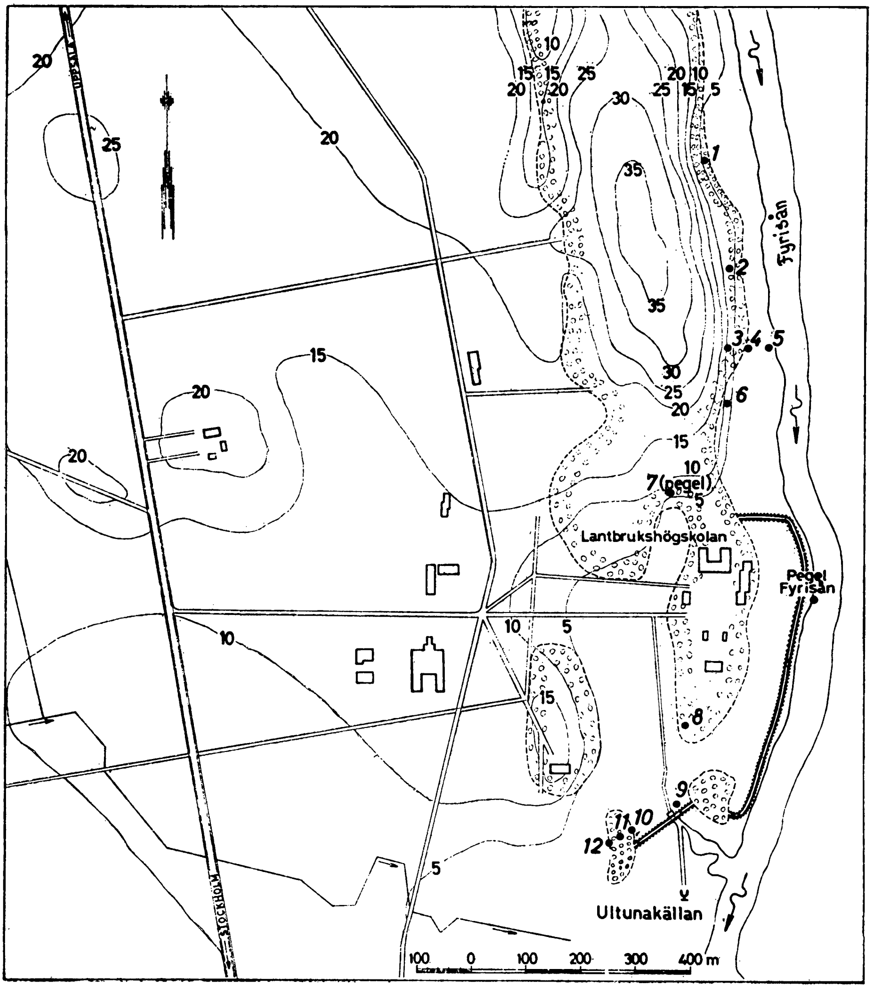

Fig. 1.
Map of the esker area. Full lines are height contour lines, figures are meters above mean sea-level. Dashed lines enclose outcrops of the esker (a large part of which is submerged in the clay sediments). Slanted figures give well number. "Pegel Fyrisån"; measuring point of River Fyris water level. "Ultunakällan", a spring draining the esker.

5 cycles/year. The frequency 5 cycles/year is, on the other hand, rather clearly indicated on all the Figures, but, aside from this, only fluctuations of frequencies lower than 2 cycles/year are found (1 cycle/year is, of course, suppressed by pretreatment of the data). Slow fluctuations, with periods of several years,

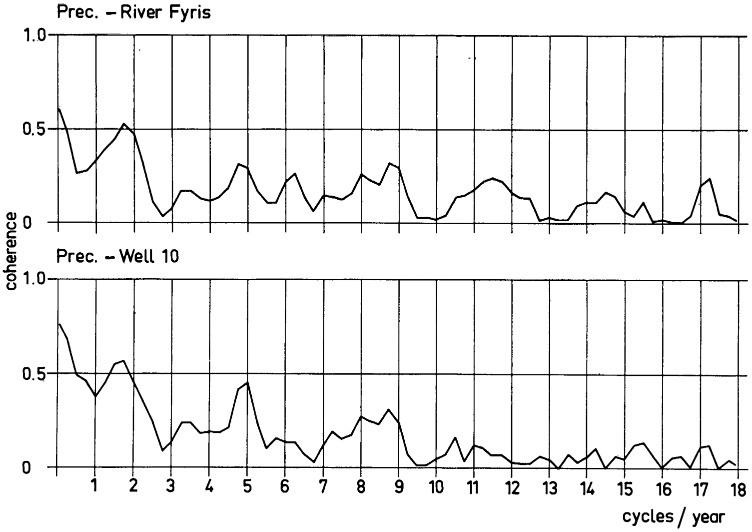

Fig. 2.

Coherence spectra for precipitation River Fyris water level (upper graph) and precipitation Well 10 (lower graph).

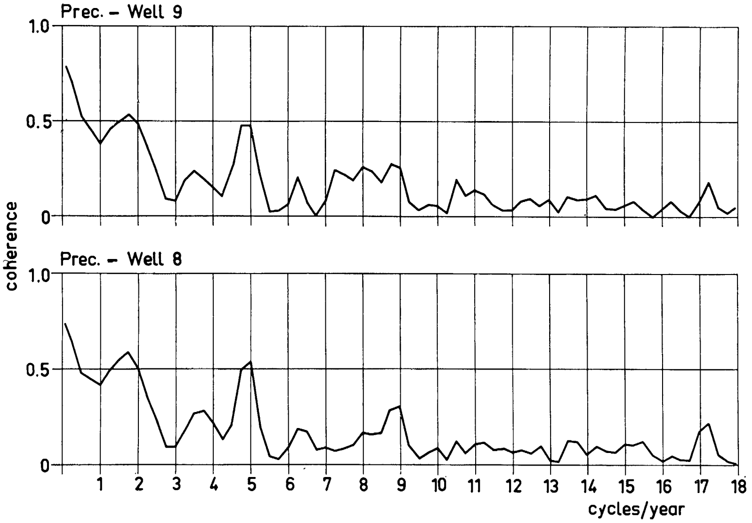

Fig. 3.

Coherence spectra for precipitation and water levels in indicated wells.

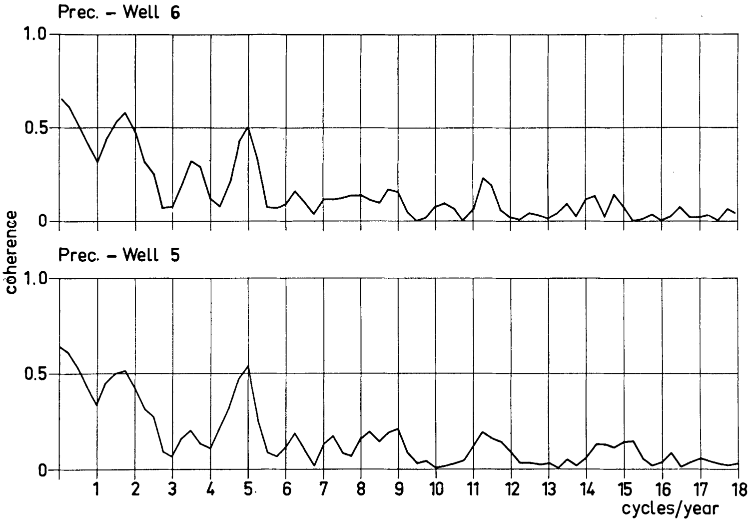

Fig. 4.
Same as Fig. 3.

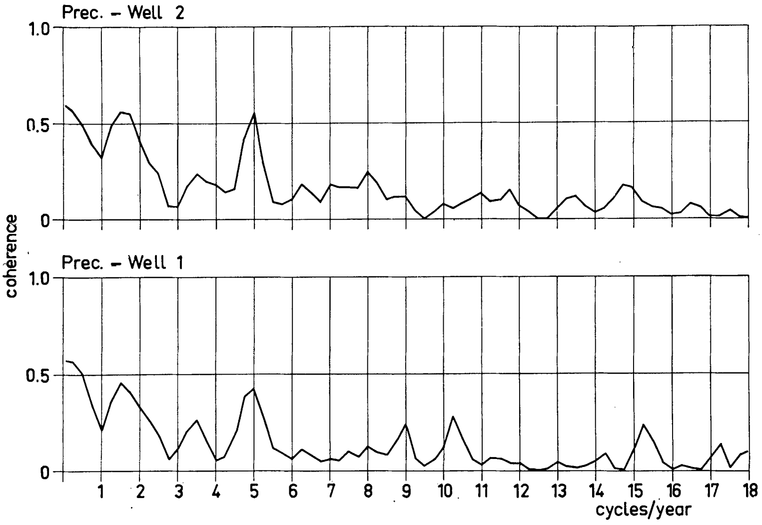

Fig. 5.
Same as Fig. 3.

are significant insofar as precipitation effects groundwater levels on that time scale. The coherence, however, does not approach unity at very low frequencies, which shows that other factors also determine long-term changes.

River Fyris water level – groundwater level

The coherence for these combinations are displayed in Figs. 6, 7, and 8. There is definitely a rather significant coherence over most of the frequency range but this should not be interpreted to mean a causual relationship between the two, rather a parallel relation, as discussed earlier. The expected relation for processes given by Eqs. (2a) and (2b), which can be called parallel processes, is, as pointed out, a coherence independent of frequency.

Although this is not strictly indicated in the Figures, it is relatively close. It is especially interesting to note that the coherence spectrum of River Fyris – Well 10 is not fundamentally different from that of River Fyris – Well 2, although Well 10 is very close to the outlet of the esker, "Ultunakällan", which at times is flooded by the river.

Another interesting feature in the Figures is the decrease in coherence when approaching very low frequencies. This would indicate that groundwater storage in this esker is more sensitive to long-term changes than the river water level.

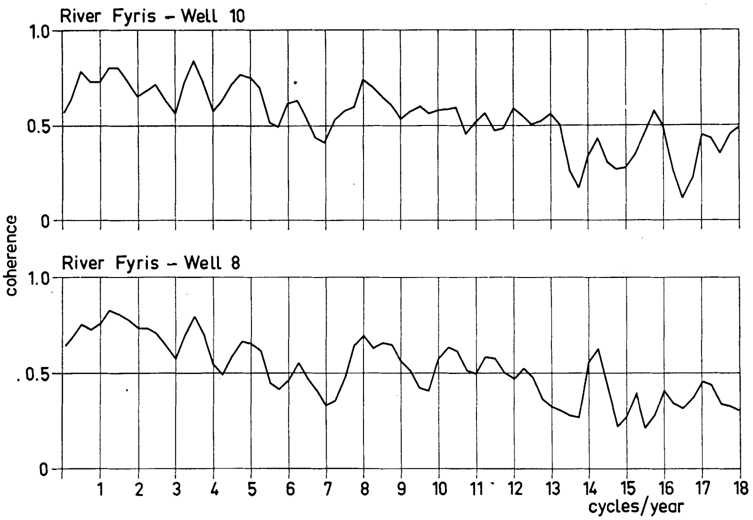

Fig. 6.

Coherence spectra for River Fyris water level and groundwater levels in Wells 10 and 8.

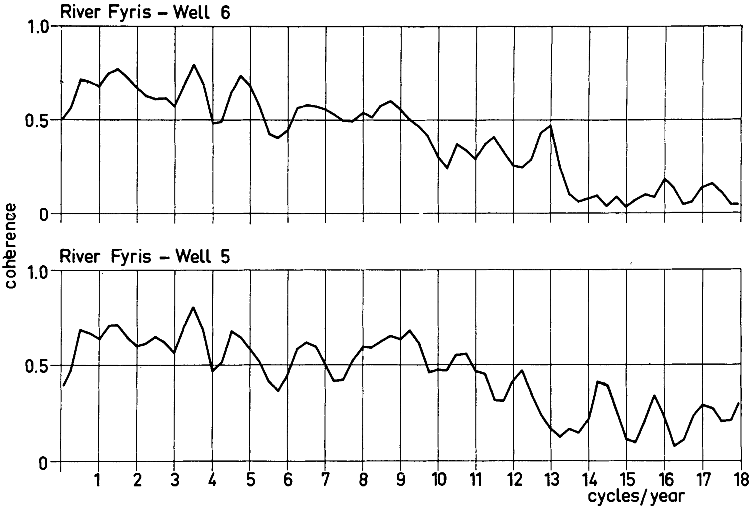

Fig. 7.
Same as Fig. 6 for Wells 6 and 5.

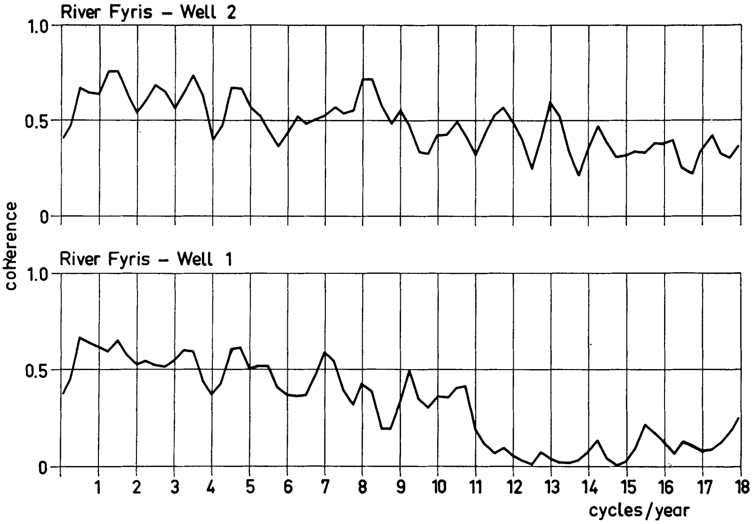

Fig. 8.
Same as Fig. 6 for Wells 2 and 1.

Interrelations between wells

Fig. 9, lower graph, shows the coherence between two fairly closely situated wells – the distance between them is about 100 m. The coherence is very high – above 0.9 – up to nearly 4 cycles/year frequency and decreasing rather regularly toward about 0.5 for higher frequencies.

The effect of the distance between wells is demonstrated in Fig. 9, upper graph, and Fig. 10 and 11. Between Wells 9 and 8 the coherence follows about the same pattern as between Wells 10 and 9, which can be expected since the distance between Wells 9 and 8 is not much greater than between Well 10 and 9. As the distance increases, the coherence drops faster with increasing frequency. This means that a well is less representative of water level changes for higher frequencies than for lower frequencies.

Slow change in groundwater levels may be well represented by a low density of observation wells but in order to record more rapid changes, for instance for water balance purposes, a denser network is needed.

One exception is found in Well 2 which seems to follow Well 9 reasonably well also at higher frequencies, at least up to 12 cycles/year. The reason for this is unknown. The combination Well 9 and 1 gives the poorest coherence.

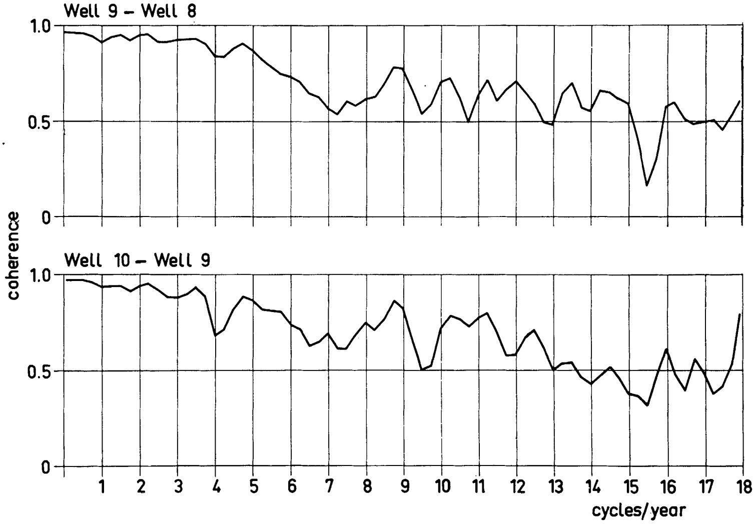

Fig. 9.
Coherence spectra between Wells 10–9 and 9–8.

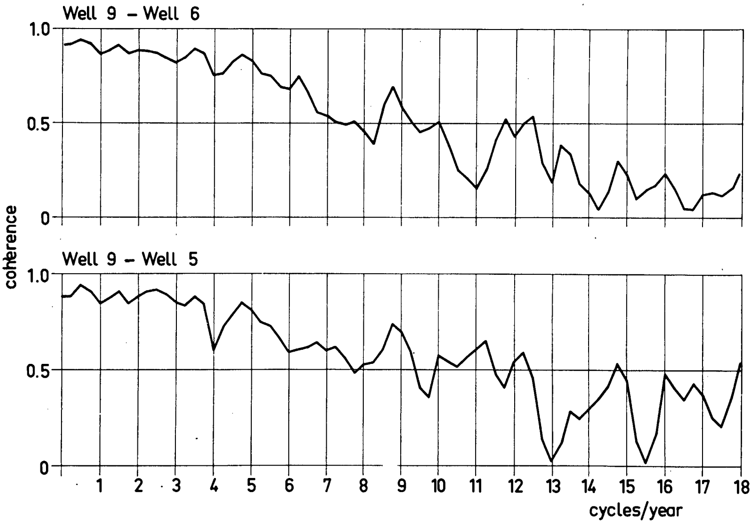

Fig. 10.

Coherence spectra between Wells 9–6 and 9–5.

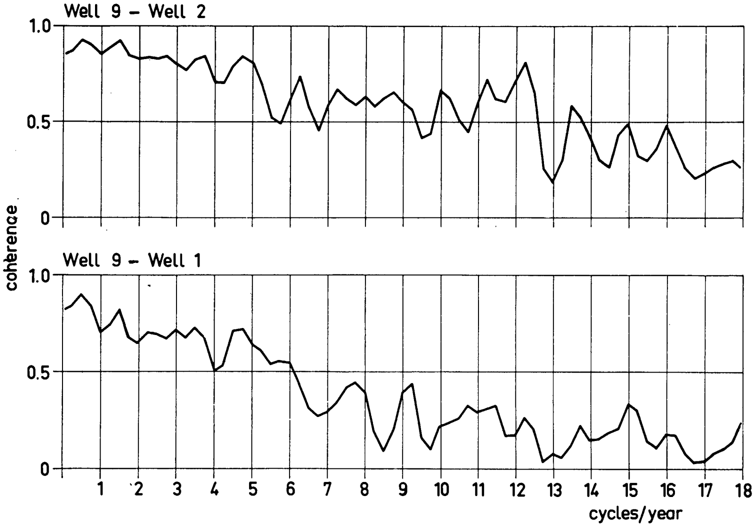

Fig. 11.

Coherence spectra between Wells 9–2 and 9–1.

Figs. 12 and 13 show the coherence for four other combinations of wells which confirm earlier conclusions. Combinations with Well 1 show the poorest coherence pictures.

Phase lags

As to phase relations, the largest phase shifts are found between precipitation and groundwater levels, in the expected direction, i. e. groundwater levels lag behind in time. The coherence is, as pointed out earlier, not too strong, not even at the lowest frequencies; phase lag estimates are therefore rather uncertain. The frequency bands around 0.25 and 0.5 cycles/year give consistent lags of about three months. This means that for such periods in the precipitation the groundwater level changes are about three months delayed. Presumably a certain storage period in the unsaturated zone for the water which is most tightly bound causes this delay; only this water can account for long-term effects in the present connection. There is, further, a small but systematic difference along the esker, the time lag being slightly smaller in the southern part than in the northern.

As to the phase relations between the River Fyris water level and the groundwater levels, the phase lag for different frequencies is shown in Table 1, the lag

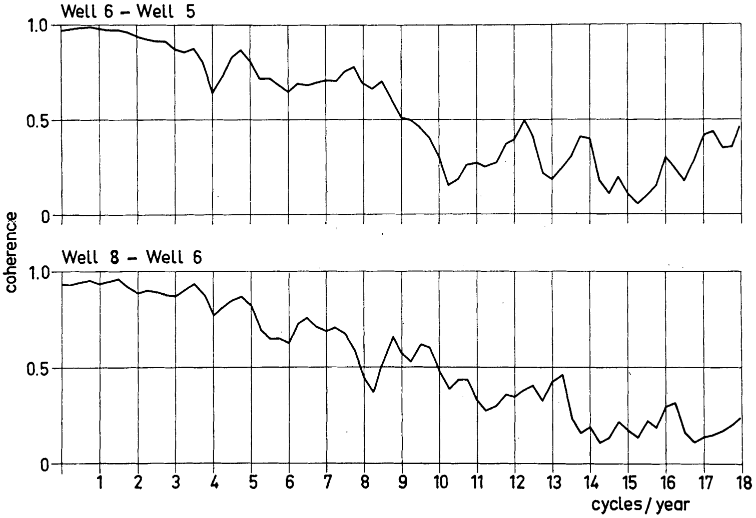

Fig. 12.
Coherence spectra between Wells 8–6 and 6–5.

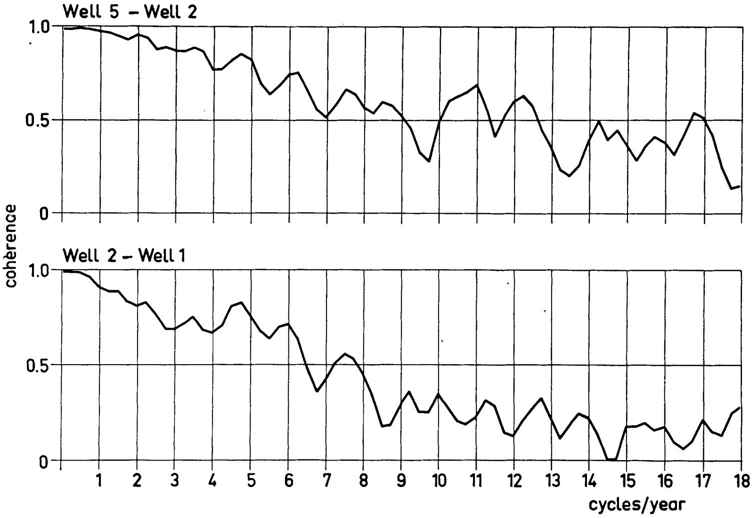

Fig. 13.
Coherence spectra between Wells 5–2 and 2–1.

being computed in days. The lags are always negative indicating that the changes in groundwater levels lag behind those of the river. Further, the lags become smaller for higher frequencies although they seem to approach asymptomatically some value close to a week.

It is interesting to note that for periods of half a year and longer, Wells 8, 9, and 10 behave very differently from Wells 2, 5, and 6 (Well 1 being different from all of these). The change from Well 8 to Well 6 is almost abrupt. The phase lag for the first group is nearly only half as long as for the latter group; according to these figures it is not likely that the river water levels influence these wells directly, e. g., at high water levels; if so, there would be clear differences among the wells in the first group, which is obviously not demonstrated by the Figures. It is possible, however, that the recharge conditions are different in the southern and in the northern areas. This can be seen already from the topography. In the northern part the esker is very high with a great distance between its surface and the groundwater surface, whereas in the southern part the groundwater surface is not very distant from the surface of the outcropping esker.

Again it is seen that the phase lag for very low frequencies is considerable, although it is shorter than in the case of interrelation between precipitation and groundwater level, which is probably not unexpected.

Table 1.

Phase lags in days between River Fyris water levels and groundwater levels for different frequencies (the minus sign indicates a lag in groundwater levels).

Frequencies in cycles/year	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3,00	6.00
Period length in days	1440	720	480	360	288	240	206	180	160	144	131	120	60
Well no.													
10	–36	–28	–18	–15	–13	–13	–11	–9	–9	–7	–5	–4	–5
9	–29	–25	–16	–14	–11	–10	–11	–11	–9	–7	–5	–5	–5
8	–34	–27	–15	–12	–12	–11	–9	–7	–8	–6	–4	–3	–4
6	–43	–41	–27	–21	–22	–21	–16	–11	–11	–10	–7	–6	–8
5	–51	–41	–24	–21	–23	–21	–16	–12	–13	–11	–8	–8	–6
2	–48	–43	–27	–22	–24	–23	–18	–13	–13	–12	–10	–9	–7
1	–62	–57	–35	–34	–41	–31	–27	–23	–19	–18	–16	–15	–

It was seen from the coherence spectra that there is a much better coherence between groundwater levels and river water levels than between groundwater levels and precipitation. The reason is most likely that nearly half the annual precipitation falls as snow; therefore, mild weather in winter causes recharge, not precipitation itself. The same is also true for precipitation supplying the river. Hence, the river stage (the river flow is not regulated by dams) gives a much better measure of recharge rate in the esker than precipitation does. There is no direct influence of the river on the groundwater; the river level and the groundwater level variations are largely parallel processes.

If the river level had some influence on the groundwater level, e. g. through seepage, one would have expected larger phase lags for higher frequencies than for lower. Since the reverse is the case here, such an influence can therefore be ruled out.

The phase lags between groundwater levels are also rather illuminating. Table 2 shows computed phase lags in days at the same frequencies as in Table 1. The first set is a comparison between Well 10 and the others. The phase lags confirm earlier conclusions rather well. There is a tendency for positive lags in the pair 10–9 which means that Well 9 is slightly ahead. This also rules out any influence of the river water level on the groundwater; Well 10 is closest to the spring "Ultunakällan" which drains into the river. Also the pair 9–8 shows

Table 2.
Phase lags in days between groundwater levels for different frequencies (a minus sign indicates that the second well lags behind the first)

Frequencies in cycles/year	0.25	0.50	0.75	1..00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	6.00
Period length in days	1440	720	480	360	288	240	206	180	160	144	131	120	60
Well Nos.													
2–1	–12	–15	–12	–12	–16	–13	–9	–9	–8	–7	–8	–7	–2
10–9	3	2	3	3	3	3	0	–1	0	1	0	0	0
10–8	–3	–1	3	3	1	1	2	2	1	1	1	0	0
10–6	–13	–15	–10	–6	–9	–9	–6	–2	–3	–4	–3	–2	–3
10–5	–16	–14	–7	–6	–11	–10	–5	–3	–4	–5	–3	–4	–1
10–2	–18	–18	–10	–7	–11	–12	–8	–3	–4	–5	–5	–4	–2
10–1	–32	–32	–21	–18	–28	–25	–17	–12	–11	–12	–13	–11	–5
9–8	–6	–3	0	0	–2	2	2	3	1	0	0	1	0
8–6	–8	–14	–13	–9	–10	–10	–8	–4	–4	–5	–4	–3	–4
6–5	–2	2	3	1	–1	–1	0	0	–1	–1	0	–2	2
5–2	–3	–4	–3	–1	–1	–2	–2	0	1	0	–1	0	–1

a slight tendency for positive lags. Otherwise the lags for these two pairs are rather small and not far from the standard deviation in the computed lags. The pairs 10–6, 10–5, and 10–2 are completely different and the lags agree well with the difference in lags between the two groups Wells 10, 9, 8 and Wells 6, 5, 2 in Table 1 for phase lags relative to the river water levels.

Finally the pair 10–1 forms a separate group as in Table 1. Moving northward, the changes in the lags are well expressed in the second set where lags between adjacent wells are shown. Here the pair 8–6 shows the above mentioned difference between the groups, and the pair 2–1 sets the limit for the second group.

It was suggested earlier that the phase lags are related to the distance from the surface of the esker to the groundwater table. It is possible that the phase lags between Wells 2 and 1 have been enhanced by the drop in the groundwater table observed at Well 1 during the period, a drop which is assumed to be due to the influence of production wells for the city of Uppsala.

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