Pion-Pion Interaction and Nuclear Forces

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Nuclear potentials arising from the pion-pion resonance in the state \( I=J=1 \) are calculated. Only the part proportional to \( r \) is considered. The resonant state is described both in the \( \rho \)-meson formalism and in the chain approximation method. The potentials are calculated explicitly in configuration space, at first in the \( \rho \)-meson formalism with its mass 600 Mev and the strength of coupling which corresponds to the width \( \sim 60 \) Mev. The potentials are found to be of the nature of the two-pion exchange potentials. In particular there appear a strong attractive \( L-S \) potential and a repulsive central potential in the \( ^3O \)-state, a strong repulsive tensor potential in the \( ^3E \)-state, and a repulsive \( L \)-dependent potential in the \( ^1E \)-state. Among these the tensor potential in the \( ^3E \)-state and the central potential in the \( ^3O \)-state can be considered to be too strong and violate some of the facts which have already been established. Therefore a careful examination of the results will offer a test for or against the assumption of the pion-pion resonance. The chain approximation method gives similar results. The electromagnetic form factors of the nucleon are reexamined and a remark is made concerning future investigation.

§ 1. Introduction

Since the work by Frazer and Fulco, who calculated the electromagnetic form factors of the nucleon, the resonant state of pion-pion scattering in the state \( I=J=1 \) has been recognized to be important. G. Takeda and his collaborators proposed an approximate method in which the resonant state is replaced by an actual particle with \( I=J=1 \), which they called a \( \rho \)-meson. This replacement was achieved by assuming the width \( I' \) of the resonance to be infinitesimal, or more precisely, retaining only the first order term (the narrow width approximation). They applied this method to pion-nucleon scattering, and found that the second and the third resonances observed around 700 Mev can be reproduced as a result of the pion-pion resonance. The mass of \( \rho \)-meson was set equal to the resonance energy, which they assumed to be 600 Mev according to Frazer and Fulco:

\[ m=600 \text{ Mev}=4.29.* \]

The width of the resonance is, on the other hand, related to the strength of the coupling between \( \rho \)-meson and pions. For later reference we shall quote their result on the interaction Hamiltonian with somewhat different notation:

* In the following we shall use units in which the pion mass is unity.
While Takeda et al. started from the assumption of the resonance in pion-pion scattering, Y. Miyamoto made an attempt to reproduce that resonance from more fundamental interaction between pions. He examined the calculation of pion-pion scattering by Chew and Mandelstam and found that their formalism is equivalent to the so-called "chain approximation" method in perturbation theory, as far as the contribution from the "right-hand cut" is concerned. He introduced the interaction Hamiltonian between pions given by
\[ H = -4\pi a(\phi_i T_{ij}^+ \phi_j)^2. \] (1.2)

This Hamiltonian is responsible for pion-pion scattering for p-waves as well as s-waves. The coupling constant \( a \) can be identified as the p-wave scattering length. The interaction Hamiltonian (1.2) need not be considered as the truly fundamental interaction. Rather it will be more reasonable to consider it as an effective interaction resulting from, for example, the virtual formation and reannihilation of a nucleon pair. Therefore we should reasonably cut off the integration of the virtual momentum by an appropriate value, which we shall assume to be the order of a nucleon mass. He found that the resonance determined by Frazer and Fulco, and used by Takeda et al., can be reproduced with 
\[ a \approx 0.2, \text{ for the cutoff } 2M. \]

The above authors showed that the pion-pion resonance in the state \( I=J=1 \), if it exists, has an appreciable effect on the electromagnetic structure of the nucleon and pion-nucleon scattering. If this is true, we should expect that the resonance will affect also nucleon-nucleon scattering. In particular this is the case with the nuclear forces in the region II (0.7 < \( r \) < 1.5). This is because the two-pion part (the pion-current part), which is considered to dominate the isovector part of the electromagnetic form factors of the nucleon, has a close correspondence to the two-pion exchange potential (TPEP) in the nuclear force. We can hardly expect that the resonance affects the electromagnetic structure appreciably while affecting the nuclear potential in the region II little.

Furthermore the two-pion exchange potentials calculated thus far do not give satisfactory agreement with observation. For example, these potentials give too weak an \( L-S \) force to account for the experiment on the depolarization in the proton-proton scattering at the energy around 150 Mev.

Miyamoto made a tentative estimation and found that the resonant state gives a very strong \( L-S \) force in the desired direction. But some care will be needed in drawing conclusions, since the resonance is expected to affect the other types.

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* \( \phi_i \) and \( \phi^{+}_k \) describe a pion of isospin index \( i \) and a \( \rho \)-meson of isospin index \( k \), respectively.
** \( M \) is the nucleon mass.
of potentials, e.g. central, tensor, and so on, which also should be compared with experiment. From the analysis which will be presented in the following, we find that the above expectation is true and some of potentials may be affected too much and violate results which have already been established. In this situation a careful examination of the results will offer a test for or against the assumption of a resonance in pion-pion scattering.

There are, of course, some uncertainties in the theoretical treatment of the resonance, which will be considered later. The results we have obtained, however, show some qualitative features which are characteristic of the assumption of the pion-pion resonance. Therefore, our results are expected to offer a basis for a further investigation of the problems both of the pion-pion interaction and of nuclear forces.

In this paper we shall confine ourselves to the calculation of that part of the potential, which is proportional to $\tau \tau$, leaving the consideration of the part, which is proportional to 1 in isospace, to a forthcoming paper. Also we shall not make a detailed comparison with the experiments, leaving that to another article.

In § 2, the outline of the calculation is described, in § 3 the explicit form of the potentials are given, in § 4 the results are discussed briefly. In § 5 the electromagnetic form factors are reexamined by the same method applied to the calculation of the potentials. In § 6 the results are summarized and a remark is made concerning future investigation.

§ 2. Outline of the calculation

The kinematics of the non-static potential in momentum representation were worked out by J. Goto and S. Machida, which we reproduce here for the sake of completeness. The calculation is made in the center of mass system of the two nucleons, and the notation is indicated in Fig. 1. The matrix element is decomposed as follows:

\[
\begin{align*}
\langle A \rangle & = \langle B \rangle + \langle C \rangle
\end{align*}
\]

Fig. 1.

\[
\begin{align*}
k & = p - p' \\
q & = (1/2) (p + p')
\end{align*}
\]

Fig. 2.
\[ V(k^2, q^2, kq) = V^1 + \tau\tau V^*, \]
\[ V^\lambda = V^\lambda_0 + i(S\cdot k \times q)V^\lambda_1 + (k\sigma)(k\sigma)V^\lambda_2 + (q\sigma)(q\sigma)V^\lambda_3 + (k \times q \cdot \sigma)(k \times q \cdot \sigma)V^\lambda_4 + (\sigma\sigma)[V^\lambda_5 + (k \times q)^2 V^\lambda_6], \]
\( \lambda = 1, \tau. \) \hfill (2.1)

We shall consider the graphs in Fig. 2. Here the graphs (A) and (B) represent the contributions arising from the resonance in the state \( I = J = 1 \), while the graph (C) comes from an interaction of the form \( \delta \cdot \phi \).

We shall assign the factor

\[ T_\mu (k^2) = \delta_\mu \cdot T(k^2), \]

\[ T(k^2) = \begin{cases} \frac{12\pi a}{D(k^2)}, & \text{in the chain approximation method,} \\ \frac{\hbar^2}{k^2 + m^2}, & \text{in the \( \rho \)-meson formalism,} \end{cases} \] \hfill (2.2)

to a heavy line in Fig. 2, which stands for the resonant state. Here \( D(k^2) \) is given by

\[ D(k^2) = 1 + \frac{2a}{\pi} \left( 1 + \frac{k^2}{4} \right) \left[ \log 2A - \sqrt{1 + \frac{4}{k^2}} \log \left( \sqrt{1 + \frac{k^2}{4}} + \sqrt{\frac{k^2}{4}} \right) \right], \] \hfill (2.3)

where \( A \) is the cutoff momentum.

It should be remarked that the method, here designated as the \( \rho \)-meson formalism, is not the one based on the vector meson theory in the rigorous meaning of a field theory, because the propagator of a vector meson is, strictly speaking, of the form

\[ \frac{\delta_\mu + k_\mu k_\nu / m^2}{k^2 + m^2}. \] \hfill (2.4)

We are considering the \( \rho \)-meson as a phenomenological substitute rather than as a true elementary particle. It should be remembered that we are able to determine only the first term \( (\delta_\mu) \) of the numerator in (2.4), leaving the second term undetermined, as long as we introduce the \( \rho \)-meson as the pole term in the scattering amplitude of the real pions. This is analogous to the situation that we encounter in the introduction of the isobaric particle for the 3-3 resonance.\(^9\) We might expect that the second term in the numerator of (2.4) will be cancelled by some other terms, at least asymptotically, if the resonance can be considered to result from interactions of the first kind, or renormalizable. From these considerations we have chosen, for convenience, the form in (2.2).

We can now construct the effective Hamiltonian for pion-pion scattering. Corresponding to the pion-pion scattering part of the graphs (A), (B), and (C) in Fig. 2, we have
\[
\langle q_1, q_2 | H_A | k_1 k_2 \rangle = -2P_1 (k_1 - k_2) \cdot (q_1 - q_2) T(k^2),
\]
\[
\langle q_1, q_2 | H_B | k_1 k_2 \rangle = P_2 \{(k_1 + q_1) \cdot (k_2 + q_2) T[(k_1 - q_1)^2] - (k_1 + q_2) \cdot (k_2 + q_2) T[(k_1 - q_2)^2] + (2P_0 - P_2) \{(k_1 + q_1) \cdot (k_2 + q_2) T[(k_1 - q_1)^2] + (k_1 + q_2) \cdot (k_2 + q_2) T[(k_1 - q_2)^2]\},
\]
\[
\langle q_1, q_2 | H_C | k_1 k_2 \rangle = 20(2P_0 - P_2) \lambda_r,
\]
where \( P \) is the projection operator for the state of the two pions with isospin \( I \).

The potential function is given by the two-nucleon expectation value of these effective Hamiltonians:
\[
V = V^A + V^B + V^C,
\]
\[
V^A = \langle p' | -p' \mid H_A \mid p - p \rangle,
\]
\[
V^B = \langle p' | -p' \mid H_B \mid p - p \rangle,
\]
\[
V^C = \langle p' | -p' \mid H_C \mid p - p \rangle.
\]

\( V^A, V^B \) and \( V^C \) correspond to the graphs (A), (B) and (C) in Fig. 2.

It is easily seen that \( V^A \) can be written as
\[
V^A = -\langle p' | \phi_i T^{k}_{ij} \partial_{\mu} \phi_j | p \rangle \cdot T(k^2) \langle -p' | \phi_m T^{k}_{mn} \partial_{\mu} \phi_n | -p \rangle,
\]
where the pion fields are evaluated at the origin of the coordinate system. Now we observe that the matrix elements \( \langle p' | \phi_i T^{k}_{ij} \partial_{\mu} \phi_j | p \rangle \) and \( \langle -p' | \phi_m T^{k}_{mn} \partial_{\mu} \phi_n | -p \rangle \) in (2.7) are of a form similar to the nucleon expectation value of the pion electric current. Indeed the electric current of a pion is given by
\[
j_\mu = -e\phi_i T^{k}_{ij} \partial_{\mu} \phi_j,
\]
the nucleon expectation value of which is given by
\[
\langle p' | j_\mu | p \rangle = \bar{u}_p \tau_5 [i \gamma_\mu G_1 (k^2) + \sigma_{\mu \nu} k_\nu G_2 (k^2)] u_p,
\]
where \( G_1(k^2) \) and \( G_2(k^2) \) are the two-pion parts (the pion-current parts) of the nucleon electromagnetic form factor. These \( G \)'s, of course, contribute to the observed isovector part of the nucleon electromagnetic form factors. By comparing (2.7), (2.8) and (2.9), we have obviously
\[
\langle p' | \phi_i T^{k}_{ij} \partial_{\mu} \phi_j | p \rangle = -(1/e) \bar{u}_p \tau_5 [i \gamma_\mu G_1 + \sigma_{\mu \nu} k_\nu G_2] u_p,
\]
from charge independence considerations. Thus, we have the simple expression for \( V^A \):
\[
V^{14} = 0,
\]
\[
V^{rA} = -(1/e^2) \bar{u}_p (i \gamma_\mu G_1 + \sigma_{\mu \nu} k_\nu G_2) u_p T(k^2) \bar{u}_{-p'} (i \gamma_\mu G_1 - \sigma_{\mu \nu} k_\nu G_2) u_{-p'}.
\]
Here it should be noted that \( G_1(k^2) \) and \( G_2(k^2) \) are the form factors with the pion-pion resonance neglected; otherwise the resonant state would be taken into account twice.

We cannot apply a similar method to \( V^B \), because the momentum appearing
in the denominator is not a constant \((k^2)\) but must be integrated over. A rigorous calculation of the graph (B) in Fig. 2 is somewhat cumbersome due to the so-called overlapping integral. In view of the approximate character of the whole problem we shall apply a simplifying method and quickly consider the qualitative or semi-quantitative features of the results. We thus make the point approximation in which the heavy line in Fig. 2 (B) is shrunk into a point, or \(T(k^2)\) is replaced by \(T(0)\). Then, we have

\[
\langle q_1 q_2 | H_B | k_1 k_2 \rangle = - P_x (k_1 - k_2) \cdot (q_1 - q_2) T(0) \\
+ (2P_y - P_z) (3k_1^2 - k_2^2 - q_1^2 - q_2^2) T(0),
\]

and

\[
V^{1B} = \left( \frac{3}{8} \right) T(0) \bar{u}_p \cdot u_p \cdot \bar{u}_{-\rho'} u_{-\rho} \left[ (3k^2 + 4) G_3(k^2) - 4C \right] G_3(k^2),
\]

\[
V^{1B} = - \left( \frac{1}{2} \right) \left( \frac{1}{e^2} \right) \bar{u}_{\rho'} \cdot (i \gamma_x G_1 + \sigma_{\rho' \nu} k_{\nu} G_2) u_{\rho} T(0) \bar{u}_{-\rho'} (i \gamma_x G_1 - \sigma_{\rho' \nu} k_{\nu} G_2) u_{-\rho},
\]

where \(G_3(k^2)\) is a form factor for the scalar interaction analogous to the vector and tensor form factors \(G_1(k^2)\) and \(G_2(k^2)\) and permits a spectral representation of the form (2.16), while \(C\) is a constant independent of \(k^2\). We shall not discuss \(V^{1B}\) in this paper. We write down, however, the formula for \(V^{0}\) for the sake of completeness:

\[
V^{1C} = \left( \frac{15}{2} \right) \lambda \bar{u}_{\rho'} \cdot u_p \cdot \bar{u}_{-\rho'} u_{-\rho} \left[ G_3(k^2) \right]^p,
\]

\[
V^{1F} = 0.
\]

We see that \(V^{1B}\) is very similar to \(V^{1A}\). Indeed we would have

\[
V^{1A} = 2V^{1B},
\]

if we applied the point approximation also to \(V^{1A}\).

We have treated \(V^{A}\) and \(V^{B}\) in different ways. There is, however, some justification for this method. Consider the matrix elements corresponding to the graphs (A) and (B) in Fig. 2, as the functions of the square of the momentum transfer \(k^2\). We shall carry on the discussion, for the moment, in the \(\rho\)-meson formalism. \(V^{A}\) has a pole at \(k^2 = -m^2\) in addition to a cut along \(k^2 \leq -4\). This means that \(V^{A}(r)\), the fourier transform of \(V^{A}(k^2)\), has a term like \(e^{-br}/r\), in addition to the contribution from the cut. As we shall see later, the cut contribution can be approximated roughly \(e^{-br}/r\) with \(b \sim 4\). If we choose \(m \sim 4\), we can expect the pole contribution to make potential appreciably wider.

\(V^{B}\) has, on the contrary, no such pole; the contribution from the heavy line lies along the cut \(k^2 \leq -4\). Therefore the deviation from the above point approximation can be considered to appear only in the relatively inner region of the
configuration space.

Now we shall calculate $G_1(k^2)$ and $G_2(k^2)$ from the graphs in Fig. 3, where the heavy line represents the isobaric particle with $I=J=3/2$. These $G$'s were partially calculated by Chew et al.\textsuperscript{10} and by Federbush et al.\textsuperscript{11} We can show that their calculation in which the rescattering correction comes only from the 3-3 resonance with the narrow width approximation is equivalent to the perturbation calculation of the graphs in Fig. 3, in which the 3-3 isobar is replaced by a Rarita-Schwinger particle with the appropriate interaction.\textsuperscript{9} We assume the spectral representations for the $G$'s:

$$G_\alpha(k^2) = \frac{1}{\pi} \int_0^{4M} \frac{g_{\alpha}(k^2)}{k^2 + k'^2} \, dk'^2, \quad (\alpha = 1, 2).$$

In accordance with the discussion of Federbush et al. we have cut off the integration with respect to the square of the energy of the two-pion state by $(2M)^2$.

We have assumed a subtraction neither for $G_1$ nor for $G_2$, though Chew et al. and Federbush et al. assumed a $G_1$ with one subtraction. The reason is that we are calculating the contribution only from the graphs in Fig. 3; the subtraction terms correspond to the other kinds of graphs which represent the wave function renormalization of the nucleon, the nucleon current contribution, and so on. In the case of the electromagnetic form factor it is necessary to include all of these contributions for gauge invariance, while this is not the case in the present calculation. The introduction of a cutoff in (2.16) was also necessary to make $G_1$ finite.

The calculated net values and the mean square radius are shown in Table I.

<table>
<thead>
<tr>
<th></th>
<th>$G_0(0)/e$</th>
<th>$2MG_0(0)/e$</th>
<th>$r_1^2$</th>
<th>$r_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pion-current contribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Born approximation</td>
<td>0.66</td>
<td>0.87</td>
<td>0.19</td>
<td>0.22</td>
</tr>
<tr>
<td>3-3 isobar included</td>
<td>0.26</td>
<td>1.63</td>
<td>0.12</td>
<td>0.17</td>
</tr>
<tr>
<td>Observed isovector part</td>
<td>(0.50)</td>
<td>1.85</td>
<td>0.33</td>
<td>0.33</td>
</tr>
</tbody>
</table>

§ 3. Potentials

Substituting the normalized Dirac spinors for $\bar{u}$ and $u$ in (2.11) and (2.13), and comparing with (2.1), we obtain the results:

$$V_{t}^{A}(k^2) = T(k^2) \bar{\psi}_t(k^2),$$

$$V_{t}^{B}(k^2) = (1/2)T(0) \bar{\psi}_t(k^2), \quad (i=0, \ldots, 6)$$

where

$$e^2 \bar{\psi}_v(k^2) = G_2^2(k^2),$$

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\[
e^2 \mathcal{V}_1(k^2) = -\frac{2}{M} \left( \frac{3}{4} \frac{1}{M} G_1 + 2G_2 \right),
\]
\[
e^2 \mathcal{V}_2(k^2) = \frac{1}{4M^2} G_1^2 + \frac{1}{M} G_1 G_2 + G_2^2,
\]
\[
e^2 \mathcal{V}_3(k^2) = \frac{1}{2M^2} e^2 \mathcal{V}_4(k^2),
\]
\[
e^2 \mathcal{V}_4(k^2) = -\frac{1}{M^2} \left( \frac{3}{16} \frac{1}{M^2} G_1^2 + \frac{3}{2} \frac{1}{M} G_1 G_2 + \frac{5}{2} G_2^2 \right),
\]
\[
e^2 \mathcal{V}_5(k^2) = -k^2 e^2 \mathcal{V}_4(k^2),
\]
\[
e^2 \mathcal{V}_6(k^2) = \frac{1}{M^2} \left( \frac{1}{8} \frac{1}{M^2} G_1^2 + \frac{1}{M} G_1 G_2 + \frac{3}{2} G_2^2 \right).
\]

In each of the \( \mathcal{V}_i \) we have neglected higher order terms with respect to \( k^2/M^2 \) and \( q^2/M^2 \).

The \( \mathcal{V}_4(k^2) \) in (3·2) turn out to be well approximated by the appropriate Yukawa potentials in configuration space. For example, \( 1/\mathcal{V}_4(k^2) \) is plotted as a function of \( k^2 \) in Fig. 4. From the gradient of the approximated straight line we can determine the intrinsic range of the equivalent Yukawa potential. Thus the \( V_i^\pi(k^2) \) can be easily transformed into potentials in configuration space. In order to transform \( V_i^\pi(k^2) \) in the similar manner, we shall do the calculation at first in the \( \rho \)-meson formalism.

In this case we can use the formula

\[
(2\pi)^{-3} \int dk e^{-ikr} \frac{1}{k^2+m^2} \frac{1}{k^2+b^2} = \begin{cases} \frac{1}{b^2-m^2} \left( \frac{e^{-mr}}{4\pi r} - \frac{e^{-br}}{4\pi r} \right), & \text{for } b \neq m, \\ \frac{1}{2m} \frac{e^{-mr}}{4\pi r}, & \text{for } b = m. \end{cases}
\]

We shall consider the case of the chain approximation in the later sections.

Defining the potentials in configuration space by

\[
V^\lambda = V^\lambda_0(r) + (\sigma \sigma) V^\lambda_\sigma(r) + (L\sigma) V^\lambda_\lambda(r) + S_\pi V^\lambda_\pi(r) + (q \sigma)(q \sigma) V^\lambda_q(r) + (L \sigma)(L \sigma) V^\lambda_\lambda(r) + (\sigma \sigma)L^2 V^\lambda_\lambda(r),
\]

we have the \( V^\nu(r) \) as follows:

\[
\]
It should be noted that these functions are valid only for \( r \gg 0.2 \), since we have justified the "straight line approximation" for \( 1/\mathcal{V}_l(k^2) \) only up to \( k \ll 5 \). It is meaningless to compare the singularities with respect to \( r \) in the potentials in (3.4) with those appearing in the static potentials, for example, the TMO potentials.\(^{12}\)
Fig. 5. Potentials calculated from (3·4). Both $r$ and $V(r)$ are measured in units in which the pion mass is unity. Curves denoted by $A$, $B$ and $A+B$ represent $V^A(r)$, $V^B(r)$ and $V^T(r) = V^A(r) + V^B(r)$, respectively.
Fig. 6. Potentials are plotted for each state of orbital and spin angular momentum. Units are the same with those of Fig. 5. Solid curves: Calculated from (3-4). Dotted curves: $m \to \infty$ is assumed with $T(0)/4\pi$ remaining unchanged (0.272). Broken curves: The one-pion exchange potentials.

$V^t_A(r)$, $V^t_B(r)$ and $V^t(r) = V^t_A(r) + V^t_B(r)$ are plotted in Fig. 5. By comparing these plots with the TMO,\textsuperscript{13} BW\textsuperscript{19} or FST potentials,\textsuperscript{14} we can con-
firm that the potentials here considered are of the nature of two-pion exchange potentials. Furthermore we can easily see the difference in the qualitative or semi-quantitative features between \( V^{TA} \) and \( V^{AB} \). The former is less singular due to the contribution from the pion-pion resonance.

In Fig. 6, various potentials are plotted for each state of orbital and spin angular momentum. The one-pion exchange potential (OPEP) is also plotted for the sake of comparison. We have used the notation:

\[
V_{TA}(r) = V_0(r) + \langle \sigma \sigma \rangle V_8(r),
\]
\[
V_{AB}(r) = \langle L \sigma \rangle \langle L \sigma \rangle V_{L\sigma}(r) + \langle \sigma \sigma \rangle L^1 V_{L\sigma}(r),
\]

where \( \langle \rangle \) denotes the expectation value of the operators involved. \( V_8(r) \) in the \( ^1S_0 \)-state is specifically plotted for the \( ^1D_2 \)-state, and that in the \( ^3S_1 \)-state for the \( ^3D_2 \)-state. The \( V_{L\sigma}(r) \) denoted by OPEP are the ones calculated by N. Hoshizaki and S. Machida, for the case of \( pv \)-coupling.

§ 4. Discussion of the obtained potentials

In this section we shall discuss in a rather crude manner the features of the potentials obtained. A detailed analysis will be made in a separate paper. Also we should keep in mind that we have calculated only the \( V^{TA} \)-part, leaving the \( V^{AB} \)-part uncalculated.

(i) \( ^1E \)-state:

We have a strong attractive central potential. It may be noted that we have a repulsive \( V_L \). At \( r=1 \), this is nearly equal to \( V_L \) from the OPEP calculated by Hoshizaki and Machida in the case of \( pv \)-coupling, though our \( V_L \) is more singular than HM.

(ii) \( ^3E \)-state:

We have a strong repulsive tensor potential. This reduces the attractive OPEP tensor potential to about half at \( r=1 \). This may destroy the success of the OPEP in explaining the quadrupole moment of the deuteron. This tensor potential will affect also the binding of the deuteron, though we must, in this case, take into account the strong attractive central potential obtained.

(iii) \( ^3O \)-state:

We have a strong repulsive central potential, which may be objectionable from the viewpoint that the OPEP together with the weak attractive TPEP gives a good fit to the observed phase shift at low energy \((\sim 4 \text{ Mev})\). We have a strong attractive \( L-S \) potential. This amounts to about a half of that proposed by Gammel and Thaler,\(^{16}\) and to about twice that of HM.\(^{17}\) We have an attractive tensor potential, which is about three times deeper than TMO.

Our potentials should be added to the TPEP without regard to the pion-pion interaction. Though there are, so far, no complete results which fully take into account the effect of recoil and the 3-3 resonance at the same time, we
can consider that the static TPEP or the non-static TPEP calculated by Hoshizaki and Machida\(^7\) give some measure of the complete TPEP without regard to the pion-pion resonance. Combining these potentials with our results we may conclude that the pion-pion resonance here considered affects the nuclear potentials in the region II too strongly and violates some of the facts already established.\(^8\) In particular, \(V_T\) in the \(^4\)E-state, and \(V_C\) in the \(^0\)O-state will serve as a test for or against the assumption of the pion-pion resonance, as it stands here.

On the other hand there are some uncertainties in the treatment of \(G_a(k^2)\) and the pion-pion resonance, which will be discussed in the next section. Therefore it seems worthwhile to see what modifications in \(G_a(k^2)\) and the pion-pion resonance will be required in order to make the potentials reasonably weaker.

In Fig. 6, we have also plotted the results when \(m\) is assumed infinitely heavy with \(h^2/m^2=T(0)\) remaining unchanged. In this case Eq. (2-15) is applied. We find that the potentials with \(m\rightarrow\infty\) are very close to those with \(m=600\) Mev at \(r\approx1\), while the values at \(r\approx1.5\) are substantially reduced (to about a half in many cases). This shows that the values of the potentials at \(r\approx1\) are governed for the most by \(T(0)=h^2/m^2\) alone. In other words the nature of the spatial extension of the potentials is governed mainly by the \(G_a(k^2)\), and we cannot make the potentials sufficiently narrow by assuming a higher resonance energy.

As will be seen later, \(T(k^2)\) in the chain approximation is of a form somewhat more singular than that in the \(\rho\)-meson formalism when they are transformed into configuration space (see Fig. 12). Therefore we can consider that the potentials in the chain approximation lie between the curves of \(m=600\) Mev and \(m\rightarrow\infty\) in the \(\rho\)-meson formalism, if the same \(T(0)\) is assumed.

The depths of the potentials are governed by the \(\mathcal{V}_i(0)\), which are the products of two \(G_a(0)\) and \(T(0)\). Here we only show in Table II what changes the \(\mathcal{V}_i(0)\) undergo when the \(G_a(0)\) are changed. We find that \(\mathcal{V}_6(0)\) and \(\mathcal{V}_4(0)\) change appreciably with a change in \(G_1(0)\), while \(\mathcal{V}_3(0),\mathcal{V}_4(0)\) and \(\mathcal{V}_6(0)\) change little. \(\mathcal{V}_2(0),\mathcal{V}_4(0)\) and \(\mathcal{V}_6(0)\) are affected greatly by a change in \(G_2(0)\).

<table>
<thead>
<tr>
<th>(G_1(0)\rightarrow(1/2)G_1(0),\ G_2(0)\rightarrow G_2(0))</th>
<th>(\mathcal{V}_6)</th>
<th>(\mathcal{V}_1)</th>
<th>(\mathcal{V}_2)</th>
<th>(\mathcal{V}_4)</th>
<th>(\mathcal{V}_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_1(0)\rightarrow 2G_1(0),\ G_2(0)\rightarrow 2G_2(0))</td>
<td>0.25</td>
<td>0.47</td>
<td>0.87</td>
<td>0.91</td>
<td>0.91</td>
</tr>
<tr>
<td>(G_1(0)\rightarrow G_1(0),\ G_2(0)\rightarrow(1/2)G_2(0))</td>
<td>4.00</td>
<td>2.22</td>
<td>1.30</td>
<td>1.18</td>
<td>1.20</td>
</tr>
<tr>
<td>(G_1(0)\rightarrow G_1(0),\ G_2(0)\rightarrow 2G_2(0))</td>
<td>1.00</td>
<td>0.54</td>
<td>0.33</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>(G_1(0)\rightarrow G_2(0),\ G_2(0)\rightarrow 2G_2(0))</td>
<td>1.00</td>
<td>1.89</td>
<td>3.46</td>
<td>3.65</td>
<td>3.62</td>
</tr>
</tbody>
</table>

\(\mathcal{V}_1(0)\) is taken as the reference value.

§ 5. Electromagnetic form factors in perturbation theory

In § 3 we have calculated the potentials in the \(\rho\)-meson formalism with the
values $m=600$ Mev and $\hbar^2=5$, which were taken from the dispersion theoretical calculation of Frazer and Fulco. But we are now calculating in perturbation theory and it is desirable to reexamine the electromagnetic form factors also in this context. In the chain approximation method Miyamoto determined the scattering length $a$ which reproduces the rms radius $r_s^\gamma$. It is also desirable to calculate the electromagnetic form factors in more detail, since the magnetic form factor has been measured only for relatively large $k^2(k^2 \gtrsim 9)$. For these reasons we shall calculate the electromagnetic form factors, especially the magnetic form factor, in some detail in perturbation theory. This will permit us to see what modifications can be applied to the results in §3 and §4.

Before entering into the consideration of the pion-pion resonance we shall reexamine the calculation of the $G_1(k^2)$.

(i) The electromagnetic form factors with the pion-pion interaction neglected

The spectral functions $g_1(k^2)$ are plotted in Fig. 7. We find that the contributions from the 3-3 isobar are very large, especially for large $\kappa^2$. This exhibits the insufficiency of the Legendre polynomial expansion as has already been pointed out.\(^{1,11}\) But it is remarkable that the calculated $G_1(0)$ with the 3-3 isobar included gives close agreement with the observed value. If we consider the graph in Fig. 8 (a), as in Miyamoto's paper, then the value of $G_2(0)$ remains unchanged with the introduction of the pion-pion resonance. In this situation the $G_3(k^2)$ given above, which includes the 3-3 isobar, will have some justification even without the rigorous validity of the method of calculation. But this very point seems to be important and will be discussed later.

We can give an intuitive explanation for the negative contribution to $G_1(k^2)$

\[
G_1(k^2) = G_1^N(k^2) + G_1^R(k^2),
\]

where $G_1^N(k^2)$ is given by

\[
g_1(k^2) = g_1^N(k^2) + g_1^R(k^2).
\]

\[
\beta \approx 15
\]

is the square of the pseudoscalar pion-nucleon coupling constant.

Fig. 7. Spectral functions. $g_1^N(k^2)$ and $g_1^R(k^2)$ ($a=1, 2$) are the contributions from the Born approximation term and the 3-3 isobar term respectively. $g_1(k^2)$ is given by $g_1(k^2) = g_1^N(k^2) + g_1^R(k^2)$. $\beta = 15$ is the square of the pseudoscalar pion-nucleon coupling constant.
from the 3-3 isobar. In the state in which a proton is dissociated into a "core" isobar and an outer pion, charge independence consideration shows that a negative pion is emitted most frequently. This is shown schematically in Fig. 8.

In any case, $G_1(0)$ cannot be compared with experiment in contrast to $G_2(0)$. Therefore we should consider mainly the quantities which are stable with the variation of $G_1(0)$. These are $V_2$, $V_4$ and $V_6$ as seen in Table II. From this point of view $V_T(r)$ and $V_L(r)$ are most suitable for a comparison between theory and experiment.

(ii) The pion-pion resonance

Let us first consider the graph Fig. 8(a) in accordance with Miyamoto. The part surrounded by the dotted line is a part of the electromagnetic form factor of a pion and is denoted by $F_\pi(k^2)$. Corresponding to the graph Fig. 8(a), we have

$$(k_1+k_2)_\mu F_\pi(k^2) - 1 = (2\pi)^{-1} T(k^2)(k_1+k_2) \mathcal{J}_\mu(k),$$

where $\mathcal{J}_\mu(k)$ is a "2-vertex" connecting a photon and the resonant state. From gauge invariance $\mathcal{J}_\mu(k)$ should be of the form

$$\mathcal{J}_\mu(k) = (k^2 \delta_\mu - k_\mu k_\nu) \mathcal{F}(k^2).$$

After separating the term directly proportional to $\delta_\mu$, which is of the same form as the self-energy of a photon, we are left with $\mathcal{F}(k^2)$ which diverges logarithmically with respect to the virtual momentum. $\mathcal{F}(k^2)$ can be written more conveniently in the form of a spectral representation:

$$\mathcal{F}(k^2) = \frac{1}{3} \int d\xi \frac{k^2}{k^2 + \xi} \left(1 - \frac{4}{\xi}\right)^{3/2},$$

where the integration can be carried out to give the form

The interaction Hamiltonian contains derivatives of the fields. Therefore, various "catastrophic terms" should be added from gauge invariance. An example is illustrated below.
\[
\mathcal{J}(k^2) = -\frac{2\pi^2}{3} \left[ \log 2A + \frac{4}{k^2} - \left(1 + \frac{4}{k^2}\right)^{3/2} \log \left(\sqrt{1 + \frac{k^2}{4}} + \sqrt{\frac{k^2}{4}}\right)\right]. 
\]

This is very similar to \(D(k^2)\) defined in (2.3). Upon substituting (5.4) and (5.2) in (5.1), we obtain

\[
F_\alpha(k^2) = \begin{cases}
\frac{D(0)}{D(k^2)}, \\
1 - \frac{\hbar^2}{6\pi} \frac{k^2}{k^2 + m^2} \left[ \log 2A + \frac{4}{k^2} - \left(1 + \frac{4}{k^2}\right)^{3/2} \log \left(\sqrt{1 + \frac{k^2}{4}} + \sqrt{\frac{k^2}{4}}\right)\right].
\end{cases}
\]

(5.5)

For the graph in Fig. 8 (a) we can write

\[
G_\alpha(k^2) = G_\alpha(0) F_\alpha(k^2), \quad (\alpha = 1, 2)
\]

(5.6)

where \(G_\alpha(k^2)\) is a form factor with the pion-pion interaction included. It is obvious that

\[
G_\alpha(0) = G_\alpha(0),
\]

(5.7)

from \(F_\alpha(0) = 1\).

In Fig. 11 we have plotted \(G_\alpha(k^2)\) for the various parameters chosen for \(F_\alpha(k^2)\), both in the \(\rho\)-meson formalism and in the chain approximation. In Fig. 11 we have also indicated the value of \(T(0)\) which is considered to govern the depth of the potentials. The curve with \(m = 600\) Mev, \(\hbar^2 = 5\) in the \(\rho\)-meson formalism is found not to reproduce the observed form factor very well. If we choose, in the \(\rho\)-meson formalism, other curves which give a better fit, \(T(0)/4\pi\) turns out to be larger (\(\gtrsim 0.54\)) than that used in § 3 (0.27). Therefore the potentials calculated in § 3 should be considered to have been underestimated by about a factor of 2, if calculated in the \(\rho\)-meson formalism.

In the case of the chain approximation, we find that the choice with \(a = 0.2\) and \(A = 2M\), used by Miyamoto gives indeed a good fit. This choice gives \(T(0)/4\pi = 0.47\), which is 1.70 times larger than the value used in § 3. In any case we cannot expect to make \(T(0)\) appreciably smaller without giving up trying to explain the electromagnetic form factors.

In order to see the nature of the singularity of the Fourier transform of \(T(k^2)\)
in the chain approximation method, we have plotted in Fig. 12 the quantity \( T(\mathbf{k}^2)/T(0) \) for several cases in which the observed electromagnetic form factor can be well reproduced. From these plots we can infer that the potentials calculated in the chain approximation method are of forms somewhat more singular than those represented in Fig. 6 by solid curves but less singular than those represented by dotted curves.

In Fig. 10 we have plotted the \( G_1(\mathbf{k}^2) = G_1(\mathbf{k}^2) F_*(\mathbf{k}^2) \) calculated by a method similar to that used with \( G_2(\mathbf{k}^2) \). We cannot obtain satisfactory agreement between theory and experiment, if we maintain the good agreement in \( G_2(\mathbf{k}^2) \). From this reason also it is desirable to consider the quantities which do not depend on \( G_2(\mathbf{k}^2) \) appreciably.

### § 6. Summary and concluding remarks

We have calculated the potentials from the graphs in Fig. 2; only the \( V^- \) part are considered in this paper. We have given the explicit forms in con-
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configuration space in the $\rho$-meson formalism with $m=600$ Mev and $h^2=5$, which were taken from the results of Frazer and Fulco, and adopted by Takeda et al. Eqs. (3·4) and the curves in Fig. 5 and Fig. 6 exhibit the qualitative or semi-quantitative features of the potentials which result from the pion-pion resonance in the state $I=J=1$. Possible changes can be inferred when the parameters are changed or the chain approximation method is adopted. Reexaming the electromagnetic structure of the nucleon both in the $\rho$-meson formalism and in the chain approximation method, we have found that the potentials given by (3·4) cannot be expected to be an overestimate. Rather the values at $r=1$ might have been somewhat underestimated.

As the main results there appear a strong attractive $L$-$S$ potential and a repulsive central potential in the $^3S$-state, a strong repulsive tensor potential and an attractive central potential in the $^3P$-state, and a repulsive $L$-dependent potential in the $^1P$-state. These are all of the nature of a two-pion exchange potential and affect the nuclear forces in the region II. Among the potentials quoted above the repulsive tensor potential in the $^3P$-state may be objectionable if we remember that the attractive one-pion exchange tensor potential succeeded in accounting for the quadrupole moment of the deuteron, though more careful analysis will be needed to draw any conclusions.

Since $G_1(k^2)$ seems to be subject to more uncertainties than $G_1(k^2)$ in the theoretical treatment, it is desirable to consider the potentials which are insensitive to a change in $G_1(k^2)$. For this reason also, the tensor potential will be most suitable for the test for or against the assumption of a pion-pion resonance in the state $I=J=1$.

Finally we shall discuss the possibility of making the potentials weaker. As stated already, we are forced to make $G_1(k^2)$ smaller in order to make the tensor potential weaker. In this connection we should remember that we have assumed that only the graph (a) in Fig. 8 contributes to the electromagnetic form factors of the nucleon. On account of this we arrived at Eq. (5·7) and consequently had to adopt a "large" $G_2(k^2)$ which gives $2MG_2(0)/e=1.63$.

On the other hand, gauge invariance considerations are particularly important when we discuss the electromagnetic form factors at $k^2=0$. Namely, other graphs in Fig. 8 should be taken into account. In this case the electromagnetic form factor of the pion can be written as follows:

$$F_\pi(k_1^2, k_2^2, k^2) = 1 + k^2 f_3(k^2) + (k^2 + 1) f_3(k_1^2, k_2^2, k^2) + (k^2 + 1) f_3(k_1^2, k_2^2, k^2),$$

which satisfies the condition

$$F_\pi(-1, -1, 0) = 1.$$

* $f_2=f_3=0$, for the graph (a) in Fig. 8.
It should be noted, however, that we have, in general,

$$F_\pi(k_1^2, k_2^2, 0) \neq 1,$$  \hspace{1cm} (6.3)

for $k_1^2 \neq -1$, or $k_2^2 \neq -1$. In the calculation of the electromagnetic form factors of the nucleon, pions are not in a free state, and we have in general

$$G_a(0) \neq G_a(0),$$  \hspace{1cm} (6.4)
due to (6.3). Frazer and Fulco obtained, in fact, $G_3(0)$ which is not equal to $G_3(0)$, although the reason is not very obvious.

There is a chance that $G_3(k^2)$ is nearly equal to that calculated in the Born approximation $(2MG_3(0)/e=0.87)$ and the difference from the observed value comes from the pion-pion resonance in the manner stated above. Then we are permitted to use a “small” $G_a(k^2)$ and would obtain the weak potentials.

This really seems to be the case, since the 3-3 isobar contribution to $G_3(k^2)$, calculated in the previous sections, seems unjustifiably large. The situation is, however, not very simple. For some reference we have calculated the potentials with the $G_a(k^2)$ in the Born approximation; the value at $r=1$ are shown in Table III. From this Table we find that the tensor potential has been reduced only a little. This is because $G_1(k^2)$ is very large in the Born approximation $(G_1(0)/e=0.66)$. Thus we must face the difficult problem of treating the $G_1(k^2)$.

<table>
<thead>
<tr>
<th>$G_a(k^2)$</th>
<th>$V_8^T(1)$</th>
<th>$V_8^S(1)$</th>
<th>$V_8^{L,S}(1)$</th>
<th>$V_7^T(1)$</th>
<th>$V_7^S(1)$</th>
<th>$V_7^{L,S}(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Born approximation</td>
<td>0.0708</td>
<td>0.0179</td>
<td>-0.0640</td>
<td>-0.0196</td>
<td>0.00161</td>
<td>-0.00140</td>
</tr>
<tr>
<td>3-3 isobar included</td>
<td>0.0143</td>
<td>0.0277</td>
<td>-0.0356</td>
<td>-0.0300</td>
<td>0.00446</td>
<td>-0.00272</td>
</tr>
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</table>

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**Note added in proof:**

(i) $V^T_{16}k_{10}q_{10}q_{20}$ should be multiplied to the left hand sides of Eqs. (2.5) and (2.12).

(ii) In $V_8^T$ and $V_7^T$ in Eqs. (3.4), we have set $\varepsilon=y$, since they are nearly equal to each other.
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