Response of a Layered Half-Space to a Moving Load

J. D. Achenbach. This writer has misgivings about the author's expectation that the dynamic response of a two-layered elastic half-space to a moving load can be approximated by the corresponding dynamic response of a two-layered fluid half-space. The author sets as a condition that \( V \) should not approach the velocity of Rayleigh surface waves \( c_R \), where presumably is meant \( c_{R1} \), because it is stated that if \( V \approx c_R \) layering must be of only secondary importance by virtue of the exponential decay of Rayleigh waves with depth. I believe that at least more stringent conditions on the load velocity are required, depending on the type of contact between elastic layer and elastic half-space. To support this opinion it is shown in this Discussion that for a certain subseismic velocity, henceforth designated as the critical velocity, where for the material properties considered \( V \approx 0.4c_R \), a resonance effect occurs in the layered elastic half-space that cannot be accounted for by the layered fluid.

We consider an elastic layer in contact with an elastic half-space as shown in Fig. 1. By assuming that the wave pattern is stationary relative to a coordinate system moving with the load, the equations governing the displacement potentials \( \varphi(r, z) \) and \( \psi(r, z) \) (where \( r = x - Vt \)) can be written as

\[
\Delta = \begin{pmatrix}
(1 + s_1^2)e^{-kq_1h} & (1 + s_2^2)e^{kq_2h} & -2e^{-kq_2h} \\
-2e^{-kq_2h} & 2e^{kq_2h} & (s_1 + s_2)e^{-kq_2h} \\
(1 + s_1^2)e^{kq_1h} & (1 + s_2^2)e^{-kq_2h} & -2e^{kq_1h} \\
-2e^{kq_1h} & 2e^{-kq_1h} & (s_1 + s_2)e^{kq_1h} \\
q_1e^{kq_1h} & q_2e^{kq_2h} & -s_2e^{kq_2h} \\
-q_1e^{kq_1h} & q_2e^{-kq_2h} & -s_1e^{kq_1h}
\end{pmatrix}
\]

where

\[
q_1^2 \frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \quad s_1^2 \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (1a, b)
\]

in which

\[
q_1^2 = 2G(1 - \nu)/p(1 - 2\nu), \quad s_1^2 = G/p \quad (3a, b)
\]

Here we consider subseismic load velocities, i.e., \( q_1^2, q_2^2, s_1^2, \) and \( s_2^2 \) all \( > 0 \). The expressions for the displacements and the stresses in terms of the displacement potentials are listed elsewhere.  

We introduce the exponential Fourier transform as

\[
\tilde{\varphi} = \int_{-\infty}^{\infty} e^{-ikr}\varphi(r, z)dr
\]

After transforming \((1a, b)\), the general solutions for \( \tilde{\varphi}_m(k, z) \) and \( \tilde{\psi}_m(k, z) (m = 1, 2) \) are easily obtained. The latter solutions contain six unknown functions of \( k \) which must be solved from the boundary conditions. For welded contact the boundary conditions are:

at \( z = h \):

\[
\sigma_{nl} = -P\delta(r), \quad \sigma_{nl} = 0 \quad (5a, b)
\]

at \( z = -h \):

\[
\sigma_{nl} = \sigma_{nl}, \quad \sigma_{nl} = \sigma_{nl} \quad u_1 = u_h, \quad u_1 = u_1 \quad (6a, b, c, d)
\]

For smooth contact \((6b, c)\) are replaced by:

at \( z = -h \):

\[
\sigma_{nl} = \sigma_{nl} \quad (7)
\]

Substitution of \( \tilde{\varphi}_m(k, z) \) and \( \tilde{\psi}_m(k, z) (m = 1, 2) \) in the Fourier transforms of \((5a, b)\) and \((6a, b, c, d)\) yields a system of six equations. By solving this system of equations, solutions are obtained of the form

\[
\sigma_{nm} = -PF_m(k, V)/\Delta(k, V) \quad (8)
\]

In \((8)\), \( F_m(k, V) \) is a certain function of the Fourier transform parameter \( k \) and the load velocity \( V \). The function \( \Delta(k, V) \) is the determinant of the coefficients of the system of six equations. For \( k > 0 \) we obtain for welded contact

\[
\begin{pmatrix}
2e^{kq_1h} & 0 & 0 \\
s_1 + s_2 & e^{kq_1h} & 0 \\
2e^{-kq_2h} & G(1 + s_2)/G_1 & 2G_2/G_1 \\
(s_1 + s_2) & e^{-kq_1h} & s_2 + s_1/G_1 \\
1 & 1 & -1/s_2
\end{pmatrix}
\]

where \( q_m \) and \( s_m (m = 1, 2) \) are defined by \((2a, b)\). A similar determinant can be written for smooth contact.

Difficulties arise in the evaluation of the inverse Fourier transforms when the integrand has poles of the second order on the real axis. In that case we have integrals of the following form

\[
\int e^{ikr} f(r)dr
\]
which do not exist, not even in the sense of principal values. For $k > 0$ the equation $\Delta(k, V) = 0$ also appears as the dispersion equation in the study of free waves in a layered half-space. For a relatively stiff and heavy layer the writer has computed the real roots of $\Delta(k, V) = 0$ for both welded and smooth contact. It is seen from Fig. 2 that for smooth contact a double zero appears for $V/c_{R} \sim 0.85$ at $kh = 0.24$. At that critical velocity we have integrals of the type (10), which blow up, and we encounter a resonance effect. This type of resonance effect does not occur in a two-layered fluid half-space. For bonded contact a similar resonance effect appears at the velocity of Rayleigh waves of the supporting half-space. Since the motion in the layer is now not exponentially decaying, we expect the influence of layering to be of little importance for load velocities close to $c_{R}$. It is seen from this example that for a relatively stiff layer the responses of a two-layered fluid and an elastic half-space can be very different.

**Author's Closure**

The modal structure of a two-layered, elastic half-space was studied originally by Bromwich and has since been studied extensively by others, notably Love. Extensive discussion and references are given by Ewing, Jardetzky, and Press. The mode studied by Achenbach is a particular case of a Love wave (it should be remarked, however, that the particular model considered by Achenbach, in which the upper layer is both denser and stiffer than the substratum, is rather unrealistic), and the minimum in the dispersion curve (wave speed versus wave number) is typical; higher modes also exist, and their dispersion curves exhibit multiple maxima and minima. The existence of these extrema implies the existence of higher-order points of stationary phase in the integral representations of the solutions to properly posed initial-value problems, but these can be handled by standard analytical devices.

The virtue, as well as the deficiency, of the liquid model, *vis-á-vis* its elastic counterpart, is that it does not exhibit a complicated modal structure and therefore permits the explicit solution of certain boundary-value problems. The solutions for such steady-state problems as that considered by the author, namely, a surface load moving at uniform speed, provide asymptotic approximations to the corresponding solutions for elastic media. These approximations cannot be expected to be uniformly valid for moving-load speeds in the neighborhoods of any of the Love waves. It seems likely that the most important of these waves is the lowest mode, which tends to the Rayleigh wave for the upper layer as the thickness of that layer becomes large compared with the wavelength; nevertheless, the author was guilty of oversimplification in referring only to this mode. The remaining modes would certainly have to be considered in the solution of the moving-load problem for a two-layered elastic half-space; indeed, it appears likely that the existence of these modes precludes an explicit solution of this problem.

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**Stresses in an Infinite Strip Containing a Semi-Infinite Crack**

J. C. RICE

While Professor Knauss' study provides a useful evaluation of a practical fracture testing configuration, no elaborate computations are required for determining the stress intensity factor and thus the singular crack tip stress state. In fact, from an independent estimate of the stress intensity factor, the results given in the paper are suggested to be in error. Irwin [1] has defined a stress intensity factor $K$ such that the stress directly ahead of the crack tip has the form (Fig. 1)

$$
\sigma(x, 0) = K(2\pi x)^{-1/2} + \text{nonsingular terms},
$$

Then, judging from Professor Knauss' equation (32), his stress intensity factor $K$ is related to Irwin's by

$$
K = \frac{(1 - \nu^2)bK_i}{(2\pi)^{1/2}F_0}
$$

where $\nu$ is Poisson ratio, $E$ is Young's modulus, $b$ is half strip width, $v_0$ is vertical displacement of clamped strip boundary. Now defining $-\partial V/\partial c$ as the potential energy per unit thickness drained out of a body by a unit crack length extension, Irwin [1] has shown that

$$
-\frac{\partial V}{\partial c} = \frac{K^2}{E}
$$

for plane stress, a factor of $(1 - \nu^2)$ appearing for plane strain. Irwin set the potential energy release equal to the work done in removing tractions from the new crack surface. The procedure is clearly valid for configurations as in Fig. 1, where boundary displacements are specified so that boundary forces do no work;

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2. Assistant Professor of Engineering, Division of Engineering, Brown University, Providence, R. I. Assoc. Mem. ASME.
3. Numbers in brackets indicate References at end of Discussion.