above the sawtooth wave. Now $e(t)$ for $0 \leqslant t \leqslant t'$ and $t' \leqslant t \leqslant T_0$ is given by a broken line $Q_1Q_2\,Q_3$ with a slope $-kB(0 < t < t')$ and $+kB(t' < t < T_0)$.
From the figure,

$$R - c_0 - kBt' = -B_s + \frac{2B_s}{T_s}(T_s - T_0 + t') \qquad (46)$$

$$\begin{aligned} e(T_0) &= R - c_0 - kBt' + kB(T_0 - t') \\ &= R - c_0 + kBT_0 - 2kBt' \end{aligned} \qquad (47)$$

Eliminating t' from equations (46) and (47)

$$e(T_0) = \left(\frac{B_s - \Delta}{B_s + \Delta}\right)(R - c_0) + \frac{2B_s\Delta}{B_s + \Delta} - \frac{2(B_s - \Delta)}{B_s + \Delta}\left(\frac{T_0}{T_s}\right) \qquad (48)$$

where $\Delta = \frac{1}{2}kBT_s$. Thus $e(T_0)$ is derived from the initial value c_0 of the output $c(t)$.

DISCUSSION

Y. Sawaragi[3]

The linearization technique used by the authors is a very interesting one which is coupled with PWM technique. The writer is not sure but a more sound mathematical background concerning the usefulness and the limit of applicability is necessary for authorizing the technique. The problem considered by the authors may be analyzed by using the dual input describing function method. Would they explain their opinion about this point?

PWM feedback control systems are, to the writer's knowledge, analyzed and synthesized as sampled-data control systems by the use of z-transformation. The writer would like to know the authors' viewpoint about the z-transformation approach to the problem treated here.

B. P. Bhattacharya[4]

This paper is an extension and generalization of Dr. Oldenburger's well-known method of signal-stabilization. As a departure from earlier works, the authors use here the output of a saw tooth generator and a relay in a manner such that the original error signal is modified into a pulse width modulated train after passage through the nonlinearity. It is interesting to note that substantial improvement of the transient response of a nonlinear system is achieved in this manner. It would be most desirable to have these results extended for higher-order systems.

It appears that the authors' method may, as well, be adapted for *designing* a system containing an incidental nonlinearity. To begin with, it would be convenient to set an "equivalent" linear performance-standard (on the basis of transient response) for the nonlinear system. Then, by following a reverse process, one might use the data presented in Table 1 of the paper to ascertain what amplitude of the saw tooth wave should be used in order to have the "equivalent" gain correspond to that required of the linear system. Future research in this direction may be rewarding.

A further application of the authors' method appears to be in those cases where a feedback system incorporates a regular functional nonlinearity. Examples of such nonlinearities arise, for example, in feedback systems employing synchros as error detectors. This is an important class of practical problems. The averaging technique used in the paper may be directly applied to evaluate the equivalent gain. The discusser is at present experimenting along these lines and the results may be published at a later date.

A. F. D'Souza[5]

In this paper, Professors Oldenburger and Ikebe have developed an ingenious method of inserting an extra sawtooth signal and an extra relay nonlinearity to modify a nonlinear system and change it into a pulse width modulated system. Hence the performance of the original system is altered but it appears that in general there may not always be an improvement in the performance. For example, with reference to the step response shown in Fig. 9, the output of the original system approaches the input and then goes into limit cycles of infinitesimal amplitude about the steady state i.e., the relay chatters. The output of the pulse width modulated system also has periodic oscillations about its steady state. However, the original nonlinear system has a much smaller rise time than the pulse width modulated system. It also appears that further investigation from stability considerations is needed in order to determine the conditions to be met such that a nonlinear system can always be correctly approximated by its equivalent linearized system. It is hoped that this paper stimulates further research in the interesting techniques proposed by the authors for the design of nonlinear systems.

A. K. Mahalanabis[6]

The paper basically employs pulse width modulation (PWM) for modifying nonlinear effects. It is shown that the average value of the modulated system output can be approximated as the output of a linear system. The derivation of this linear system for some common nonlinearities is also outlined.

Though PWM has been fairly widely employed in the control field as a means of obtaining sampled-data operation (see for example footnote 7 and other references listed in it), this seems to be the first paper to call attention to the facts stated above. Obviously many details (e.g., error between the actual PWM system output and the linearized system output, constraints on the sawtooth signal frequency, etc.) are yet to be looked into. Before that, it might be instructive to consider the relative costs of this and other linearizing schemes, e.g., use of either signals, and to justify the additional cost involved in the present case.

T. Nakada[8]

My question is concerned with future research.

Some nonlinear elements of actual control systems have the characteristics of a time dependent nonlinearity—for example, $\frac{k}{Ts + 1}$ followed by a time independent nonlinearity.

Is there any upper limit to the frequency of the extra saw tooth wave for linearizing the above mentioned nonlinear element?

S. Ochiai[9]

This paper has brought a new valuable approach for the study of signal stabilization. In previous works in this field only the extra signal was inserted into feedback control systems with nonlinear elements to improve the performance of the systems.

[3] Professor, Department of Applied Mathematics and Physics, Kyoto University, Kyoto, Japan.

[4] Professor, Electrical Engineering Department, Bengal Engineering College, W. Bengal, India.

[5] Assistant Professor, Department of Mechanical and Aerospace Engineering, Illinois Institute of Technology, Chicago, Ill. Assoc Mem. ASME.

[6] Department of Electronics and Communication Engineering, University of Roorkee, Roorkee, U. P., India.

[7] F. R. Delfeld and G. J. Murphy, "Analysis of Pulse Width Modulated Control Systems," IRE *Trans. on Automatic Control*, AC-6, pp. 283–292 September, 1961.

[8] Professor, Research Laboratory of Precision Machinery and Electronics, Tokyo Institute of Technology, Tokyo, Japan.

[9] Research and Development Engineer, Fibers Technical Center, Fiber Industries, Inc., Charlotte, N. C. Assoc. Mem. ASME.

The effect of the extra signal was that in general it reduced the gain of the nonlinear elements in the sense of time average. There, obtaining the magnitude of the extra signal which yielded the required system performance was the primary concern.

By using their basic concepts of the time independent nonlinearity, the authors of this paper have succeeded in using the pulse width modulation to linearize time independent nonlinear elements in the sense of time average. This is the first use of the PWM method in the signal stabilization problems and probably so in the automatic control field. The use of the authors' concepts of the nonlinearity along with the PWM method should certainly be extended to problems which were not covered in this paper. For example, the authors' method of treating the closed loop systems with lower order linear elements can be used for the systems with higher order elements, although the complexity of mathematical treatments may increase to some extent.

Authors' Closure

The points brought out by Professor Sawaragi are well taken and should be of interest to all engineers who use the approach of this paper. The dual input describing function was tried by the authors. They found that overwhelming mathematical complications arise for the systems of this paper. These difficulties melt away when the concepts of "equivalent gain," "equivalent nonlinearity," "representative output wave," and "pseudo describing function" introduced by R. Oldenburger (see the paper of reference [2] with Boyer) are employed. The accuracy of the simplified approach of R. Oldenburger is adequate for engineering purposes. The dual input describing function approach is more accurate but its value is questionable when the mathematical complications are considered.

The systems treated in this paper are nonlinear, while the sampling times t_{n1} and t_{n2} are variable. The Z-transform is valid for linear systems and constant sampling times. It therefore does not apply to the systems of this paper.

The authors agree with Professor Bhattacharya and Dr. Ochiai that the theory developed here for low order systems should be extended to higher order systems. This step is necessary if Bhattacharya's suggestion of employing the method for system design is to be implemented for general engineering use. Publication of his work on synchros should be a valuable contribution to the literature.

Professor D'Souza is correct in pointing out that in the example of Fig. 9 oscillations are not completely removed. The saw-tooth extra signal will result in a steady oscillation, but this can be made small compared to the amplitude of the limit cycle that arises because of the instability of the original system. The authors agree that much more remains to be done to determine when a nonlinear system can be correctly approximated by an equivalent linear system.

Professor Mahalanabis emphasizes the importance of several details, such as constraints, that must be looked into before the theory can be considered to be complete. The authors agree with this point of view. Also an analysis of the cost in using PWM should be made. Ultimately, the value of the theory will depend on the amount that it is successfully employed in building hardware.

Professor Nakada's point is well taken. There will be upper and lower limits on the frequency of the extra saw-tooth wave, depending on the nature of the components of the given system. The lower limit is established by the fact that the frequency f_s of the saw-tooth wave should be at least $10/T$ for the system lag T. Because of physical considerations there is always an upper limit to f_s. Thus if f_s is high enough it is difficult to produce a pure saw-tooth wave of sufficient amplitude B_s.