Note on the Integral Representation of Commutators

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The difference between Dyson’s integral representation and Deser et al.’s one of commutators is discussed. It is shown that we cannot determine the vanishing region of the weight function for the latter representation from the spectral condition. Hence, we cannot prove the forward dispersion relation for nucleon-nucleon scattering in Deser et al.’s sense from the integral representation alone.

§1

Recently, Deser, Gilbert and Sudarshan (DGS) have found an integral representation as a function of invariants for the following quantity,*

\[ f(q) = \int dx \exp iqx \langle \alpha | [A(x/2), B(-x/2)] | \beta \rangle. \]

(1)

Here both \( |\alpha\rangle \) and \( |\beta\rangle \) are statevectors which describe the states of one particle with the same energy-momentum \( p \), or one of them is the state of one particle with the energy-momentum \( p \) and the other is the vacuum state. The same representation has been found by Ida for the Bethe-Salpeter amplitude under the stability condition.

This representation makes use of the information derivable from the Lorentz invariance, the local commutativity of the operators and the spectral condition. From the expression (1) and the spectral condition, we see that \( f(q) = 0 \) in some region in \( q \)-space. We denote such a region by \( R \).

The representation they have found is

\[ f(q) = \int d\mu \ d\beta \ H(\mu, \beta) \epsilon (pq + \beta \beta^*) \delta (q^2 + 2pq\beta - \mu), \]

(2)

where \( \epsilon (x) \) is the sign function and \( H(\mu, \beta) \) is a weight function. Let us consider the straight line

\[ q^2 + 2pq\beta - \mu = 0 \]

(3)

in the \((q^2, pq)\) plane for fixed \( \mu \) and \( \beta \). We call such a straight line “admissible” if this line does not pass the region in the \((q^2, pq)\) plane which corresponds to \( R \). This determines an admissible maximum region \( S \) in \((\mu, \beta)\) plane such that

* In the first paper of DGS, the arguments of operators \( A \) and \( B \) are \( x \) and zero.
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every point in \( S \) gives rise to an admissible line. The main result due to DGS is that \( H(\mu, \beta) = 0 \) at the outside of \( S \).

Using this representation, DGS have proved that there exists an analytic function which coincides with the forward nucleon-nucleon scattering amplitude on real axis of the energy larger than the rest mass of the nucleon and which is regular in the upper half plane of the energy of the nucleon in the laboratory system. However, so far as we use Dyson’s representation instead of DGS’s one, we never come to the same conclusion in spite of the use of the same information, that is, the Lorentz invariance, the local commutativity, the spectral condition and the number of the constant momenta (in this case only one \( p \)).

In order to make clear the above stated contrariety, we shall discuss the difference between the two kinds of representations. The conclusion of this paper is that the result of DGS \( (H(\mu, \beta) = 0 \) at the outside of \( S ) \) is incorrect. At the present stage of our knowledge, it is impossible to incorporate the spectral conditions into the integral representation (2). Hence, we cannot prove the forward dispersion relation for nucleon-nucleon scattering in DGS’s sense from the integral representation (2) alone.

\[ \text{§2} \]

Dyson’s representation is

\[ f(q) = \int_0^\infty d\kappa \int d\mathbf{u} \int d\mathbf{u}_0 \epsilon(q_0 - u_0) \delta[(q_0 - u_0)^2 - (q - \mathbf{u})^2 - \kappa^2] \varphi(u, u_0, \kappa^2) \]  \( (4) \)

instead of (2) for the matrix element of the commutator between arbitrary states. The information on the statevector is included in the invariant weight function \( \varphi(u, u_0, \kappa^2) \). Let us consider the hyperboloid

\[ (q_0 - u_0)^2 - (q - \mathbf{u})^2 - \kappa^2 = 0 \]  \( (5) \)

in \( q \)-space for fixed \( u, u_0 \) and \( \kappa^2 \). We call such a hyperboloid admissible if both sheets do not invade the region \( R \). This determines an admissible maximum region \( S \) in \( (\kappa^2, \mathbf{u}, u_0) \)-space such that every point in \( S \) gives rise to an admissible hyperboloid. The main result due to Dyson is that \( \varphi(u, u_0, \kappa^2) = 0 \) at the outside of \( S \).

In what follows, we shall investigate the matrix element between special states as discussed in DGS’s representation and without loss of generality we choose the special Lorentz frame where the space components of the four-vector \( p \) are zero. If we put

\[ \varphi(u, u_0, \kappa^2) = \delta(u)\psi(u_0, \kappa^2), \]  \( (6) \)

we can easily obtain DGS’s representation for the same special frame. The weight function \( H(\mu, \beta) \) is expressible in terms of \( \psi(u_0, \kappa^2) \). DGS’s representation is a special case of Dyson’s one. Because the representation (2) with the weight func-
tion $H(\mu, \beta)$ and Dyson's representation with the specialization (6) of the weight function (namely the representation with the weight function $\phi(u_0, \kappa^2)$) are equivalent, we shall use the more convenient one between the two, as the case may be, for the discussion of DGS's representation.

In configuration space, DGS's one is

$$f(x) = \int_0^\infty dx^2 A(x, \kappa) \tilde{\phi}(x_0, \kappa^2). \quad (7)$$

Dyson's one is

$$\tilde{f}(x) = \int_0^\infty dx^2 A(x, \kappa) \tilde{\phi}(x, x_0, \kappa^2). \quad (8)$$

Here $A(x, \kappa)$ is usual invariant commutator function of the free field operator with mass $\kappa$. $f(x)$ depends only on $x^2$ and $x_0 = px/(p^2)^{1/2}$, and the integral representation (7) is always possible, because this is essentially the Fourier Bessel transform. Therefore, if we can conclude that for every $f(q)$ which vanishes in $R$ it is possible to put $\phi(u_0, \kappa^2) = 0$ at the outside of $S$, DGS's representation is correct. Thus we shall discuss the possibility of putting $\phi(u_0, \kappa^2) = 0$ at the outside of $S$.

§ 3

The first proof by DGS on the possibility of putting $H(\mu, \beta) = 0$ is not precise. It should be noted that $\tilde{\phi}(x_0, \kappa^2)$ is not uniquely determined because of arbitrariness of $\tilde{f}(x)$ in the unphysical region $x^2 = x_0^2 - x_2 > x_0^2$. Then, even if it is possible to put $\phi(u_0, \kappa^2) = 0$ at the outside of $S$, $\phi(u_0, \kappa^2) = 0$ is not a necessary condition for $f(q)$ to vanish in $R$. The problem we must solve is whether a set of the functions, whose weight function $\phi(u_0, \kappa^2)$ or $H(\mu, \beta)$ vanishes at the outside of $S$, covers all the functions which vanish in $R$ and whose Fourier transforms are zero in $x^2 < 0$.

An answer to this problem has been given in the appendix by DGS. Their proof is made in two stages: First they assert that if $f(q) = 0$ in $R'$ (which is contained in $R$), then $H(\mu, \beta) = 0$ at the outside of the intersection of $S'$ and $S''$, where the region $R'$ is defined by $(q - p)^2 < 0$ and $q^2 < 0$, and $S'$ and $S''$ are defined by $0 \geq \beta \geq -1$ and $\mu \geq -\beta^2 p^4$ respectively. No assumption concerning the mass spectrum is involved in the requirement of the vanishing of $f(q)$ in $R'$. In the next stage they have incorporated the mass spectral condition by using a lemma of Jost and Lehmann. If the first stage of their proof is correct, we can come to the conclusion that $H(\mu, \beta) = 0$ at the outside of $S$. Now, in the first stage, the assertion that one can set $H(\mu, \beta) = 0$ at the outside of $S''$ is obviously unquestionable in view of the relation $\kappa^2 = \mu + \beta^2 p^4$ and Eq. (7). Hence the important result in the first stage is that, even though $H(\mu, \beta) = 0$ at the outside of $S'$, a
set of \( f(q) \) can cover all the functions which vanish in \( R' \) and their Fourier transform is zero for \( x^a < 0 \).

However, this conclusion is not correct. In the proof of the first stage, they have used the following identity given by Dyson,\(^7\)

\[
\int du \Delta(q - \bar{u}, m) \delta(u^2 - b^2) = \pi \int dv I_0(m[b^2 - v^2]^{1/2}) \Delta(q - \bar{v}, m), \tag{9}
\]

where \( \bar{u} \) is a four-vector with the vanishing time component and the space components \( u \), and \( \bar{v} \) is a four-vector with the vanishing space components and the time component \( v \). Now we consider the following integral,

\[
\int dm \int dv \chi(m) I_0(m[b^2 - v^2]^{1/2}) \Delta(q - \bar{v}, m) \tag{10}
\]

where \( \chi(m) \) is an arbitrary function. We cannot invert the order of integration in (10), since the asymptotic behaviour of \( I_0(x) \) is exponential. In the proof of the first stage, there appears the integral like (10). The range \( 0 \geq \beta \geq -1 \) (\( S' \)) of \( \beta \) corresponds to the range \( b \geq v \geq -b \) of \( v \) in (10). We cannot come to the conclusion of the first stage without requiring the inversion of the order of the integration like (10). Therefore the discussion in the appendix\(^1\) is incorrect.

By using the integral representation of the double commutator,\(^7\) Ida\(^9\) has arrived at the same conclusion as that of DGS. However, as was revised by Dyson himself,\(^8\) the integral representation of the double commutator is incorrect by the same reason as that of the defect of the proof of the first stage in DGS.

In conclusion, the integral representation (2) is correct, however, we cannot put \( H(\mu, \beta) = 0 \) at the outside of \( S \). At the present stage of our knowledge, it is impossible to incorporate the spectral conditions into the integral representation (2). Hence, we cannot prove the forward dispersion relation for nucleon-nucleon scattering in DGS's sense from the integral representation (2) alone.

After completion of this work, we find the similar work.\(^9\) However, the assertion and contents seem to be somewhat different from ours. Recently, Nakanishi\(^10\) has pointed out that for the vertex function \( H(\mu, \beta) = 0 \) at the outside of \( S \) is valid in every order of perturbation theory.

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References

8) F. J. Dyson, Phys. Rev. 117 (1960), 1616.
10) N. Nakanishi, preprint.