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Algebraic Solution of the Horton-Izzard Turbulent Overland Flow Model of the Rising Hydrograph

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In the differential equation of the overland turbulent flow which was first postulated by Horton, Eq.(6), the value of c equals 5/3. For this value of c, the flow equation could not be integrated algebraically. Horton solved the equation for c = 2 and believed that his solution was valid for mixed flow. The flow equation with c = 5/3 is solved algebraically herein. It is shown elsewhere (Gill 1976) that the flow equation can indeed be integrated for any rational value of c.

Introduction

One of the earliest algebraic solutions of the unsteady turbulent overland flow was proposed by Horton (1938). The same problem was subsequently considered by Izzard (1946) who obtained a solution for the unsteady laminar overland flow. A description of overland flow in terms of general equations developed by Horton and later used by Izzard is now sometimes called the Horton- Izzard approach or model (Dooge 1973). The solution obtained by Horton was regarded to be valid for »mixed« (a transitional regime between laminar and fully turbulent flows) flow because an algebraic solution to the supposedly fully turbulent flow equation could not be obtained at that time. Such a solution is presented herein.

Subsequent to the approach proposed by Horton, a kinematic wave solution was proposed by Henderson and Wooding (1964). The Horton approach and the kinematic wave postulation are both somewhat oversimplified solutions. Woolhiser and Liggett (1967) considered the same problem more rigorously and obtained solutions by characteristics which embody features of both the Horton and the kinematic wave solutions. Inspite of the oversimplified nature of the Horton approach, Horton's solution has widely been used in hydrologic design of airport drainage with satisfactory results.

Description of the Horton-Izzard Model

The Horton- Izzard model has been adequately described by Dooge (1973). A brief outline is given herein essentially on the same lines as described by Dooge. The overland flow model is schematically shown in Fig. 1. A rainfall of intensity r per unit area is occurring over a catchment of length L_0. The rainfall intensity is assumed to be constant in space and time. It was noted by hydrologists that a simple power relationship existed between the equilibrium discharge at the downstream end and the corresponding surface storage over the catchment. This observation was a result of analysis of experimental data from a number of catchments. In mathematical terms,

$$q_e = a S_e^c \tag{1}$$

where q_e is the discharge at the downstream end at equilibrium condition, S_e is the corresponding storage and a and c are empirical coefficients. Equilibrium condition is defined to be the one at which q_e equals rL_0. Although Eq.(1) was derived from data

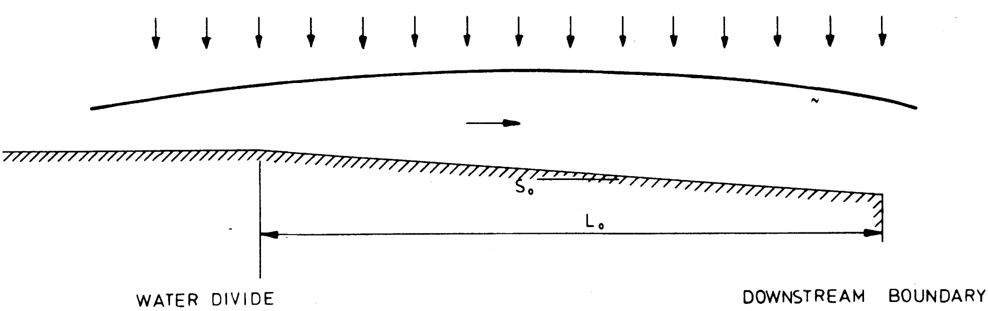

Fig. 1. Schematic diagram of overland flow.

corresponding to equilibrium conditions, Horton assumed that such a relationship holds at all stages of flow with the result that

$$q = a\,S^{c} \tag{2}$$

where q and S are time variant discharge, at the downstream end, and storage. This assumed relationship between outflow and storage plays a pivotal role in hydrology. The theory of unit hydrograph is based on this assumption. In the theory of linear unit hydrograph c equals 1 and the flow is routed through a number of linear channels and reservoirs. Although the assumption embodied in Eq(2) may be regarded somewhat severe from the viewpoint of hydraulics which seeks a fuller treatment of the problem in terms of the fundamental dynamic equation and associated initial and boundary conditions, it is not so severe in the hydrological sense in view of the observations stated above. According to equation of continuity,

$$rL_o - q = \frac{dS}{dt} \tag{3}$$

which can now be written as

$$q_e - a\,S^{c} = \frac{dS}{dt} \tag{4}$$

The solution of Eq.(4) is given by

$$t = \frac{1}{a\,S_e^{(c-1)}} \int \frac{d(\frac{S}{S_e})}{1-(\frac{S}{S_e})^{c}} \tag{5}$$

or in terms of q and q_e as

$$t = \frac{1}{a^{1/c}\,(q_e)^{(c-1)/c}} \int \frac{d(\frac{q}{q_e})^{1/c}}{1-\frac{q}{q_e}} \tag{6}$$

According to Dooge (1973), Eq.(6) can be solved analytically for values of c= 1, 2, 3, or 4, and also for ratios of these values, that is, for c= 3/2 or 4/3. Horton solved the equation of the rising hydrograph for c= 2 and believed that the flow for this value of c is mixed turbulent (transitional between laminar, c= 3, and fully turbulent, c= 5/3). Izzard obtained the solution for c= 3. It is shown herein that Eq.(6) can in fact be solved for any rational value of c and a solution for fully turbulent flow is given.

Solution for Fully Turbulent Flow

Eq.(6) is very much similar to the gradually varied flow function developed by Bakhmeteff (1932) and later by Chow (1959). The gradually varied flow function has been integrated by Chow and the results are given in the form of comprehensive tables for various values of c. It was recently shown by the author (Gill 1976) that the gradually varied flow function can indeed be integrated in closed form for any practical and rational value of c. Such solutions are generally lengthy depending on the value of c. For fully turbulent flow, that is when $c = 5/3$, Eq.(6) integrates to

$$a^{\frac{3}{5}} q_e^{\frac{2}{5}} t = -0.60 \ln[1-(\frac{q}{q_e})^{\frac{1}{5}}] + 0.48541 \ln[(\frac{q}{q_e})^{\frac{2}{5}} - 0.61804(\frac{q}{q_e})^{\frac{1}{5}}+1]$$
$$- 0.18541 \ln[(\frac{q}{q_e})^{\frac{2}{5}} + 1.61804(\frac{q}{q_e})^{\frac{1}{5}} + 1]$$
$$- 0.70534 \tan^{-1}[\frac{(\frac{q}{q_e})^{\frac{1}{5}} - 0.30902}{0.95106}]$$
$$+ 1.14126 \tan^{-1}[\frac{(\frac{q}{q_e})^{\frac{1}{5}} + 0.80902}{0.58779}] - 1.29720 \qquad (7)$$

The constant of integration was found to be equal to -1.2972 for $q/q_e = 0$ at $t = 0$.

Time to Equilibrium

In the Horton- Izzard approach equilibrium condition is reached only at infinite time. In order to have some practical measure, it was assumed by Izzard that the time at which the downstream flow attains a value of 0.97 of the equilibrium discharge is the equilibrium time. Izzard's solution of Eq.(6) for $c = 3$ is given by

$$a^{\frac{1}{3}} q_e^{\frac{2}{3}} t = \frac{1}{6} \ln[\frac{1+(\frac{q}{q_e})^{\frac{1}{3}}+(\frac{q}{q_e})^{\frac{2}{3}}}{\{1-(\frac{q}{q_e})^{\frac{1}{3}}\}^2}] + \frac{1}{\sqrt{3}} \tan^{-1}[\frac{2(\frac{q}{q_e})^{\frac{1}{3}}+1}{\sqrt{3}}] - 0.30230 \qquad (8)$$

Setting $t = t_e$ for $q/q_e = 0.97$, it can be shown that Eq.(8) gives

$$t_e = 2.017 \frac{S_e}{q_e} \tag{9}$$

which is Izzard's equation for equilibrium time. Using dimensionless parameters, Eq.(8) can now be written as

$$t_* = 0.49579 \left[\frac{1}{6} \ln\left\{ \frac{1 + q_*^{\frac{1}{3}} + q_*^{\frac{2}{3}}}{(1-q_*^{\frac{1}{3}})^2} \right\} + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1 + 2q_*^{\frac{1}{3}}}{\sqrt{3}} \right) - 0.30230 \right] \tag{10}$$

where $t_* = t/t_e$ and $q_* = q/q_e$. Horton's equation

$$\frac{q}{q_e} = \tanh^2(t\sqrt{a\, q_e}\,) \tag{11a}$$

or

$$t = \frac{1}{2\sqrt{a\, q_e}} \ln\left(\frac{1+\sqrt{\frac{q}{q_e}}}{1-\sqrt{\frac{q}{q_e}}} \right) \tag{11b}$$

can be written in dimensionless terms as

$$t_* = 0.20502 \ln\left(\frac{1+\sqrt{q_*}}{1-\sqrt{q_*}} \right) \tag{12}$$

Similarly Eq.(7) can be written in dimensionless terms as

$$\begin{aligned} t_* \equiv\ & 0.37147 \Big[0.48541 \ln(q_*^{\frac{2}{5}} - 0.61804\, q_*^{\frac{1}{5}} + 1) - 0.6 \ln(1-q_*^{\frac{1}{5}}) \\ & - 0.18541 \ln(q_*^{\frac{2}{5}} + 1.61804\, q_*^{\frac{1}{5}} + 1) - 0.70534 \tan^{-1}\left(\frac{q_*^{\frac{1}{5}} - 0.30902}{0.95106} \right) \\ & + 1.14126 \tan^{-1}\left(\frac{q_*^{\frac{1}{5}} + 0.80902}{0.58779} \right) - 1.27920 \Big] \end{aligned} \tag{13}$$

Equations for equilibrium time for Horton's mixed flow and fully turbulent flow are obtained as follows.

$$t_e = 2{:}439 \frac{S_e}{q_e} \quad \text{mixed flow} \tag{14}$$

$$t_e = 2.692 \frac{S_e}{q_e} \quad \text{fully turbulent flow} \tag{15}$$

Eqs.(10), (12), and (13) are plotted in Fig. 2 on t_* and q_* coordinates. These equations are compared with the earlier numerical results of Woolhiser and Liggett (1967). The results of Woolhiser and Liggett were obtained for F_O= 1 and k= 3 where $F_O = V_O/(gH_O)^{1/2}$, $k=S_O L_O/H_O F_O^2$, V_O and H_O= velocity and depth of uniform flow at the downstream end, g= acceleration due to gravity, S_O= bed slope, and L_O= the length of the catchment, Fig. 1. The agreement between Horton's equation, Eq.(12), and the results of Woolhiser and Liggett is close for t_* from 0 to 0.6 and there are noticeable deviations for larger values of t_*. The fully turbulent flow equation, Eq.(13), shows good agreement with the results of Woolhiser and Liggett for t_*= 0.7 and onwards. There is a need for testing these theoretical models with the experimental results. The

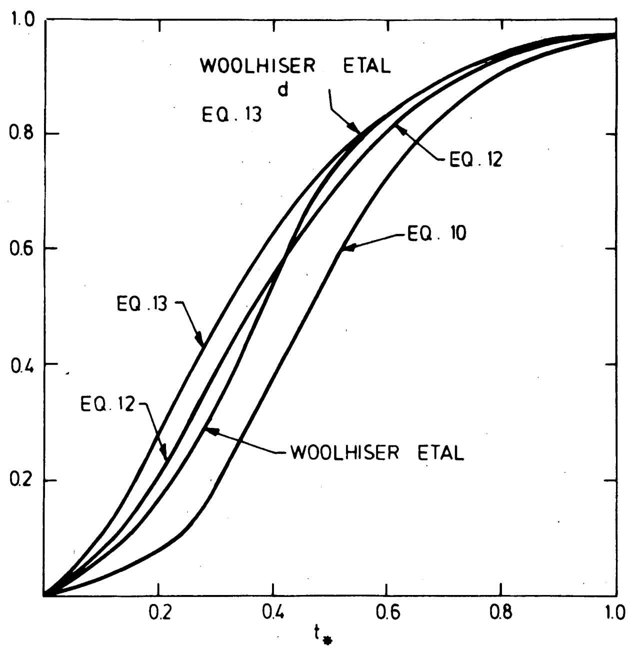

Fig. 2. Comparison of theoretical results of Eqs. (10), (12), and (13) with the earlier results of Woolhiser and Liggett.

testing could not be done for lack of suitable data. The theoretical results, Eqs.(12) and (13) are however regarded to be significant because they yield predictions which are close to other independent results e.g. the results of Woolhiser and Liggett. One fact may however be mentioned. Initially (for small values of t_*) the flow will presumably vary from laminar to transitional regime. Only after the lapse of sufficient time will the flow become fully turbulent. There is thus a need to integrate the results for laminar, mixed, and fully turbulent flows together in order to obtain satisfactory results. Such a task can be undertaken after comparing the theoretical results with the experimental data.

Conclusions

Solution of fully turbulent overland flow is obtained algebraically making assumptions similar to those of Horton. Equations for equilibrium time are given for Horton's transitional and the fully turbulent flows. Theoretical results are compared with the earlier numerical results of Woolhiser and Liggett.

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