The number of terms to be retained in the approximation (83) will be determined by comparing the quasi-steady solution to results by the integral method.

Four and five parameter quasi-steady profile approximations were computed by the integral method with results shown in Fig. 16. We note the agreement in location of peak and zero values between the quasi-steady solution and the five parameter approximation.

Velocity Profiles. The essential features of the second-order quasi-steady solution can be described by five parameter profiles, with profile coefficients determined by the integral method. Thus, we assume general second-order mean flow profiles of the form:

\[ u_i = e^{-\Phi(\psi_1 + \psi_2^3 + \psi_3^4 + \psi_4^5)} \]  

(84)

In the quasi-steady limit the \( \zeta \)-dependence of the form parameters is known, and equation (84) reduces to the form equation (83). We compute the five form parameters in equation (84) by the integral method using three integral relations and two compatibility conditions.

The compatibility conditions yield the following algebraic equations:

\[
\begin{align*}
\frac{\psi_1}{\psi_1} &= 1/\phi \\
\frac{\psi_2}{\psi_1} &= 1/\phi^2 \\
\frac{\psi_3}{\psi_1} &= 1/\phi^3 \\
\frac{\psi_4}{\psi_1} &= 1/\phi^4 \\
\frac{\psi_5}{\psi_1} &= 1/\phi^5
\end{align*}
\]  

(85)

We reduce the integral relations to a set of three ordinary differential equations by following the steps employed for the first-order solution; that is (a) introduce matrix notation, (b) carry out indicated differentiation, (c) transform integrals over \( y \) to integrals over \( \eta \) when the integrand depends on \( \eta \) alone, (d) define the \( \phi \)-functions for the integration sums, (e) write the equations in standard form for first-order ordinary differential equations. The resulting set of differential equations was solved numerically with results shown graphically in Fig. 17.

The development of the velocity component along the plate is shown in Fig. 18. The general behavior is spreading of the affected range of \( \eta \) values, and decrease in peak magnitudes, with increasing values of \( \zeta \) along the plate.

Summary and Conclusions

The integral method of analysis was investigated for unsteady laminar boundary layer flows, and detailed computations were made for the specific case of oscillating flow over a flat plate. Some general observations are then made based on the results of that example.

The integral analysis of the special case revealed a procedure which is generalized for unsteady flows as follows: (a) Obtain asymptotic solutions for limiting cases of the flow under consideration. (b) Assume velocity profiles with sufficient generality to include the asymptotic solutions. (c) Apply integral relations and compatibility conditions, as required, to determine the profile form parameters. The flow over a flat plate with a free-stream having small harmonic fluctuations about a uniform mean value was considered in detail. The basic solution for the steady flow was shown to be the familiar Blasius solution. Small local velocity fluctuations (of order \( \epsilon \)) were found to exhibit a phase lead with respect to the free-stream oscillations. The second-order (\( \epsilon \)) velocity was shown to consist of a time-independent contribution plus fluctuation at twice the fundamental frequency. The first-order fluctuations provide the "input" force to drive the second-order flow. The contribution of the steady second-order flow was found to diminish with increasing distance along the plate.

References


Discussion

P. D. Richardson* 

Dr. Miller and Professor Han have presented a clear description of the extension of integral methods to use with unsteady laminar boundary layers. They have also drawn attention to the utility of asymptotic solutions in guiding the choice of approximation functions for application of the method. It is worthwhile to emphasize that there are two distinct matters described in this paper: (1) integral equations for use in analysis of unsteady laminar boundary layers, these equations being used whatever the functional approximations for the velocity profiles may be; (2) examples of application, using specific assumptions for the functional approximations. It is to the second matter that the discusser wishes to draw attention. One bothersome feature of integral methods is that the accuracy of the results is tied up with the choice of the approximating function, and one does not have an infallible system for knowing how the

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accuracy will fare at the point where one has to choose an approximating function (unless the exact solution is known, so that the question is specifically academic). For example, there are approximations for the Blasius profile in the literature which range from a modest polynomial (which does quite well) to a completely transcendental approximation (which proves more accurate); the authors' choice, equation (40), is a hybrid form. The price that is often paid for improved accuracy is greatly increased effort in evaluating integrals for a more complicated approximation, and improved accuracy is not always achieved with more complicated approximations. Have the authors carried out calculations limiting themselves to purely polynomial profile approximations where possible, e.g., to replace equation (40) etc., and what were the results by comparison with those presented in the paper? Alternatively, did the authors find that use of the exponential decay factor in equations (40), (49), (83), and (84) is essential to the achievement of reasonable success? It is useful to know what is the most simple yet realistic approximation which might be used, because the effort of using any approximations must be weighted against the effort of an accurate numerical solution, and knowing the simplest approximation gives a bound for this decision.

Do the authors have some sample calculations to illustrate how the method and their approximations fare with configurations other than the flow with fluctuations over a flat plate, e.g., the problem of fluctuating flow transverse to a circular cylinder, a problem with well-known analytic solutions by Schlichting [18] extended by Stuart [19]? There is considerable interest in unsteady laminar boundary layers at present so this paper may prove very useful.

Additional References

A. H. Stenning

The authors present an interesting integral method for analyzing oscillatory laminar boundary layers which gives good agreement with experimental results for the case of oscillations superimposed on a Blasius flow. It would be useful to investigate whether the method is equally effective when the oscillations are superimposed on a mean flow with a pressure gradient. Can the method be extended to analyze the response of the boundary layer to traveling waves in the free stream, that is to free stream oscillations of the form $e^{i(ax - ft)}$?

Authors' Closure

Professor Richardson poses some basic questions which are directed not only to the authors of this paper, but also to all investigators utilizing the integral approach to solve boundary layer problems. There are legitimate questions one must address himself to whenever an approximate method is used in lieu of an exact solution.

Our motivations for using transcendental expressions (we regard the expression equation (40) as transcendental since each term is of the type $ae^{-ax}$) are two-fold. First, it does not seem to have been fully explored; presumably for steady-state problems the complications appear to be greater than the benefits afforded. The second reason for choosing this expression over a completely polynomial expression is the fact that the conditions to be satisfied at the edge of the boundary layer are automatically taken care of by the exponential-decay function. Thus more degrees of freedom can be used to specify the inner boundary condition and integral relations.

An additional advantage of the transcendental expression over a purely polynomial one is the former's capability to satisfy the asymptotic conditions at large $x$. In this region, the flow response is completely of the Stokes type, i.e., $u_0 \sim (1 - e^{-\sqrt{x}})$ and this profile cannot be approximated by purely polynomial expressions. The early work by Lighthill cited in the paper was based on a fourth-order polynomial and it appeared pointless for the present to merely duplicate what has been done in the past.

We believe that we have answered Professor Richardson's questions to a large extent. We consider our approach essential and realistic in light of the problem we wish to consider, and we certainly hope Professor Richardson will see our views.

To Professor Stenning we can only say that we have not taken up your suggested disturbance function in sufficient detail to formulate a firm response. This type of traveling wave oscillation is obviously of great technical importance and can be, we believe, treated by the method outlined in this paper.

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5 Numbers 18 and 19 in brackets designate Additional References at end of discussion.
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