On an Extension of the Mathematical Framework of the Quantum Theory

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Extending the usual framework of the quantum theory, we have succeeded in getting a new mathematical framework. The most essential character of the usual theory is preserved in the new theory, that is, the set of physical quantities can be represented by an operator algebra in a separable Hilbert space. In the new theory, on the other hand, we can rigorously treat the interaction Hamiltonians, which cannot be treated in the usual framework. In our new framework there are always solutions of the Schrödinger equation for any Hamiltonian. However, the most serious defect of our theory is in the fact that we have no guarantee of the uniqueness of solutions.

§ 1. Introduction

The outline of the mathematical framework of the usual quantum theory is as follows. For any physical system there is a separable Hilbert space $\mathcal{H}$, and every physical quantity is represented by an operator $a$ in $\mathcal{H}$ and every state by a vector $\psi \in \mathcal{H}$ of norm 1. The expectation value of $a$ in a state $\psi$ is given by the inner product $(\psi, a\psi)$. The change of the system with time can be described by a one-parameter group $U(t)$ of unitary operators, and in the Schrödinger representation $\psi(t)$ is given by

$$\psi(t) = U(t)\psi(0). \quad (1.1)$$

Stone’s theorem indicates that there is a self-adjoint operator $H$ in $\mathcal{H}$ such that

$$i(d/dt)U(t) = HU(t), \quad U(t) = \exp(-iHt), \quad (1.2)$$

and $H$ is called the Hamiltonian of this system [ref. 1], p. 598]. Since $U(t)$ are unitary operators in $\mathcal{H}$, it is possible to use the Heisenberg representation and consider that physical quantities change with time in accordance with the equation

$$a(t) = U^{-1}(t)a(0)U(t), \quad (1.3)$$

and that states are unchanged.

On the one hand the usual mathematical framework of the quantum theory is in harmony with the physical contents of the theory, and at first sight it seems to be almost impossible to separate these two aspects of the theory.
But, on the other hand, there is little room for doubt that the improvement of the framework is inevitable in order to deal rigorously with the quantum theory of interacting fields. This is due to the fact that, because the usual interaction Hamiltonian $H$ cannot be an operator in $\mathcal{H}$, $U(t)$ cannot be a one-parameter group of unitary operators in $\mathcal{H}$. In fact, if $U(t)$ is to be physically meaningful, it should be continuous in some sense. But, since $\mathcal{H}$ is separable, even the weak measurability of $U(t)$ implies the strong continuity of it. Namely, if $(\mathcal{F}_1, U(t)\mathcal{F}_2)$ is measurable for any two vectors $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{H}$, $U(t)\mathcal{F}$ is strongly continuous in $\mathcal{H}$ for any $\mathcal{F} \in \mathcal{H}$ [ref. 1, pp. 74, 598], and there is a self-adjoint operator $H$ in $\mathcal{H}$ which satisfies Eq. (1.2). This means that, when we deal with the interacting fields, $U(t)$ in general cannot be weakly measurable, and the incompetence of the usual framework is obvious.

Now we must seek for an extended framework which, on the one hand, preserves most of the essential character of the usual theory and, on the other hand, enables us to overcome the above difficulties. In our opinion, that the set of physical quantities at a time $t$ forms a *-algebra is the most essential part of the usual theory, and this almost inevitably implies that the *-algebra is represented by an operator algebra $\mathfrak{A}$ in a separable Hilbert space $\mathcal{H}$. Therefore, in this paper, we accept this assumption from the outset, and furthermore we consider the set of bounded operators instead of $\mathfrak{A}$ for brevity's sake. It is well-known that $\mathfrak{B}$ forms a $\mathcal{B}^*$-algebra [ref. 1, pp. 22, 160; ref. 3].

As to the definition of "state," however, there is much room for improvement. The most essential role of a state is that it is a linear functional on $\mathfrak{B}$, and it is rather secondary in importance that a state $f$ can be represented as

$$f(a) = (\mathcal{F}, a\mathcal{F}), \quad a \in \mathfrak{B}, \quad \mathcal{F} \in \mathcal{H}, \quad \|\mathcal{F}\| = 1. \quad (1.4)$$

With due attention to this point, we assume that a state $f$ is a positive linear functional on $\mathfrak{B}$ but not necessarily written as in Eq. (1.4). Further we assume that a state $f$ is of norm 1, namely

$$\sup_{a \in \mathfrak{B}, \|a\| \leq 1} |f(a)| = 1.$$ 

This is nothing but to assume that $f$ is continuous. Let the set of all these functionals be $\tilde{S}$. Then it is clear that functionals in $\tilde{S}$ are not necessarily written as in Eq. (1.4), but it is known that the set $S$ of linear functionals of the form

$$f(a) = \sum_{i=1}^{\infty} \langle \mathcal{F}_i, a\mathcal{F}_i \rangle, \quad \sum_i \|\mathcal{F}_i\|^2 = 1, \quad (1.4')$$

is everywhere dense in $\tilde{S}$ in some topology [ref. 4, p. 300]. Therefore it is very natural to assume that a state can be represented as a limit of a sequence of functionals $f_n(a) = \langle \mathcal{F}_n, a\mathcal{F}_n \rangle, n=1, 2, \ldots$, in some sense.
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In order to treat the extended notion of states on the line parallel to the usual one, we introduce a non-separable Hilbert space $\mathfrak{H}$, in which $\mathfrak{S}$ is contained. Then we can show that $\mathfrak{B}$ is isomorphic to a $B^*$-algebra of bounded operators in $\mathfrak{H}$, and that every state $f$ can be represented by a vector $\Psi \in \mathfrak{H}$ as

$$f(a) = (\Psi, a\Psi), \quad a \in \mathfrak{B}, \quad \Psi \in \mathfrak{H}, \quad \|\Psi\| = 1. \quad (1.4'')$$

Consequently it is possible to say that our new framework is a natural extension of the usual one.

All the above arguments refer to a system at a time $t$. Now it is necessary to introduce a one-parameter group $U(t)$ of unitary operators which satisfies Eq. (1.2) in some appropriate sense. As stated above, it will be impossible to define such $U(t)$ in $\mathfrak{S}$, but the obstacle disappears in $\mathfrak{H}$. In fact, even if $U(t)$ is weakly measurable, namely, even if $(\Psi, U(t)\Psi)$ is measurable, $U(t)$ is not necessarily strongly measurable, and Stone's theorem does not hold. Hence, though $U(t)$ is an operator in $\mathfrak{H}$, $H$ in Eq. (1.2) is not so in general.

The fact that $U(t)$ is an operator in $\mathfrak{H}$ and not in $\mathfrak{S}$ is most significant in our new theory. This implies that the unitary operator $U(t)$, which describes the change of the system with time, does not belong to $\mathfrak{B}$, that is, $U(t)$ is not a physical quantity and not "observable" in the original sense. Therefore, in our theory, the Schrödinger representation is more adequate than the Heisenberg one. In fact, since

$$a(t) = U^{-1}(t)a(0)U(t) \quad (1.3')$$

is an operator in $\mathfrak{H}$ but cannot belong to $\mathfrak{B}$, the problem becomes troublesome in the Heisenberg representation.\footnote{In this paper we use symbols of bold face letters in order to indicate that they refer to $\mathfrak{H}$ and not to $\mathfrak{S}$.} On the other hand, the equation

$$\Psi(t) = U(t)\Psi(0) \quad (1.1')$$

has a clear-cut meaning in our theory and causes no trouble.

§ 2. Non-separable Hilbert space $\mathfrak{H}$

As a preliminary we begin with a definition of a $\mu$-Limit of any bounded sequence. Let $\xi$ be a bounded sequence of complex numbers, that is,

$$\xi = \{\alpha_n; n=1, 2, \ldots\}, \quad \sup_n |\alpha_n| < +\infty,$$

and $m$ be the set of all these sequences. By defining addition and scalar multiplication by

$$\xi + \gamma = \{\alpha_n \gamma_n \}, \quad \alpha \xi = \{\alpha \alpha_n \},$$

and $\mathfrak{m}$ be the set of all these sequences. By defining addition and scalar multiplication by

$$\xi + \gamma = \{\alpha_n \gamma_n \}, \quad \alpha \xi = \{\alpha \alpha_n \}, \quad (2.1)$$

\footnote{When we consider the Lorentz invariance of the theory, it becomes clear that the interaction representation is most adequate. But, in this paper, we do not enter into this problem.}
and the norm of $\xi$ by

$$
\|\xi\| = \sup_n |\alpha_n|,  \quad (2.2)
$$

$m$ becomes a Banach space [ref. 5], pp. 11, 53]. Let $c$ be the set of all elements of $m$ each of which has a finite limit $\lim_{n \to \infty} \alpha_n$. Then $c$ is also a Banach space and is a closed linear subspace of $m$ [ref. 5], p. 11.

In this paper we use the following conventional notation:

$$
0 = \{0, 0, \ldots\}, \quad 1 = \{1, 1, \ldots\},
$$

$$
|\xi| = \{|\alpha_n|\}, \quad \xi^2 = \{(\alpha_n)^2\}, \quad \bar{\xi} = \{\bar{\alpha}_n\},
$$

where $\bar{\alpha}$ is the conjugate complex number to $\alpha$. Further we write

$$
\tilde{M} = \{\tilde{\xi}; \tilde{\xi} \in M\}
$$

for any subset $M$ of $m$. $\xi = \{\alpha_n\}$ is said to be real if all $\alpha_n$ are real. It is said to be positive and we write $\xi \geq 0$, if $\alpha_n \geq 0$, $n = 1, 2, \ldots$. If a linear functional $\mu$ on $m$ or $c$ satisfies the condition that $\xi \geq 0$ implies $\mu(\xi) \geq 0$, then $\mu$ is said to be positive.

Put

$$
\mu_0(\tilde{\xi}) = \lim_{n \to \infty} \alpha_n, \quad \tilde{\xi} = \{\alpha_n\} \in c.  \quad (2.3)
$$

Then $\mu_0$ is a positive linear functional on $c$ of norm 1, and, by extension, we can get a linear functional $\mu$ on $m$ of norm 1 by virtue of Hahn-Banach's theorem [ref. 6), p. 111]. Now we shall show that $\mu$ can be taken to be positive.

Theorem 1. There is a positive linear functional $\mu$ on $m$ which is an extension of $\mu_0$ and of norm 1.

Proof. Putting

$$
B_1 = B + 1, \quad B = \{\tilde{\xi}; \|\tilde{\xi}\| < 1, \tilde{\xi} \in m\},
$$

and

$$
M_0 = \{\tilde{\xi}; \mu_0(\tilde{\xi}) = 0, \tilde{\xi} \in c\},
$$

we can show that $B_1$ is an open convex set, and that $B_1 \cap M_0 = \phi$. Let $m^R, B_1^R$ and $M_0^R$ be the subsets of all real elements of $m, B_1$ and $M_0$, respectively. It is easy to show that $m^R$ becomes a Banach space over the real number field, and that $M_0^R$ is a linear subspace of it. Furthermore, $B_1^R$ is open and convex in $m^R$ and $B_1^R \cap M_0^R = \phi$. Hahn-Banach's theorem indicates that there is a closed hyperplane $M^R$ which contains $M_0^R$ and does not intersect with $B_1^R$ [ref. 6), p. 69], that is,

$$
M^R \supset M_0^R, \quad B_1^R \cap M^R = \phi.  \quad (2.4)
$$

Let $M$ be the linear subspace in $m$ which is generated by $M^R$. Then $M$ is a closed hyperplane in $m$, and $\bar{M} = M$. Further, Eq. (2.4) implies that

$$
M \supset M_0, \quad B_1 \cap M = \phi.  \quad (2.4')
$$
In other words, there is a continuous linear functional $\mu$ on $m$ which is an extension of $\mu_0$, and $M$ is defined by the equation $\mu(\check{\xi}) = 0$. It is obvious that, if $\check{\xi}$ is real, $\mu(\check{\xi})$ is real, too, and that

$$\mu(\check{\xi}) > 0 \text{ on } B_1. \quad (2.5)$$

Put

$$\beta_n(\check{\xi}) = \min \{\max((\alpha_n/\|\check{\xi}\|), \varepsilon), (1-\varepsilon)\}$$

for some $\check{\xi} \geq 0$, and $1/2 > \varepsilon > 0$. Then

$$\eta(\varepsilon) = [\beta_n(\varepsilon)] \in B_1^R,$$

and $\eta(\varepsilon) \rightarrow \check{\xi}/\|\check{\xi}\|$ as $\varepsilon \rightarrow 0$. From Eq. (2.5) and the continuity of $\mu$, it follows that $\check{\xi} \geq 0$ implies $\mu(\check{\xi}) \geq 0$, that is, $\mu$ is positive.

Lastly we shall prove that $||\mu|| = 1$. As $\mu(1) = 1$, we have $||\mu|| \geq 1$. Conversely, if $\check{\xi} \in B$, and $|\alpha| = 1$, then $1 - \alpha \check{\xi} \in B$, and

$$|\mu(1 - \alpha \check{\xi})| = |1 - \alpha \mu(\check{\xi})| > 0,$$

namely $|\mu(\check{\xi})| \neq 1$ on $B$. This means that $|\mu(\check{\xi})| < 1$ on $B$, that is, $||\mu|| \leq 1$. Q.E.D.

Now we can define a $\mu$-Limit of any bounded sequence by the equation

$$\mu\text{-Lim } a_n = \mu\text{-Lim } (\check{\xi}) = \mu(\check{\xi}), \quad \check{\xi} = \{a_n\}, \quad (2.6)$$

and this $\mu$-Limit plays an essential role in the following.

Lemma 1. If $\check{\xi} \geq 0$, then the two equations $\mu\text{-Lim } (\check{\xi}) = 0$ and $\mu\text{-Lim } (\check{\xi})^2 = 0$ are equivalent to each other.

Proof. We can assume that $||\check{\xi}|| \leq 1$ without loss of generality. Since $1 \geq a_n \geq 0$, $\alpha \geq (\alpha_n)^2$ and $\mu\text{-Lim } (\check{\xi}) \geq \mu\text{-Lim } (\check{\xi})^2$. Thus, $\mu\text{-Lim } (\check{\xi}) = 0$ implies $\mu\text{-Lim } (\check{\xi})^2 = 0$. Conversely, if $\mu\text{-Lim } (\check{\xi}) = 0$, then for any real $\alpha$

$$0 \leq (1 - \alpha \check{\xi})^2 = 1 - 2\alpha \check{\xi} + \alpha^2 \mu\text{-Lim } (\check{\xi})^2 = 1 - 2\alpha \mu\text{-Lim } (\check{\xi})^2,$$

and

$$0 \leq \mu\text{-Lim } (1 - \alpha \check{\xi})^2 = 1 - 2\alpha \mu\text{-Lim } a_n + \alpha^2 \mu\text{-Lim } a_n = 1 - 2\alpha \mu\text{-Lim } a_n.$$ 

Thus $\mu\text{-Lim } a_n = 0$. Q. E. D.

Lemma 2.

$$\mu\text{-Lim } ||\check{\xi}|| \geq |\mu\text{-Lim } (\check{\xi})|.$$ 

Proof. Let the closure of $B_1$ be $(B_1)^c$. If $\check{\xi} \in (B_1)^c$, then the real part of $\mu\text{-Lim } (\check{\xi})$ is not negative. If $||\check{\xi}|| < 1$, we can show that $||\check{\xi}|| - \check{\xi} \in (B_1)^c$. Put

$$\mu\text{-Lim } (\check{\xi}) = |\mu\text{-Lim } (\check{\xi})| \exp(i\theta).$$

Then $|\check{\xi}| - \check{\xi} \exp(-i\theta)$ belongs to $(B_1)^c$, and

$$\mu\text{-Lim } |\check{\xi} - \check{\xi} \exp(-i\theta)| = \mu\text{-Lim } |\check{\xi}| - |\mu\text{-Lim } (\check{\xi})|$$

is real. Therefore,
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\[ \mu\text{-Lim}(|\xi| - \xi \exp(-i\theta)) \geq 0, \]

that is,

\[ \mu\text{-Lim}(|\xi|) \geq |\mu\text{-Lim}(\xi)|. \]

Q. E. D.

Now we shall show that we can define a non-separable Hilbert space \( \mathfrak{H} \), which can be regarded as an extension of the separable Hilbert space \( \mathcal{H} \). Let \( \omega \) be a bounded sequence of vectors in \( \mathfrak{H} \), namely

\[ \omega = \{ \mathcal{F}_n; n = 1, 2, \cdots \}, \sup_n \| \mathcal{F}_n \| < +\infty, \]

and \( \Omega \) be the set of all these sequences. By defining

\[ \omega^1 + \omega^2 = \{ \mathcal{F}_n^1 + \mathcal{F}_n^2 \} = \{ \mathcal{F}_n^1 + \mathcal{F}_n^2 \}, \]

\[ \alpha \omega = \alpha \{ \mathcal{F}_n \} = \{ \alpha \mathcal{F}_n \}, \]

(2.7)

\( \Omega \) becomes a linear space.

Lemma 3. Put

\[ \Omega_0 = \{ \omega; \| \omega \| = 0 \}, \quad \| \omega \|^2 = \mu\text{-Lim}(\mathcal{F}_n, \mathcal{F}_n). \]

Then \( \Omega_0 \) is a linear subspace of \( \Omega \), and \( \Omega/\Omega_0 \) is also a linear space.

Proof. Lemma 1 shows that, if \( \omega \in \Omega_0 \), then \( \mu\text{-Lim}\| \mathcal{F}_n \| = 0 \). Further, from Lemma 2 and the positiveness of \( \mu \), it follows that

\[ |\mu\text{-Lim}(\mathcal{F}_n^1, \mathcal{F}_n^2)| \leq \mu\text{-Lim}(\mathcal{F}_n^1, \mathcal{F}_n^2) \]

\[ \leq \mu\text{-Lim}(\| \mathcal{F}_n^1 \| \cdot \| \mathcal{F}_n^2 \|) \leq (\mu\text{-Lim}\| \mathcal{F}_n^1 \|) \cdot \sup \| \mathcal{F}_n^2 \|. \]

Thus, for every pair of \( \{ \mathcal{F}_n^1 \}, \{ \mathcal{F}_n^2 \} \in \Omega_0 \),

\[ \mu\text{-Lim}(\mathcal{F}_n^1, \mathcal{F}_n^2) = 0, \]

and

\[ \mu\text{-Lim}(\mathcal{F}_n^1 + \mathcal{F}_n^2, \mathcal{F}_n^1 + \mathcal{F}_n^2) = 0. \]

Q. E. D.

Theorem 2. Let \( \tilde{\omega} \in \Omega/\Omega_0 \) be the class containing \( \omega \). Then we can define an inner product \( \langle \tilde{\omega}^1, \tilde{\omega}^2 \rangle \), and \( \Omega/\Omega_0 \) becomes a non-separable pre-Hilbert space.

Proof. The proof of Lemma 3 shows that, if \( \{ \mathcal{F}_n^1 \} \in \Omega_0 \), then

\[ \mu\text{-Lim}(\mathcal{F}_n^1, \mathcal{F}_n^2) = 0 \]

for every \( \{ \mathcal{F}_n^2 \} \in \Omega \). Therefore, we can uniquely define an inner product by the equation

\[ \langle \tilde{\omega}^1, \tilde{\omega}^2 \rangle = \mu\text{-Lim}(\mathcal{F}_n^1, \mathcal{F}_n^2), \quad \{ \mathcal{F}_n^1 \} \in \tilde{\omega}^1, \{ \mathcal{F}_n^2 \} \in \tilde{\omega}^2, \]

(2.9)

and we can show that \( \langle \tilde{\omega}^1, \tilde{\omega}^2 \rangle \) is an inner product in a proper sense [ref. 1, p. 17], that is, \( \Omega/\Omega_0 \) is a pre-Hilbert space. The non-separability of \( \Omega/\Omega_0 \) can be easily shown. Q.E.D.

Let the completion of \( \Omega/\Omega_0 \) be \( \mathfrak{B} \). Then \( \mathfrak{B} \) is a non-separable Hilbert space, and we write \( \mathfrak{B}_\mu \) with a subscript \( \mu \) in order to express explicitly that
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\( \mathcal{H} \) is defined by using the functional \( \mu \). In the following we write \( \Psi = \{ \mathcal{F}_n \} \) when \( \Psi \in \mathcal{H} \) is a class containing the sequence \( \{ \mathcal{F}_n \} \) for the sake of simplicity. Since the set of sequences \( \{ \mathcal{F}, \mathcal{F}, \ldots \} \), \( \mathcal{F} \in \mathcal{H} \), is isomorphic to \( \mathcal{F} \), \( \mathcal{H} \) can be regarded as a subspace of \( \mathcal{H} \). For any element \( a \) in \( \mathcal{H} \), we can define an operator \( a \) in \( \mathcal{H} \) by the equation

\[
 a \Psi = \{ a \mathcal{F}_n \},
\]

and it is easy to show that \( a \) is also a bounded operator and that \( \|a\| = \|a\| \). We denote the set of these operators by \( \mathfrak{B} \), and it follows that \( \mathfrak{B} \) is isomorphic to \( \mathcal{F} \). It is obvious that any state \( f \), which is the limit of a sequence of functionals \( \{ (\mathcal{F}_n, a \mathcal{F}_n) \} \) in a sense, can be written as in Eq. (1.4). Thus we have succeeded in getting a new framework as stated in § 1.

§ 3. Physical consideration

In this section we shall consider the main features of the physical contents of our new theory. A state \( f \) which is given by Eq. (1.4') differs essentially from that in Eq. (1.4) in the following sense. Since \( f \) is a continuous linear functional on \( \mathfrak{B} \), it can be split uniquely into two parts \( f_1 \) and \( f_2 \), where \( f_1 \) is a linear functional on a subset \( \mathfrak{A} \subset \mathfrak{B} \) of all completely continuous operators, and \( f_2 \) is orthogonal to \( \mathfrak{A} \) [ref. 3), p. 49; ref. 7), p. 396], namely

\[
 f = f_1 + f_2, \quad f_1 \in \mathfrak{A}^\prime, \quad f_2 \in \mathfrak{A}.
\]

Since \( f_1 \) can be represented by Eq. (1.4'), it belongs to the same category as the states in Eq. (1.4), that is, \( f_1 \) is the same state as those in the usual theory. On the other hand, \( f_2 \) is of very different nature, and the following example shows the difference clearly. Let \( e \) be the unit element of \( \mathfrak{B} \), \( \{ \mathcal{F}_i; i=1, 2, \ldots \} \) be a complete orthonormal system of \( \mathfrak{H} \), and \( P_i \) be a projection on each one-dimensional subspace generated by \( \mathcal{F}_i \). Then

\[
 f_2(e) = \| f_2 \|, \quad f_2(P_i) = 0, \quad i=1, 2, \ldots.
\]

Therefore, though \( e = P_1 + P_2 + \cdots \),

\[
 f_2(e) = \frac{1}{\omega} \sum_{i=1}^{\omega} f_2(P_i).
\]

At first sight this seems to be very curious. Nevertheless it is possible to consider that this feature is rather appropriate for the representation of the state of a particle accompanied with the cloud of the so-called proper field. For example, there is a state \( \Psi \), in which we may say that all states \( \mathcal{F}_i, i=1, 2, \ldots \), are equally probable.

When a Hamiltonian \( H \) is not an operator in \( \mathfrak{H} \), a one-parameter group \( U(t) \) of unitary operators can be defined as follows. In general, \( H \) can be represented by the equation
where \( h_i \) is a self-adjoint operators in \( \mathcal{S} \). Put

\[
U_n(t) = \exp(-iH_n t), \quad H_n = \sum_{i=1}^{n} h_i. \tag{3.3}
\]

Then we can define \( U(t) \) by the equation

\[
U(t) \Psi = \{U_n(t) \mathcal{F}_n\}, \tag{3.4}
\]

and it is easy to show that \( U(t) \) is a one-parameter group of unitary operators in \( \mathcal{S} \). By formal differentiation, we get

\[
i(d/dt) (U(t) \Psi, aU(t) \Psi) = (U(t) \Psi, [a, H] U(t) \Psi), \tag{3.5}
\]

where

\[
\mathcal{H}\Psi = \{H_1 \mathcal{F}_1, H_2 \mathcal{F}_2, \ldots\}. \tag{3.6}
\]

Thus we may say that \( U(t) \) satisfies Eq. (1.2) in a sense and that it is the required one. It is, however, necessary to note the following two points. First, in order that Eq. (3.5) may hold, the two operations \( i(d/dt) \) and \( \mu\text{-Lim} \) must be commutative. Secondly, \( \mathcal{H}\Psi \) does not belong to \( \mathcal{S} \) in general, and the right-hand side of Eq. (3.5) is merely a formal inner product.

When the renormalization is necessary, the situation becomes more complicated, but the essential character of the problem remains unchanged. Let \( H_{\text{free}}(m_0) \) and \( H_{\text{int}}(e_0) \) be the free Hamiltonian and the interaction one respectively. Putting

\[
H_n = H_{\text{free}}(m_0) + H_{\text{int}}(e_0),
\]

and

\[
\lim_{n \to \infty} H_{\text{int}}^n(e_0) = H_{\text{int}}(e_0),
\]

we get the sequence \( \{U_n(t)\} = \{\exp(iH_n t)\} \). Now we must renormalize the mass and the charge in each \( U_n \), and each \( U_n \) should be written by the observed mass \( m \) and the observed charge \( e \), instead of \( m_0 \) and \( e_0 \). Then we must take the \( \mu\text{-Lim} \) of this renormalized \( \{U_n\} \), and we get the desired one-parameter group \( U(t) \) of unitary operators in \( \mathcal{S} \). As \( m \) and \( e \) are definite, \( m_0 \) and \( e_0 \) should have definite values \( m^n_0 \) and \( e^n_0 \) corresponding to each \( H_n \). But it is not necessary that the sequences \( \{m^n_0\} \) and \( \{e^n_0\} \) have finite limits which are not equal to zero. In fact, it is very plausible that \( e^n_0 \to 0 \) as \( n \to \infty \), that is, \( H_{\text{int}}^n(e_0) \to 0 \). If this is the case, we cannot write the Hamiltonian in an explicit form. But this is no matter to us, because \( U(t) \) is defined completely without using such a Hamiltonian.

Moreover, our theory has several formal advantages. For example, in Dirac's positron theory, two kinds of vacua \( \mathcal{F}_0 \) and \( \mathcal{F}_0 \) coexist in \( \mathcal{S} \). As a matter of fact, if \( \mathcal{F}_0 \in \mathcal{S} \) is the vacuum in which no electron exists in any posi-
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tive and negative energy state, then the vacuum $\Psi_0 \in \mathcal{H}$ in which all the negative energy states are occupied can be represented by

$$\Psi_0 = \{ \Psi_0, \alpha_1 a_1 \Psi_0, \alpha_2 a_2 a_1 \Psi_0, \ldots \},$$

where $a_i, i=1, 2, \ldots$, is a creation operator of an electron in a negative energy state and $\alpha_i$ is a normalizing factor. There is another example. As the Hamiltonian $H$ is not necessarily an operator in $\mathcal{H}$, the so-called zero-point energy causes no difficulty. In fact, even if $H=\alpha_1+\alpha_2+\ldots = +\infty$,

$$U(t) = \exp(-iH_n t), \quad H_n=\alpha_1+\alpha_2+\ldots+\alpha_n,$$

properly defines a one-parameter group of unitary operators in $\mathcal{H}$.

All the above arguments show that our new framework is wide enough to contain the present quantum theory of fields. Unfortunately, it is too wide, and it contains many unphysical contents. Before we can apply our theory to a concrete physical problem, we must solve several problems: Which of the infinitely many $\mathcal{H}_n$ has a physical meaning [see the Appendix]?; do the different representations of $H$ in Eq. (3.2) give the same result to a physical problem?; and so on. All these problems will be studied on another occasion.

Appendix

In this Appendix we shall show an example of reasonable extensions of $\mu_0$, which is uniquely defined in a subspace $\pi \subseteq \mathfrak{m}$. We believe that every physically meaningful functional should coincide with this extension on $\pi$.

Let $\tau$ be an operator on $\mathfrak{m}$ such that

$$\tau \xi = \tau(\alpha_1, \alpha_2, \alpha_3, \ldots) = (\alpha_2, \alpha_3, \ldots).$$

(A.1)

Then $\tau$ is a linear operator and $\|\tau\|=1$. Put

$$\xi_k = (1/k) \sum_{l=1}^{b} \tau^l \xi$$

(A.2)

for any $\xi \in \mathfrak{m}$. If $\{\xi_k\}$ has a limit in $\mathfrak{m}$, we denote it by $\xi_\infty$, namely $\xi_k \to \xi_\infty$ in $\mathfrak{m}$ as $k \to \infty$. Let $\mathfrak{n}$ be the set of all elements each of which has a corresponding $\xi_\infty$. Then we can show that $\mathfrak{n}$ is a closed linear subspace of $\mathfrak{m}$, and that $\mathfrak{n} \subseteq \mathfrak{m}$. We can also show that, if $\xi$ is “periodic” or at least “almost periodic,” it belongs to $\mathfrak{n}$. Since $\|\tau\|=1$, $\xi_k \to \xi_\infty$ implies $\tau \xi_k \to \tau \xi_\infty$, that is, $\xi \in \mathfrak{n}$ implies $\tau \xi \in \mathfrak{n}$, and $\tau \xi_\infty = \xi_\infty$, because

$$\|\tau \xi_k - \xi_k\| = (1/k) \| \sum_{l=1}^{b} \tau^l+1 \xi - \sum_{l=1}^{b} \tau^l \xi \| = (1/k) \| \tau^{l+1} \xi - \tau^l \xi \| \to 0.$$ 

Therefore, $\xi_\infty$ is constant, and we write

$$\xi_\infty = \{ \xi_\infty, \xi_\infty, \ldots \} = \xi_\infty \cdot 1.$$ 

(A.3)

Putting

$$\mu_\infty (\xi) = \xi_\infty, \quad \xi \in \mathfrak{n},$$

(A.4)
we can define a functional $\mu_1$ on $\mathfrak{n}$, and it is easy to show that $\mu_1$ is an extension of $\mu_0$. $\mu_1$ is a positive linear functional of norm 1 on $\mathfrak{n}$, and we can extend it on $\mathfrak{m}$, getting such a functional on $\mathfrak{m}$. It is obvious that $\mu_1$ is a reasonable extension of $\mu_0$, and we may rather say that $\mu_1$ is a "mean" of $\xi$, instead of a "limit." In fact, $\mu_1$ corresponds to the so-called Hölder's summation $(H, 1)$. We shall be able to utilize any other method of summation, but we do not know which of these methods is most suitable for our purpose.

References