The Effect of the Pion-Pion Interaction on the Pion-Nucleon Scattering

Kin-ichi ISHIDA

Research Institute for Fundamental Physics, Kyoto University, Kyoto*

(Received January 11, 1961)

It is suggested that the low-energy pion-nucleon p-wave phase shifts will be well understood by taking account of the effects of not only the pion-pion P-wave interaction but also the pion-pion S-wave interaction into the equations for the p-wave scattering amplitudes derived by Chew, Goldberger, Low and Nambu.

§ 1. Introduction

Chew, Goldberger, Low and Nambu (CGLN) have derived the equations for the partial wave amplitudes of the pion-nucleon scattering and have shown that they nearly reproduce the experimental p-wave phase shifts at the low energies, except that the theoretical $|\alpha_{11}|$ is about 3~4 times as large as the experimental one. According to the investigations by many authors, however, the effect of the pion-pion interaction** $(\pi+\pi\rightarrow N+N)$ has been neglected there. Recently, Bowcock, Cottingham and Lurie have shown that the pion-pion P-wave interaction has the effects to make $\alpha_{11}$ and $\alpha_{31}$ close to each other at the low energies and to make $\alpha_{11}$ larger than $\alpha_{31}$ at the intermediate energies, which are consistent with the experiments.

In this paper, we will show that the introductions of not only the pion-pion P-wave interaction but also the pion-pion S-wave interaction are rather favorable for understanding the pion-nucleon p-wave phase shifts at the low energies. The reason is as follows: The pion-pion S-wave interaction produces an isospin-independent pion-nucleon interaction, then if one assumes it to be attractive, one can make the rather smaller p-wave phase shifts predicted by CGLN plus the correction from the pion-pion P-wave interaction close to the experiments. In §2 we will reproduce the p-wave phase shifts by the method introduced by CGLN and describe the corrections caused from the possible pion-pion interactions. In §3 we will discuss how the p-wave phase shifts predicted by CGLN can be made close to the experimental ones by the assumed effects of the pion-pion interactions.

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* On leave from Faculty of Liberal Arts and Science, Yamagata University, Yamagata.

** This has been naturally introduced in order to explain the electromagnetic structure of nucleon and also has been shown to be very suitable to understand the second and third resonance-like behaviors of the pion-nucleon scattering. See references 4) and 5).
§ 2. Theory of CGLN and the corrections from the pion-pion interactions

We will first reproduce the pion-nucleon $p$-wave scattering amplitudes, following the procedure given by CGLN. They can be derived from the following equations:

\[
\frac{f_{\pi\Lambda}^{(\pm)}}{q^2} = \frac{2}{3} \frac{E+M}{2W} \left\{ (A^{(\pm)'}(W^2, 0) - 2q^2 A^{(\pm)''}(W^2, 0)) + (W-M)(B^{(\pm)'}(W^2, 0) - 2q^2 B^{(\pm)''}(W^2, 0)) \right\} / 4\pi, \tag{2.1}
\]

\[
\frac{f_{\pi\Lambda}^{(\pm)} - f_{\pi\Lambda}^{(\mp)}}{q^2} = \frac{1}{E+M} \left\{ (B^{(\pm)}(W^2, 0) - 2q^2 B^{(\pm)'}(W^2, 0)) / 4\pi, \tag{2.2}
\right.
\]

where the superscript ' represents the derivative with respect to $t (= -4\kappa^2)$, at $t=0$.

In (2.2), very small contributions from the pion-nucleon $s$- and $d$-waves to the left-hand side have been neglected. Here, we will examine to use the different equation (2.2) from the corresponding one of CGLN, for, if one used this instead of (2.2), it would be necessary to use the dispersion relation for $A^{(\pm)}(W^2, 0)$, which, as to the convergence of the related dispersion integral, is the worst of the $A^{(\pm)}(W^2, 0)$, $B^{(\pm)}(W^2, 0)$ and these derivatives, and further, extra information about the effect of the pion-pion interaction on the $A^{(\pm)}(W^2, 0)$ would later become necessary. By the use of the dispersion relations for $A^{(\pm)}$ and $B^{(\pm)}$ given by CGLN, from (2.1) and (2.2) we have the $p$-wave scattering amplitudes as follows:

\[
\text{Re} f_{33}/q^2 = \frac{4}{3} \frac{f^2}{\omega} + \frac{1}{\pi} \int_1^\infty \frac{d\omega'}{1+\omega'/M} \left( \frac{E+M}{E'+M} \right) \Im f_{33}(\omega') \frac{1}{q^2(\omega'-\omega)}
\]

\[
+ \frac{1}{9\pi} \int_1^\infty \frac{d\omega'}{q^2(\omega'+\omega)},
\]

\[
\text{Re} f_{33}/q^2 = -\frac{2}{3} \frac{f^2}{\omega} + \frac{4}{9\pi} \int_1^\infty \frac{d\omega'}{q^2(\omega'+\omega)},
\]

\[
\text{Re} f_{33}/q^2 = -\frac{2}{3} \frac{f^2}{\omega} - \frac{f^2}{M} + \frac{1}{\pi} \int_1^\infty \frac{d\omega'}{q^2} \Im f_{33}(\omega') \left( \frac{4}{9} \frac{1}{\omega'+\omega} + \frac{2}{3M} \right),
\]

\[
\text{Re} f_{33}/q^2 = \frac{8}{3} \frac{f^2}{\omega} + \frac{2f^2}{M} + \frac{1}{\pi} \int_1^\infty \frac{d\omega'}{q^2} \Im f_{33}(\omega') \left( \frac{16}{9} \frac{1}{\omega'+\omega} + \frac{2}{3M} \right).
\]

(2.3)

* In this paper, the notations of CGLN will be used, unless specially mentioned, and the conventional units $\hbar = c = 1$ and pion mass $\mu = 1$ will be used.
These equations are derived by expanding the various terms which appear in the original equations with respect to \( \omega/M \) and \( \omega'/M \) and by neglecting these second orders. The last two equations in (2.3) are different from the corresponding ones of CGLN in the order of \( f^2/M \). But, if one remembers the approximate relation \( f^2 \approx 2/3 \pi \int_0^{\infty} d\omega' \text{Im} f_{\mu\alpha}(\omega')/q^3 \), these differences are found to be not so numerically distinctive.*

Next, we will describe the contributions from the pion-pion interactions. The S- and P-wave pion-pion interactions contribute to \( A^{(\pm)}(W^2, t) \) and to \( A^{(-)}(W^2, t) \) and \( B^{(-)}(W^2, t) \), respectively. It will be found that there appear no contributions to the \( B^{(\pm)}(W^2, t) \) as long as we omit the higher partial waves other than S- and P-waves. Then, from (2.1) and (2.2), the pion-pion contributions will be written as

\[
\delta(f_{\mu\alpha}/q^2) = \alpha(q),
\delta(f_{\mu\alpha}/q^2) = \epsilon(q),
\delta(f_{\mu\alpha} - f_{\mu\alpha}^{(-)})/q^2 = b(q),
\delta(f_{\mu\alpha}^{(\pm)} - f_{\mu\alpha}^{(-)})/q^2 = 0,
\]

where \( \alpha(q) \) represents the contribution from the S-wave pion-pion interaction and \( \epsilon(q) \) and \( b(q) \) represent the contributions from the P-wave pion-pion interaction. Then, using the relations

\[
f^{(\pm)} = \frac{1}{3} (f^{(1)} + 2f^{(3)}), \quad f^{(-)} = \frac{1}{3} (f^{(3)} - f^{(3)}),
\]

the contribution to each \( p \)-wave amplitude becomes as follows:

\[
\delta(f_{\mu\alpha}/q^2) = \alpha(q) - \epsilon(q), \quad \delta(f_{\mu\alpha}/q^2) = \alpha(q) + 2\epsilon(q),
\delta(f_{\mu\alpha}/q^2) = \alpha(q) - b^*(q), \quad \delta(f_{\mu\alpha}/q^2) = \alpha(q) + 2b^*(q),
\]

\[
b^*(q) = b(q) + \epsilon(q),
\]

which may be summed up as

\[
\delta = 3(p \cdot q)\alpha(q)\delta_{\alpha\beta} + [3(p \cdot q)\epsilon(q) + (\sigma \cdot p)(\sigma \cdot q)b(q)]\tau_{\alpha\beta}/2,
\]

where \( q \) and \( p \) are the initial and final pion momenta and \( \beta \) and \( \alpha \) are the initial and final pion isospin indices. In the next section, we will try to estimate these parameters \( \alpha(q) \), \( b^*(q) \) and \( \epsilon(q) \) by comparing with the experiments at the low energies.

* This will be very natural from the following consideration. The difference between (2.2) and that of CGLN is essentially in whether one uses the experimental or the theoretical s-wave scattering amplitudes. (And in (2.2) the contributions from the s-waves have been neglected, as they are estimated to be very small by using the experimental s-wave amplitudes.) However, the theoretical s-wave amplitudes predicted by CGLN have been shown to nearly reproduce the experiments without the pion-pion effect. Then, the results should be nearly equal to each other.
§ 3. Numerical estimation of the pion-pion effects. Discussion

We shall here restrict our attention to the pion-nucleon scattering within the low energy limit. The $p$-wave phase shifts predicted by (2·3) are numerically calculated as is shown in Table I, one taking the coupling constant $f^2 \approx 0.08 \pm 0.01$. The experiment reported by Puppi in the 1958 Annual International Conference on High Energy Physics at CERN is also listed in Table I. Owing to the large experimental inaccuracy, in addition to the limitations on the theory of Cini and Fubini applied here, one might not be able to derive so decisive conclusions on the pion-pion effects, but we wish to derive as much information on these effects as possible.

Table I. The pion-nucleon $p$-wave phase shifts at the low energies are calculated by using (2·3). The experiment is taken from the report by Puppi in the 1958 Annual Conference at CERN.

<table>
<thead>
<tr>
<th>$f^2$</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
<th>experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{33}/q^3$</td>
<td>0.196</td>
<td>0.210</td>
<td>0.223</td>
<td>0.234±0.019</td>
</tr>
<tr>
<td>$\sigma_{31}/q^3$</td>
<td>-0.035</td>
<td>-0.041</td>
<td>-0.048</td>
<td>-0.039±0.022</td>
</tr>
<tr>
<td>$\sigma_{31}/q^3$</td>
<td>-0.037</td>
<td>-0.045</td>
<td>-0.054</td>
<td>-0.044±0.005</td>
</tr>
<tr>
<td>$\alpha_{11}/q^3$</td>
<td>-0.110</td>
<td>-0.132</td>
<td>-0.157</td>
<td>-0.038±0.038</td>
</tr>
</tbody>
</table>

From Table I it can be seen that the $\alpha_{33}$ and $\alpha_{31}$ are consistent with experiment, but (i) the $|\alpha_{31}|$ is about 3-4 times as large as the experimental one, (ii) the $\alpha_{33}$ has a tendency to be smaller than the experimental one at least within $f^2 \approx 0.08 \pm 0.01$, though this is not so remarkable as (i). In this paper, we will assume these discrepancies to be due to the neglect of the effects of the pion-pion interactions. These discrepancies might be considered to be due to the neglect of the effects of the higher partial waves of the pion-nucleon scattering on the dispersion integrals, but these effects will be found to be approximately included in the assumed pion-pion effects at least if one assumes that the two-pion exchange interaction will be dominant at the low energies.

Table II. The possible pion-pion effects $a(q)$, $\epsilon(q)$ and $b^*(q)$ in (2·6) and (2·7) are estimated.

<table>
<thead>
<tr>
<th>$f^2$</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ from (2·6)</td>
<td>0.024±0.015</td>
<td>0.017±0.015</td>
<td>0.010±0.015</td>
</tr>
<tr>
<td>$a$ from (2·7)</td>
<td>0.019±0.013</td>
<td>0.032±0.013</td>
<td>0.046±0.013</td>
</tr>
<tr>
<td>$b^*$</td>
<td>0.026±0.013</td>
<td>0.031±0.013</td>
<td>0.036±0.013</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>-0.014±0.010</td>
<td>-0.007±0.010</td>
<td>-0.001±0.010</td>
</tr>
</tbody>
</table>
Comparing the \( p \)-wave phase shifts predicted by \((2\cdot3)\) with the experiment, we can then estimate \( a(q) \) and \( \epsilon(q) \) from \((2\cdot6)\) and can estimate \( a(q) \) and \( b^*(q) \) from \((2\cdot7)\). The two ways to estimate \( a(q) \) are, of course, due to the last equation of \((2\cdot4)\) which comes from the assumption of the \( S \)- and \( P \)-wave interaction dominance. The results are shown in Table II.

The necessity of the positive \( b^* \) can be found from \((2\cdot7)\), as the positive \( b^* \) can make the \( \alpha_{st} \) and \( \alpha_{tt} \) (which are predicted as \( \alpha_{st} > \alpha_{tt} \) in \((2\cdot3)\)) close to each other. But from this effect the \( \alpha_{st} \) becomes rather smaller than the experimental one, then the positive \( a(\approx b) \) becomes necessary, which is reflected in the third line of Table II. On the other hand, the positive \( a \) from \((2\cdot6)\) comes from the fact that, without this, the \( \alpha_{st} \) is rather smaller than the experimental one. As to the \( J=3/2 \) states only (\( d \pi J=1/2 \) states), if one properly took the coupling constant \( f^2 \) as the order of 0.10 (or 0.06), one could nearly reproduce the experiments without the pion-pion \( S \)-wave effect \( a(q) \), but then one could not reproduce the experiments of the other \( J=1/2 \) states (or \( J=3/2 \) states). This circumstance is illustrated in Fig. 1. From Fig. 1 we can estimate the coupling constant \( f^2 \) and the pion-pion \( S \)-wave effect \( a(q) \) so as to coincide the \( a(q) \) from \((2\cdot6)\) with the \( a(q) \) from \((2\cdot7)\). Then the results become as follows:

\[
f^2 = 0.073 \pm 0.013, \quad a = 0.022 \pm 0.013. \tag{3\cdot1}
\]

Thus, the attractive isospin-independent interaction \( a(q) > 0 \) is found to be very favorable for understanding the low-energy \( p \)-wave phase shifts. Further, the above estimated \( f^2 \) is found to be close to the usual \( 0.08 \pm 0.01 \), which is considered to support our assumption of the pion-pion \( S \)- and \( P \)-wave interactions dominance. We can then calculate \( b^*(q) \) and \( \epsilon(q) \) tentatively by taking \( f^2 = 0.073 \) as

\[
b^* = 0.028 \pm 0.013, \quad \epsilon = -0.012 \pm 0.010, \tag{3\cdot2}
\]

where the error in \( b^* \) arises almost from the large inaccuracy in \( \alpha_{tt} \), but the \( a - b^* \) will be fairly accurately determined from the data of \( \alpha_{tt} \) within \( \pm 0.005 \). These \( b^* \) and \( \epsilon \) are consistent with the values calculated by Bowcock et al. in reference 3), where the pion-pion \( P \)-wave resonant interaction is assumed. The pion-pion \( S \)-wave effect \( a(q) \) in this paper is about 1/2 of the one previously estimated from the different method.\(^6\) Hence, the pion-pion \( S \)-wave scattering
length \( \alpha_s \) will be revised as the order of \( 1/2 \mu \) which is one half of the one previously estimated. However, these values of \( \alpha_s \) might not be so reliable by the following two reasons: (i) In the treatment of the integral equation for \( \pi^+\pi^-\rightarrow N+N \) in reference 6), the imaginary part of the pion-nucleon scattering amplitude has been approximated by the partial waves for \( t \gtrsim 4\mu^2 \) as has been made in the treatment of the electromagnetic structure of the nucleon by Federbush et al.,\(^6\) which may, however, be a drastic assumption in view of the treatment of the electromagnetic structure of the nucleon by Frazer and Fulco,\(^4\) and (ii) in our "phenomenological treatments", the assumed attractive force will essentially represent all the contributions other than the one from the dominant (3-3) state to the dispersion relation for \( A^{1+} \).

We will finally try to estimate the pion-pion corrections at the intermediate energies, for instance, at \( q=1 \) so as to reproduce the behaviors of the small phase shifts given by Chiu and Lomon,\(^7\) taking \( f^2 = 0.08 \). It then follows that

\[
\begin{align*}
  a(q) & \simeq 0.028, \quad b^*(q) \simeq 0.015, \quad \epsilon(q) \simeq +0.007, \\
  \alpha_{13} \simeq +0.015 q^3, \quad \alpha_{11} \simeq \alpha_{31} \simeq -0.018 q^2 \quad \text{at} \quad q=1.
\end{align*}
\]  

From these, \( a(q) \) and \( b^*(q) \) may be slowly varying functions with the energies. However, in order to reproduce the positive \( \alpha_{13} \) at the intermediate energies, it will be convenient if \( \epsilon(q) \) rapidly increases from negative to positive as \( q \) goes to the intermediate energies. At any rate, from our analysis, it can be found that the assumed effects of the pion-pion \( S \)- and \( P \)-wave interactions are very favorable for understanding at least the low-energy \( p \)-wave pion-nucleon scattering. In order to see the pion-pion effects on the partial waves other than the \( p \)-waves, however, one has to know the explicit structures of these effects on the invariant scattering amplitudes \( A^{(\pm)} \) and \( B^{(\pm)} \). Here, we have not referred to this problem.

The author wishes to thank Professors K. Nakabayasi and I. Sato for their interest shown in this work and continual encouragement. He wishes to thank also the members of Kyoto and Tohoku Groups for their valuable discussions.

References

4) K. Ishida, Prog. Theor. Phys. 119 (1960), 1429, etc.