Quantum theory of point-like systems is established by extending the concept of relativistic particle in some respects: A point-like system means a one-parameter series of events \( x_p(t) \) with substantial internal degrees of freedom concentrated upon \( x_p \), and indefinite metric in Hilbert space is generally taken as to the internal degrees. The theory corresponds to an extension of the usual local field equations, suitable to obtaining a unified theory of elementary particles. The rest-mass, \( m^2 = -p_p^2 \) (with \( p_p \) as momentum-energy vector) becomes a dynamical quantity of the system with its possible eigenspectrum, leading to uncertainty relations between rest mass value and space-time localization. The internal angular momentum tensor \( S_{pq} \) is another basic dynamical quantity of the system and is responsible for spin and Zitterbewegung. Also defined is the instantaneous velocity operator \( v_p \), which is not generally colinear with \( p_p \) and must be restricted by certain kinematical conditions. Three different criteria about these conditions on \( v_p \) make point-like systems classified into various types. For "normal class" of systems, \( \rho = -v_p^2 \) is an absolute invariant with eigenvalue 1 or 0 and is regarded to represent baryon number. Especially important are point-like systems of the first kind, i.e. the ones in which \( v_p \) commute with the position \( x_p \) and thus mean internal variables. Such a system generally has, besides rest mass, spin and \( \rho \), three self-adjoint commuting invariant quantities formed out of \( p_p, v_p \) and \( S_{pq} \) only, which are to be identified eventually with the intrinsic properties of elementary particles (isospin, hypercharge, etc.). Systems are further divided into "classical models", where velocity components are commutable (an example being relativistic rotator), and "non-classical models" where they are not \( \{v_p, v_p\}_{\neq 0} \), to derive general characteristics for each of them. Dirac and Kemmer particles are special simple examples of the latter, where system has no substantial internal degrees of freedom apart from \( v_p \).

It thus becomes a matter of great importance to set up new dynamical systems and see if they will better describe the atomic world.

P.A.M. Dirac

§ 1. Introduction

The usual theory of elementary particles is based upon field equations corresponding to originally structureless point particles with arbitrary rest mass values with the hypothesis that an elementary particle cannot have a substantial internal structure apart from that due to interactions in so far as it must be a geometrical point. On such hypothesis possible relativistic wave equations are enumerated according to irreducible representations of the Lorentz group, e. g. by the method of Dirac, Fierz and Pauli or that of Wigner. Each of such wave equations is suitable to representing one kind of elementary particles with
definite spin and rest mass values. This kind of method, however, affords only a partial explanation for the existence of various elementary particles associated with that variety of apparently irreducible intrinsic properties.

On the other hand there are several lines to attempt at the unification of elementary particles. One of the lines is to find out the "unified equation" which describes not the elementary particles of a certain kind but all possible elementary particles. This will mean to renounce the above-mentioned hypothesis of structureless point particle in the ordinary theory and to assume a certain unique dynamical system with some sort of internal degrees of freedom in such a way that various elementary particles be reproduced as various possible forms of motion of that system which are mutually distinct with respect to its internal states. Introducing internal degrees of freedom within the framework of relativistic causality and quantum principles would raise a delicate problem but is by no means forbidden. Instead it will open ample possibilities beyond the conventional wave equations.

In this standpoint the wave function is a reducible representation as far as the usual Lorentz group is concerned, but each elementary-particle state corresponds to an irreducible representation with respect to the totality of the Lorentz group and the various transformation groups relevant to internal motions. This means on the one hand that spin and rest mass are not the parameters fixed for each system as in the usual theory but represent themselves dynamical quantities of a single basic system, taking their eigenvalue spectra. It means on the other hand that internal states of the system are not specified completely by rest mass and spin alone but we need some additional quantum numbers. Evidently, both points are necessary to have a unified theory along the present line; in particular the latter point is required to account for the intrinsic properties of elementary particles other than spin and rest mass (isospin, hypercharge, etc.).

Indeed a typical example of this kind of approach was given by Yukawa in his non-local theory. However, in order to proceed by a less radical extension of the conventional theory we will take the hypothesis that particle is essentially local but nevertheless maintains certain degrees of freedom of internal motion. Also we start with one-particle theory following the case of Dirac equation, which was historically found out as a relativistic Schrödinger equation for the state amplitude $\psi(x)$ for a single particle. From the original Dirac's standpoint the $x$ in $\psi(x')$ corresponded to position* of a particle. We extend such notion of particle in the following basic respects, and call it "point-like system", or "corpuscle".

(i) By point-like system we mean first of all a one-parameter series of events $x_n$ in the quantum-mechanical sense. An event is an elementary relativistic concept and is to be regarded as a set of observables $x_n$'s quantum-

* This is different from the position operator in the sense of Newton-Wigner.
mechanically, where $t = x_0$ is also an observable on the same footing as $x_\alpha$. They are observed by scale and clock, and assuming that these observations do not mutually interfere, we put

$$[x_\alpha, x] = 0. \quad (1)$$

(ii) A point-like system is, however, not merely events $x_\alpha$ but means a certain localizable dynamical system which otherwise has certain degrees of freedom* $\xi_\alpha$ describing certain internal motion concentrated upon the $x_\alpha$.

First we remark that since $t$ is itself an observable in our theory we need a time-ordering parameter $\tau$, which is essentially a $c$-number, in order to describe the time evolution of the system. We shall call it "intrinsic time" or "instant parameter" to distinguish it from the observable time $t$. Both times must be correlated in a certain manner, but the correlation itself may differ according to the characteristics of each point-like system.

The internal motion must be concentrated upon $x_\alpha$ since we assume that the real spatial extension of the system is reduced, indeed, to a single point $x_\alpha$. This assumption is required in order to be consistent with (1). Thus $x_\alpha$ cannot be supposed, for instance, to mean center of mass coordinates $X_\alpha$ of a certain extended continuous matter, since the latter components do not mutually commute:

$$[X_\alpha, X_\beta] \neq 0,$$

as long as the center of mass is defined to be independent of reference frames.*** The problem of how it is physically possible and consistent for a system to be point-like but nevertheless to maintain the degrees of freedom of internal motion (rotation, etc.) is investigated in detail in Part III, where we shall consider in particular the "condition of reality" and the "condition of original rest-mass null". In this paper, however, being content with the fact that the idealization of point-like system is consistent at least on formal grounds, we shall refer only casually to those conditions.

(iii) The third important point is that we generally admit an indefinite metric in Hilbert space, because once we introduce internal degrees of freedom we may allow negative probabilities for internal configurations which are not directly observable as such.

Now the first general property of our point-like systems is that the momentum-energy 4-vector $p_\alpha$ of the system become canonical conjugates of the position coordinates $x_\alpha$:

$$[x_\alpha, p_\beta] = i\hbar \delta_{\alpha \beta}. \quad (2)***$$

* The suffix $\alpha$ in $\xi_\alpha$ does not mean a vector suffix but simply runs over all internal degrees of freedom.

** Compare also with Eq. (74).

*** In this paper Greek suffixes range from 1 to 4, and imaginary time coordinate $x_t = ix_0 = it$ is used, together with the unit system where $c = 1$. Latin suffixes run over 1 to 3. Summation convention is applied to repeated indices unless otherwise stated.
This covariant commutation relations contain, besides the usual commutation rule between coordinate and momentum, the relation

\[ [x_t, p_t] = -[x_0, p_0] = -[t, E] = i\hbar. \]  

(3)

Thus time \( t = x_0 \) and energy \( E = p_0 = p_0 / i \) are regarded as canonical conjugates like \( x_k \) and \( p_k \) (apart from negative sign).

That this is possible is related to the previously mentioned fact that in our theory the rest mass \( m \) (or more exactly the squared rest mass \( P \)) is not a fixed c-number parameter but is rather a dynamical quantity that is defined by the very relation

\[ -p^2 = P = m^2, \]  

(4)

(which is of course a constant of motion for a closed system), and can take any possible value (i.e. the eigenvalues) as the result of the "unified" wave equation, and (4) itself does not mean any condition. This is a situation different from the usual theory, where the rest mass \( m_0 \) appearing in the wave equation is a fixed parameter restricting the momentum and energy of the particle by

\[ -p^2 = E^2 - p^2 = m_0^2. \]  

(5)*

In our theory, on the other hand, all \( p_\mu \)'s represent independent quantities so that (2) does not involve any contradiction only if we regard \( t \) as dynamical variable on the same footing as \( x_k \). The commutation relation (3) accounts for the uncertainty relation between \( E \) and \( t \):

\[ \Delta E \Delta t \geq \hbar, \]

on the same basis as for \( \Delta p \Delta x \geq \hbar \).

Our equation (4) simply expresses the fact that the rest mass is usually measured through the equation \( m^2 = -p^2 = E^2 - p^2 \) in such state as is approximately a simultaneous eigenstate of \( (p_\mu, E) \). On the other hand if particle is sharply localized the rest mass is generally subject to certain indeterminacy due to the relations (2) and (4) (cf. Eq. (60)). This is a new effect but seems not to conflict with observational evidences.

The next important point for any point-like system is the existence of the internal angular momentum tensor \( S_{\mu\nu} = -S_{\nu\mu} \) as a direct consequence of (1) and (2), whose components are functions of internal variables of the system \( \varepsilon \) alone and work as generators of homogeneous Lorentz transformations for any internal variable, with definite algebraic properties. This \( S_{\mu\nu} \) produces the spin of the system as well as Zitterbewegung in the orbital motion \( x_\mu \) (§ 4). We thus have the set of basic quantities

* This is the relation to result from the wave equation and is to be regarded as subsidiary condition to be fulfilled for a permissible state. So it is better to be written as \( -p^2 = m^2 \), meaning the Klein-Gordon equation, \( (p^2 + m_0^2)\phi = 0 \). If one regards (5) as a relationship between dynamical quantities, energy \( E \) becomes a definite function of momentum \( p_\mu \) and it cannot satisfy \([E, x_\mu] = 0\) contained in (2), nor (3).
Internal Degrees of Freedom and Elementary Particles. I

\[(x_\mu, p_\mu, S_\mu)\]  \hspace{1cm} (6)

for any point-like system.

The time-evolution of the system is described, as stated already, by means of its intrinsic time \(\tau\). New independent quantities to be supplied by the intrinsic-time derivation of the dynamical variables (6) are “4-velocity” components:

\[v_\mu = \frac{dx_\mu}{d\tau} = \dot{x}_\mu\]

only, on account of the basic conservation laws (cf. § 5). In (relativistic) Newtonian mechanics one tacitly imposes a stringent condition that \(v_\mu\) be colinear with \(p_\mu\), but for our point-like system we generally have essential disparity between \(p_\mu\) and \(v_\mu\) in accordance with the existence of the internal degrees. Indeed \(v_\mu\) means the instantaneous velocity of the orbital motion which contains Zitterbewegung and is intimately connected with the internal motion of the system. Especially the (negative) relativistic magnitude of the \(v_\mu\)-vector,

\[\rho = -v_\mu^2, \quad \text{(i.e. } d\dot{x}_\mu^2 = -\rho d\tau^2)\]

represents a new scalar dimensionless quantity which is generally independent of the rest mass, \(P = -p_\mu^2\).

Thus we always have the set of basic variables

\[(x_\mu, p_\mu, S_\mu, v_\mu),\]  \hspace{1cm} (7)

irrespective of the properties of the pure internal variables \(\xi_a\), although \(S_\mu\)’s are certain functions of \(\xi_a\) (and \(v_\mu\)), and the interrelations and behaviours of the variables (7) determine already the gross characteristics of point-like systems (§§ 4 and 5). Naturally, in order to specify the system completely it is generally required to treat all internal degrees of freedom \(\xi_a\) explicitly,* whose kinematical and dynamical properties with their physical significances are essential in defining the system. Simply speaking, they are characterized by the structure of the self-adjoint invariant hamiltonian \(H\) of the system, which is different from \(p_\mu = E\) and means the intrinsic-time displacement operator determining the equation of motion for any dynamical quantity \(F\) in the “Heisenberg picture” by

\[i\hbar \frac{dF}{d\tau} = [F, H] + i\hbar \frac{\partial F}{\partial \tau}.\]

However, before entering into the precise treatment of the whole internal properties by the explicit use of \(\xi_a\), it is necessary to define the kinematical properties of \(v_\mu\).

* In some simple cases we need not treat \(\xi_a\) explicitly. This may happen when the independent internal degrees of freedom implied by \(\xi_a\) is less than six and can completely be represented by \(S_\mu\)’s.
The commutation relations within the set of the quantities (6) are universally given for point-like systems in general. On the contrary those concerning $v_\mu$ are not given in advance. Thus to have a definite theory we must always postulate certain inhomogeneous kinematical conditions on $v_\mu$ which determine the commutation rules concerning $v_\mu$ and fix the definition of $\tau$ at the same time. The condition must be such that the inequality

$$\langle v_\tau \rangle = \frac{d\langle t \rangle}{d\tau} > 0,$$

is ensured consequently, where $\langle \cdot \rangle$ designates the expectation value. (8) means that $\tau$ goes on monotonously with the expectation value of $t$ even though the reading of clock has a quantum-mechanical dispersion in general.

The possible form of the above envisaged condition for $v_\mu$ allows us just to classify point-like systems into various types. The following three criteria are important for this purpose.

(A) Whether or not $\rho = -v_\mu^2$ is an absolute invariant (i.e. commutable with all other quantities). We shall call the former case “the normal class”.

(B) Whether or not the commutators $[v_\mu, v_\nu]$ vanish identically.

(C) Whether or not the commutators $[v_\mu, x_\nu]$ vanish identically. We shall call the former case the “system of the first kind” and the latter “the second kind”.

Now, for normal class, since $\rho$ is practically a $c$-number, we can always normalize it such that $\rho = 1$ through a suitable redefinition of the parameter $\tau$, only if $\rho \neq 0$. Therefore $\rho$ is essentially a dichotomic variable which takes only two possible values, 1 or 0. Evidently, states belonging to $\rho = 1$ and $\rho = 0$ have inherently different internal properties, and for the latter some of the internal degrees of freedom necessarily degenerate. Between both states no transition is allowed. We may consider $\rho = 1$ and $\rho = 0$ as corresponding to baryon and lepton states, respectively ($\S$ 5).

Next let us consider the criterion (B). For the former case:

$$[v_\mu, v_\nu] = 0,$$

$v_\mu$ can be regarded as the usual 4-velocity and then $\tau$ becomes a proper time in the usual sense. This case corresponds to corpuscles with realistic internal structure in general, having its direct classical analogue. We therefore call them “classical models”, although the treatment is always made quantum-mechanically. On the other hand we call a system for which $[v_\mu, x_\nu] \neq 0$ “non-classical model”.

Finally we consider the criterion (C). For example, a Newtonian particle belongs to the second kind because $[v_\mu, x_\nu]$ does not vanish on account of the direct connection between $v_\mu$ and $p_\mu$ together with the basic commutation relation (2).

Our main interest, however, is placed on systems of the first kind. Here velocity components $v_\mu$ become parts of the internal variables of the system. Also, the third scalar quantity to be formed of $p_\mu$ and $v_\mu$: ...
\[ \varepsilon = -v_p p, \]

(the first being \( P = -p^2 \) and the second \( \rho = -v^2 \)), plays an essential role for the motion. This is because the invariant Hamiltonian for such system has the general form

\[ H = -\varepsilon + \mathcal{K} = p + \mathcal{K}. \]  \hspace{1cm} (9)

Here \( \mathcal{K} \) is a scalar function of the internal variables \( (S, v, \xi) \) alone, while the term \(-\varepsilon\) means the existence of the coupling between internal and external motions. The \( \varepsilon \) and \( \mathcal{K} \) are not separately conserved in general.

In this case the Schrödinger equation,

\[ i\hbar \frac{d\psi}{dz} = H \psi = (p + \mathcal{K}) \psi, \]  \hspace{1cm} (10)

becomes a linear differential equation with respect to \( x_p \) since \( p \rightarrow \hbar/\partial \partial x_p \) due to (2). The properties of (10) depends on the kinematical characters of \( v_p \) already mentioned and the structure of \( \mathcal{K} \), but with a suitable prescription for them Eq. (10) has appropriate properties to comprise different elementary-particle states by its eigen-solutions. Then Eq. (10) means our “unified equation”. It refers to one-particle states, but it provides the starting point for proceeding to the treatment of interactions by means of the subsequent second-quantization or other procedures. Such approach has an evident advantage that the theory maintains particle character clearly.

In § 6 we deduce general properties of the first kind systems, and especially we conclude the existence of six mutually commuting conserved invariant quantities which are formed of the variables \( (p, S, v) \) alone. They include, besides rest mass and the magnitude of spin, four quantities that may eventually be identified with baryon number, magnitude of isospin, and hypercharge.

Evidently, systems of the first kind are divided, according to the criterion (B), into two categories, classical and non-classical models. A typical example* belonging to the former is the relativistic rotator given in a previous paper.\(^5\) On the other hand an important type of non-classical models is defined by the condition

\[ [v_p, v] = \frac{i}{\hbar} S_p. \]

It includes as special examples various familiar systems such as Bhabha particles.\(^6\)

The precise treatment of both of the above types will be given in Part II and III. The main purpose of this paper (Part I) is to set up the notion

\* Various examples are investigated in T. Takabayasi, Soryusiron-Kenkyu (Mimeographed circular in Japanese) 21 (1960), 723.
of point-like system and establish the general foundation of theory in order to clarify what will generally emerge, once we introduce substantial internal degrees of freedom into relativistic quantum mechanics in the level of one-particle theory.

§ 2. Theoretical framework

First in this section we outline the general foundation of the theory of point-like systems. As stated in the Introduction, it means a one-parameter series of events $x_\alpha(\tau)$ satisfying $[x_\alpha(\tau), x_\beta(\tau)]=0$, associated with possible internal degrees of freedom $\xi_\alpha$.

2a. Let us consider in the "Schrödinger picture", where $x_\alpha$ and other dynamical quantities do not depend on $\tau$ while the state vector $\psi$ changes with $\tau$. If we take the representation in which $x_\alpha$ are diagonal, $\psi$ is represented as a function

$$\psi(x_\alpha, \theta_\alpha, \tau),$$

where $\theta_\alpha$ signifies the set of independent parameters* corresponding to the variables $\xi_\alpha$ of internal motion (rotation, etc.) of the corpuscle. We are dealing not only with the external motion (represented by $x_\alpha$) but also with the internal motion at the same time, but for the latter the internal configuration $\theta_\alpha$ cannot be measured as such and hence we may admit negative probabilities for them. Thus, introducing an indefinite metric in general, we assume that the normalization is given by

$$\psi^* \gamma \psi,$$

where $\gamma$ is a $\tau$-independent hermitian operator:

$$\gamma = \gamma^*, \quad \dot{\gamma} = 0,$$

and is related to internal variables alone. In the representation of (11),

$$\phi^*(x_\alpha, \theta_\alpha, \tau) \gamma \psi(x_\alpha, \theta_\alpha, \tau) D$$

means the (hypothetical) relative probability density that the corpuscle appears as an event at $(x_\alpha, x_\beta=1)$ with the internal configuration around $\theta_\alpha$ at an intrinsic time $\tau$. The factor $D$ means the weight function for the parameters $\theta_\alpha$. Accordingly, the expectation value of any observable $F$ is given by

$$\langle F \rangle = \phi^* \gamma F \psi = \int \phi^*(x_\alpha, \theta_\alpha, \tau) \gamma F \psi(x_\alpha, \theta_\alpha, \tau) d^3x dt D d\theta_\alpha.$$

The adjoint of an operator $F$ is defined by

* The $\xi_\alpha$ themselves may be subject to some constraints in general.
Clearly, 

\[ (F')^\dagger = F, \quad (FG)^\dagger = G^\dagger F^\dagger. \]

A physical variable must correspond to self-adjoint operator, since its expectation values must be real. Since \( \gamma \) is independent of \( x_\rho \), we have 

\[ [x_\rho, \gamma] = 0, \]

so that for \( x_\rho \) the hermitian conjugate and the adjoint are identical: \( x_\rho^\dagger = x_\rho^* \). On the other hand, \( x_\kappa^* = x_\kappa \), \( x_4^* = -x_4 \). We shall call a \( q \)-number vector "self-adjoint vector" if its space components are self-adjoint while its time component is anti-self-adjoint. The expectation value of such a vector is real vector (namely a vector with real space components and pure-imaginary time component). Evidently, \( x_\rho \) is a self-adjoint vector.

Now a state vector \( \psi \) develops with \( \tau \) by the Schrödinger equation:

\[ i\hbar \frac{d\psi^{(s)}}{d\tau} = H^{(s)} \psi^{(s)}. \]  

(14) *

Since \( \tau \) is a real scalar parameter, the invariant hamiltonian \( H^{(s)} \) must be a scalar, and self-adjoint:

\[ \gamma H = H^* \gamma. \]

(15)

The latter fact is necessary in order that the norm (12) be conserved with \( \tau \), as is verified by employing (14) and its adjoint:

\[ -i\hbar \frac{d\psi^{(s)}_{\dagger}}{d\tau} = \psi^{(s)}_{\dagger} H^{(s)}, \quad (\psi^s = \psi^s_\gamma). \]

(16)

The invariant hamiltonian represents the structure of the point-like system and we assume that it does not involve \( \tau \) explicitly for closed system. Under this assumption there still remains certain arbitrariness in the definition of \( \tau \) which will be discussed later. The evolution of the expectation values of a not explicitly \( \tau \)-dependent quantity \( F \) is given by

\[ \frac{d}{d\tau} \langle F \rangle = \frac{d}{d\tau} \left( \psi^{(s)}_{\dagger} F^{(s)} \psi^{(s)} \right) = \frac{i}{\hbar} \psi^{(s)}_{\dagger} (H^{(s)} F^{(s)} - F^{(s)} H^{(s)}) \psi^{(s)} \]

\[ = \frac{i}{\hbar} \langle HF - FH \rangle. \]

(17)

A conserved quantity (or constant of motion) in our theory is a quantity whose expectation value for any state does not change with \( \tau \); i.e.

* In this section we associate the index S or H to distinguish between the quantities in Schrödinger picture and those in Heisenberg picture wherever confusion may arise.
\[ \frac{d}{dz} \langle F \rangle = 0, \quad \text{so} \quad \frac{d\langle f(F) \rangle}{dz} = 0. \]

Eq. (17) means that if \( F \) commutes with \( H \) and is not explicitly \( \tau \)-dependent it is a conserved quantity. Clearly, \( H \) is a conserved quantity for closed system.

2b. Next we consider a linear transformation in the Hilbert space which leaves the norm (12) as well as the hermiticity of \( \gamma \) invariant:

\[
\psi' = A\psi, \\
\gamma' = A^\dagger \gamma A^{-1}, \quad \text{so} \quad \phi' = \phi A^{-1}. \tag{18}
\]

At the same time an observable transforms by

\[
F' = AF A^{-1} \tag{19}
\]

to leave expectation values invariant. We shall call such transformation "canonical" if \( A \) is restricted by the condition

\[
A' A = AA' = 1, \tag{20}
\]

which means

\[
A^* \gamma A = \gamma, \quad \text{i.e.} \quad \gamma' = \gamma \text{= invariant.} \tag{20'}
\]

In this case

\[
(F')^\dagger = AF A^{-1} = (F')',
\]

so that the self-adjoint (or anti-self adjoint) property of any observable is maintained by a canonical transformation. If \( A \) is self-adjoint \( A = e^{i\mathcal{H}} \) means a canonical transformation.

The theory of 2a can now be transferred to the "Heisenberg picture" by a canonical transformation with

\[
A = e^{(i/\hbar)\mathcal{H} \tau}, \tag{21}
\]

namely,

\[
\psi^{(\text{II})} = e^{(i/\hbar)\mathcal{H} \tau} \psi^{(\text{I})}, \\
\gamma^{(\text{II})} = \gamma^{(\text{I})}, \\
F^{(\text{II})} = e^{(i/\hbar)\mathcal{H} \tau} F^{(\text{I})} e^{-(i/\hbar)\mathcal{H} \tau}, \quad (H^{(\text{II})} = H^{(\text{I})}). \tag{22}
\]

We then have

\[
\dot{\psi}^{(\text{II})} = 0, \quad \dot{\gamma}^{(\text{II})} = 0, \tag{23}
\]

and

\[
\dot{F}^{(\text{II})} = \frac{i}{\hbar} [H, F^{(\text{II})}], \tag{24}
\]

for a not explicitly \( \tau \)-dependent quantity \( F \). Eqs. (23) and (24) reproduce (17):
From $x_\mu$ we can always define the 4-velocity $v_\mu$ by (24):

$$v_\mu = \frac{dx_\mu}{d\tau} = \frac{i}{\hbar} [H, x_\mu].$$

(25)

Since $x_\mu$ is a self-adjoint vector, so is $v_\mu$. However,

$$[v_\mu, \gamma] \neq 0$$

in general, hence $v_\mu$ and $v_\nu = v_\lambda / i$ are not necessarily hermitian.

Next we consider inhomogeneous Lorentz transformations. If we take the “Schrödinger standpoint”, $\psi$ changes under such transformation by a canonical transformation:

$$\psi' = A \psi,$$

$$A' A = A A' = 1,$$

so

$$\gamma' = \gamma = A^* \gamma A,$$

(26a)

while physical quantities remain unchanged:

$$F = \text{invariant.}$$

(26b)

Evidently the Lorentz invariance of the norm (12) and of the self-adjoint characters of physical quantities is ensured by (26). On the other hand, from the “Heisenberg standpoint” the same situation is represented by

$$\phi = \text{invariant,} \quad \gamma = \text{invariant,}$$

$$F' = A \Gamma A^{-1},$$

(27)

with the same $A$. The transformations of expectation values are identical in both cases. In any case $A$ forms a representation (satisfying the canonical condition (20)) of the inhomogeneous Lorentz group, and the set of infinitesimal operators corresponding to $A$ defines the components of momentum-energy vector, $p_\mu$, and of the total angular momentum skew-tensor around the origin, $J_\mu$, of the system. They satisfy the well-known commutation relations for the generators of the inhomogeneous Lorentz group:

$$[p_\mu, p_\nu] = 0,$$

$$[J_\mu, p_\nu] = i \hbar \delta_{\mu \nu} p_\nu,$$

$$[J_\mu, J_\nu] = i \hbar (\delta_{\mu \nu} J_\rho + \delta_{\mu \rho} J_\nu).$$

(28)

(29)

(30)

First we consider about Lorentz transformations: $x_\mu' = a_\mu x_\mu$. The expectation values of any scalar quantity $S$ and of any vector quantity $V_\mu$ must, respectively, transform such that

$$\langle S \rangle' = \phi^* \gamma S \phi' = \phi^* \gamma A S A^{-1} \phi$$

$$= \langle S \rangle = \phi^* \gamma S \phi,$$
\[ \langle V_s \rangle' = \psi' \eta V_s \psi' = \psi \eta V S A^{-1} \psi \]
\[ = a_{a_s} \langle V_s \rangle = a_{a_s} \eta \psi V_s \psi, \]

(where we are considering from the Schrödinger standpoint), so that \( A \) satisfies
\[
\begin{align*}
ASA^{-1} &= S, \\
AV_s A^{-1} &= a_{a_s} V_s,
\end{align*}
\]

and analogous relations for tensor quantities. If we consider an infinitesimal rotation in the \((\rho \sigma)\)-plane:
\[ a_{(\rho \sigma)} = \delta_{\rho \sigma} + \epsilon (\delta_{\rho \sigma} \delta_{\rho \sigma} - \delta_{\rho \sigma} \delta_{\rho \sigma}), \]

the corresponding operator is written as
\[ A^{(\rho \sigma)} = 1 + \frac{i}{\hbar} \epsilon J_{\rho \sigma}. \]

Taking into account that \( \epsilon \) is real if \((\rho \sigma) = (ij)\) but is pure imaginary if \((\rho \sigma) = (k4)\), the canonical condition (20) leads to
\[ J_{ij} \eta = \eta J_{ij}, \quad J_{k4} \eta = -\eta J_{k4}, \]

namely \( J_{\rho \sigma} \) has self-adjoint space components and anti-self-adjoint space-time components. We call such tensor the "self-adjoint tensor". On the other hand the relations (31) yield the commutation rules:
\[
\begin{align*}
[ [J_{\rho \sigma}, S] &= 0, \\
[ [J_{\rho \sigma}, V_p] &= i \hbar \delta_{\rho \sigma} V_p. 
\end{align*}
\]

The latter is the generalization of (29).

Next we consider space-time displacement by an amount \( a_{\rho \sigma} \). From the fact that \( x_{\rho \sigma} \) are observables representing the space-time location of the particle, the corresponding transformation matrix in the Hilbert space, \( A \), must satisfy
\[ A x_{\rho \sigma} A^{-1} = x_{\rho \sigma} - a_{\rho \sigma}. \]

On the other hand \( A \) is written
\[ A = 1 + \frac{i}{\hbar} a_{\rho \sigma} p_{\rho \sigma}, \]

if we assume the displacement \( a_{\rho \sigma} \) to be infinitesimal. Eq. (35) together with (36) gives the canonical commutation relation formerly mentioned:
\[ [x_{\rho \sigma}, p_{\rho \sigma}] = i \hbar \delta_{\rho \sigma}, \]

which is characteristic for point-like systems in general. Like \( x_{\rho \sigma} \), the \( p_{\rho \sigma} \) mean external variables, hence
\[
\begin{align*}
[ [p_{\rho \sigma}, \gamma] &= 0, \\
p_{\rho \sigma}^\dagger &= p_{\rho \sigma}^* = g(\rho) p_{\rho \sigma},
\end{align*}
\]
with
\[
g(\mu) = \begin{cases} 
1 & \text{for } \mu = 1, 2, 3, \\
-1 & \text{for } \mu = 4.
\end{cases}
\]

Finally, an intrinsic-time displacement is independent of translations and Lorentz transformations for a closed system and this means
\[
\begin{aligned}
p_\mu &= 0, \\
\dot{p}_\mu &= 0.
\end{aligned}
\]
These represent in our theory the covariant conservation laws of the total momentum-energy and the total angular momentum.

2d. Particularly simple are "stationary states", which are the eigenstates of \( H \) given by
\[
H \psi = \lambda \psi, \quad \langle \psi | H | \psi \rangle = \lambda \langle \psi | \psi \rangle.
\]
Since \( H \) is self-adjoint, \( \lambda \) is real provided that the state is not "isotropic" (i.e. \( \psi^* \eta \psi \neq 0 \)), and then the state vector has the simple \( \tau \)-dependence:
\[
\psi^{(\tau)}(\tau) = e^{-\mu(H\lambda \tau)} \psi^{(0)}(0).
\]
Accordingly, the expectation value of any not explicitly \( \tau \)-dependent quantity \( F \), \( \langle F \rangle_\tau \), remains unchanged with \( \tau \). Thus
\[
\langle \hat{F} \rangle_\tau = 0.
\]
This theorem, however, does not apply to an unbounded operator such as \( x_\tau \), so that \( \langle \hat{x} \rangle_\tau = \langle v_\tau \rangle_\tau \) does not necessarily vanish.* This fact is important especially for \( \langle dt/d\tau \rangle_\tau = \langle v_0 \rangle_\tau \). Indeed, \( \langle v_\tau \rangle_\tau \) must be positive definite in order that \( \tau \) really means a time-ordering parameter (cf. (8)). We have the general relation (cf. § 5):
\[
\langle v_\tau \rangle_\tau = \left\langle \frac{\mathcal{E}}{m^2 k} P_{\tau} \right\rangle_\tau.
\]

2e. The physical definition of \( \tau \), however, is fixed sufficiently only when we supplement the basic equations (14) or (24) with certain inhomogeneous kinematical conditions on \( v_\tau \). To make it explicit we here mention two simple possibilities.

(i) As mentioned in § 1, a simple possibility is
\[
\begin{aligned}
[v_\tau, v_\nu] &= 0, \\
\rho &= -v_\nu^2 = 1,
\end{aligned}
\]

* The situation is analogous to the case of the elementary non-relativistic theory of a free particle, where "stationary state" (eigenstate of the energy) includes, for instance, a plane wave \( e^{i(k \cdot x - \omega t)} \) with \( \omega = \hbar k^2/2m \). For this state the expectation value of the velocity \( dx/dt = p_i/m \) does not vanish but rather equals \( \hbar k_i/m \).
which define systems with classical analogue. In this case $\tau$ becomes proper time in the usual sense and $v_\mu$ becomes unitary velocity, its components taking continuous spectrum between $\pm 1$. Consistent with (41) we may put the condition

$$\langle v_0 \rangle \geq 1,$$

in accordance with the condition (8). The ordinary 3-dimensional velocity components $V_\alpha$ can be defined by

$$V_\alpha = \frac{v_\alpha}{v_0}, \quad v_0 = (1 + v_0^2)^{1/2}.$$

(ii) Another simple possibility for the kinematical condition on $v_\mu$ is given by

$$\{v_\mu, v_\nu\} = -2 \delta_{\mu\nu}. \tag{42}$$

For this case we must take

$$\gamma = \frac{v_0}{i} = v_\mu/i \tag{43}$$

as the indefinite metric operator, so that the $v_\mu$ under (42) forms a self-adjoint vector, and also fulfills (8):

$$\langle v_0 \rangle = \psi^* \gamma v_0 \psi = \psi^* \psi \geq 0.$$

The solution of (42) is expressed as

$$v_\mu = i \gamma_{\mu\nu} \tag{44}$$

with the use of Dirac matrices $\gamma_{\mu\nu}$'s which can all be taken hermitian simultaneously, and then

$$\gamma = i \gamma_{\nu} \tag{43'}$$

(As a dynamical quantity $\gamma_\nu$ is constant in the Schrödinger picture which we denote $\gamma_\nu^{(0)}$, and $\gamma$ must be taken to equal this constant $\gamma_\nu^{(0)}$ even in the Heisenberg picture.) The velocity is quantized in this case and $\rho = 1$.

If we take $\mathcal{K} = 0$ in Eq. (10) for simplicity, the self-adjoint invariant hamiltonian becomes

$$H = -\varepsilon = v_\mu p_\mu = i \gamma_\mu p_\mu, \tag{45}$$

with the Schrödinger equation

$$i \hbar \frac{d\psi}{dt} = i \gamma_{\mu\nu} p_\mu \psi.$$

A corresponding stationary state to be given by

$$H \psi = i \gamma_{\mu\nu} p_\mu \psi = \lambda \psi \tag{46}$$

represents a Dirac particle, since (46) is exactly the Dirac equation with $\lambda$ for the rest-mass constant. In fact for the present case an eigenstate of $H$ is always

\* \{ \} means the anticommutator.
a simultaneous eigenstate of rest mass since the squared rest mass operator $P$ satisfies $P=H^2$ identically.

Various other possibilities for the condition on $v_\mu$ will be systematically considered in later sections.

§ 3. Preliminary considerations about conserved quantities

On the basis of the theoretical foundation given in § 2 we shall in this section summarize some preliminary considerations about conserved quantities of relativistic dynamical systems.

The existence of ten basic dynamical quantities $(p_\mu, J_\mu)$ satisfying the commutation rules (28)–(30) is merely the general consequence of special relativity, i.e. the invariance under inhomogeneous Lorentz transformations: We shall first give the consequences following therefrom. The particular situation of a point-like system is the fact that $p_\mu$ and $J_\mu$ (and other possible dynamical quantities $F$) are quantities assigned to one-particle state by the nature of the theory and that their conservation means $dF/d\tau=0$ (cf. (38)) instead of $dF/dt=0$. Those points, however, do not essentially influence the general arguments to follow.

3a. First we write

$$P \equiv -p_\mu^2 = m^2 = \text{const.}, \quad (4')$$

then $m$ represents the rest mass of the system, since it signifies the total energy in the reference frame in which $p_\mu=0$. Such state and frame exist, unless $m=0$, owing to (28), and means an inertial frame because of (38a). We call this frame "mean rest frame" and denote it as $I$.

Next we introduce the pseudovector

$$W_\mu = \vec{J}_\mu, \quad (W_\mu p_\mu = p_\mu \vec{J}_\mu = \text{const.}), \quad (47)$$

or

$$\omega_\mu = \frac{1}{m} \vec{J}_\mu, \quad (\omega_\mu p_\mu = \text{const.}), \quad (47')$$

provided that $m \neq 0$. Here the tilde signifies the dual for a skew-tensor, such that

$$\vec{J}_{\mu\nu} = \frac{1}{2i} \epsilon_{\mu\nu\lambda} J_{\lambda\kappa},$$

with the alternating symbol $\epsilon_{\mu\nu\lambda}$. The commutation rules for $W_\mu$, which are derived from (28)–(30), read
As is known, \( w_\mu \) gives the covariant definition \(^6\) of the spin of the total system for case* \( m \neq 0 \), since it is independent of the origin (cf. Eq. (48)), and in the \( II \)-frame its space components coincide with the angular momentum components while its time component vanishes:

\[
(w_\mu)^{(\tau)} = (J_{2\mu}^{(\tau)}, J_{3\mu}^{(\tau)}, J_{1\mu}^{(\tau)}, 0).
\]

Indeed it is well-known that (4') and

\[
\Sigma = w_\mu^2 = \text{const.,}
\]

represents the two group-invariants of the inhomogeneous Lorentz group generated by the basic quantities \( p_\mu \) and \( J_\mu \), and also that \( \Sigma \) takes integer and half-integer eigenvalues:

\[
\hbar^2 \sigma (\sigma + 1), \quad \sigma = 0, \frac{1}{2}, 1, \ldots.
\]

Generally, for the product of non-commuting quantities we need to take a symmetrized one to have the result self-adjoint. The \( W_\mu \) of (47) is a self-adjoint vector since \([J_\mu, p_\nu] = 0\) by (29).

3b. We have seen the existence of rest mass and spin as dynamical quantities, which are particle quantities in our theory. However, in order to attain a unified theory of elementary particles, the problem is immediately raised whether we can really find out a point-like system which has, besides rest mass and spin, just a set of additional invariant conserved quantities \( T^{(1)}, \ldots, T^{(n)} \) that are identifiable with various known intrinsic properties of elementary particles (isospin, strangeness, etc.), so that the quantum states of the system may just be classified in terms of a set of quantum numbers taken by those quantities.

Now in order that such identification be possible for certain dynamical quantities, they must satisfy the following general requirements:

(i) They must be self-adjoint.

(ii) They must be invariant quantities (under Lorentz transformations and translations), being commutable with all \( p_\mu \) and \( J_\mu \) components. Thus, for instance, \( J_\mu^2 \) and \( J_\mu J_\nu \) are not such quantities since they depend on the origin, being non-invariant under displacement. Since we know that we can form only two such invariants (4') and (51) as far as we use the basic dynamical quantities \( p_\mu \) and \( J_\mu \), alone, the other invariant quantities, if any, must be constructed

* The case with \( m = 0 \) needs special consideration.
by using the kinematical variables of the system such as $v_\mu$ and $\xi_\mu$ as well as $p_\mu$ and $J_\mu$.

(iii) They must be conserved at least for closed system. We are calling the quantities satisfying the conditions (ii) and (iii) invariant conserved quantities. The existence of such quantities, besides rest mass and spin, is made possible if the configuration and dynamics of the system involve any "internal" symmetry property independent of the inhomogeneous Lorentz transformations.

(iv) They must satisfy due commutation rules. In particular the quantities $T^{(r)}$, ($r=1, 2, 3$), to be identified with isospin components must satisfy $[T^{(r)}, T^{(s)}]=i\varepsilon_{rst}T^{(t)}$, and those corresponding to hypercharge and baryon number must commute with each other and with $T^{(r)}$. (They, of course, commute with $p_\mu$ and $w_\mu$ by the requirement of (ii)).

(v) By the above commutation rules the isospin certainly takes integer and half-integer eigenvalues, but it is further required that the other quantities also have discrete eigenvalues. In particular the quantities corresponding to hypercharge and baryon number need to take eigenvalues of preferably limited integer values, and the eigenvalues of the rest mass $m$ need to agree qualitatively with the empirical mass spectrum of elementary particles.

§ 4. Dynamical quantities of point-like systems

In this section we shall further analyse the general properties of dynamical quantities of point-like systems. The analysis in the velocity operator will be given separately in the next section.

The defining property of point-like system was the existence of the position coordinate $x_\mu$ as kinematical variables satisfying

$$[x_\mu, x_\nu]=0, \quad (1)$$

and this led to the identity of the momentum-energy vector of the system with the canonical momentum conjugate to $x_\mu$:

$$[x_\mu, p_\nu]=i\hbar \delta_{\mu\nu}, \quad (2)$$

as far as the system is closed from external interactions. By point-like system we are considering a system of which spatial extension is really reduced to a single point $x_\mu$. However, such coordinates $x_\mu$ under (1) exist, e.g. for a bi-local system, and so the arguments in this section may formally be applicable to such systems also, in so far as interactions are disregarded.

4a. The immediate consequence of the above basic relations, (1) and (2), is the existence of the internal angular momentum skew-tensor $S_\mu$, as is defined by

* Compare, e.g. with D. R. Yennie, Phys. Rev. 85 (1952), 877.
\[
J_{\mu\nu} = S_{\mu\nu} + x_{(\mu} p_{\nu)} = S_{\mu\nu} - p_{(\mu} x_{\nu)} = \text{const.,}
\]
which satisfies the commutation rules:
\[
\begin{align*}
[S_{\mu\nu}, p_{\rho}] &= 0, \\
[S_{\mu\nu}, x_{\rho}] &= 0, \\
[S_{\mu\nu}, S_{\rho\sigma}] &= i\hbar (\delta_{\nu\rho} S_{\mu\sigma} + \delta_{\mu\rho} S_{\nu\sigma}),
\end{align*}
\]
as the result of (28)-(30), (1) and (2). Eqs. (53) indicate that \( S_{\mu\nu} \)'s are independent of \( x_{\mu} \) and \( p_{\mu} \) and are therefore functions of internal degrees of freedom \( \xi_{\alpha} \) only. We shall call such quantities \textit{proper internal quantities}.*

Furthermore, (53) and (54) indicate that \( S_{\mu\nu} \) represent the generators of homogeneous Lorentz transformations for any proper internal tensor variable. Thus, noting (34), for a proper internal scalar quantity \( s \) we have
\[
[J_{\mu\nu}, s] = [S_{\mu\nu}, s] = 0,
\]
and for any possible proper internal vector quantity \( \xi_{\mu} \),
\[
[J_{\mu\nu}, \xi_{\rho}] = [S_{\mu\nu}, \xi_{\rho}] = i\hbar \delta_{\nu\rho} \xi_{\sigma}.
\]
Eq. (52) gives the covariant separation of \( J_{\mu\nu} \) into mutually commutable parts, the orbital part \( x_{(\mu} p_{\nu)} \) and the internal part \( S_{\mu\nu} \). Each part, however, is not separately conserved in general.** Also \( S_{\mu\nu} \) must be a self-adjoint tensor.

We observed that the fundamental commutation relations for a point-like system split into the two independent sets, the canonical commutation rules (28), (1), and (2) for external quantities \( (x_{\mu}, p_{\mu}) \), and the commutation rules (54) and (56) for internal quantities \( (S_{\mu\nu}, \xi_{\mu}) \). Thus the problem of setting up a new point-like system may be regarded as finding a new solution of (54) and (56) by constructing \( S_{\mu\nu} \) as suitable functions of internal variables \( \xi_{\alpha} \).

We here write some relations following from (54) and (56) for future references:
\[
\begin{align*}
[S_{\mu\nu}, \tilde{S}_{\rho\sigma}] &= \hbar \varepsilon_{\mu\nu\kappa\lambda} (\delta_{\rho\sigma} S_{\kappa\lambda} + \delta_{\kappa\lambda} S_{\rho\sigma}), \\
[\tilde{S}_{\mu\nu}, \tilde{S}_{\rho\sigma}] &= -i\hbar (\delta_{\mu\rho} S_{\nu\sigma} + \delta_{\nu\rho} S_{\mu\sigma}), \\
[S_{\mu\nu}, S_{\rho\sigma}] &= 2i\hbar S_{\lambda\delta}, \\
[S_{\mu\nu}, \tilde{S}_{\rho\sigma}] &= 2i\hbar \tilde{S}_{\lambda\delta}, \\
[S_{\mu\nu}, \tilde{S}_{\rho\sigma}] &= 0 \text{ (N.S.***)} ; [S_{ij}, S_{kl}] = 0, \\
& \quad (i \neq k, j \neq k)
\end{align*}
\]

* A \textit{proper} internal quantity commutes with both \( p_{\mu} \) and \( x_{\mu} \) but can be a tensor or spinor quantity or an even more complicated parameter (such as an Eulerian angle). It must not be confused with the notion of the \textit{invariant quantity} previously mentioned in (ii) in 3b. The latter is externally invariant, i.e. independent of translations and Lorentz transformations and may therefore be considered as \textit{internal} quantity of the system, but it need not commute with \( x_{\mu} \).

** In fact it is always possible to split the tensor \( J_{\mu\nu} \) into the orbital part and spin part for any relativistic dynamical system. The main feature for the case of point-like systems is rather the fact that they are particle quantities.

*** N. S. indicates that summation convention is not applied to repeated indices.
4b. Although we eventually adopt a system which has originally null rest mass, it takes in ordinary states non-zero rest mass in virtue of its internal motion. For simplicity* we shall therefore consider states with \( m \neq 0 \). We can then employ the spin pseudovector \( \mathbf{w}_\mu \), which is now written
\[
\mathbf{w}_\mu = \frac{1}{m} \mathbf{S}_\mu \mathbf{p}_\mu = \text{const.}
\] (47'')

This contains two mutually commuting quantities. One is its magnitude (51'), which is written
\[
\Sigma = \mathbf{w}_\mu^2 = \frac{1}{m^2} (\mathbf{S}_\mu \mathbf{p}_\mu)^2,
\] (51')

while the other may be taken to be either the fourth component:
\[
\omega_4 = \omega_i / i = \sum_{\epsilon=0} S_{ij} p_j / m
\]
representing the helicity, or else the third component in the II-frame:
\[
\omega_3^{(II)} = S_{33}^{(II)}.
\]

In any case the latter depends on the reference frame employed** and does not correspond to an intrinsic property of elementary particles. It means a property to distinguish between different states of the same elementary particle, which just works to specify the space-quantization of the internal motion.

As stated before the rest mass and the magnitude of spin, (4') and (51'), are two invariant conserved quantities commuting with all \( \mathbf{p}_\mu \) and \( \mathbf{J}_\mu \). The rest mass commutes furthermore with \( \mathbf{S}_\mu \). However, the fact that \( m \) is a dynamical quantity and not a mere \( c \)-number appears in the fact that it does not commute with \( x_\mu \). We obtain
\[
[P, x_\mu] = 2i \hbar p_\mu, \quad [m, x_\mu] = \frac{i \hbar}{m} p_\mu,
\] (59)

which imply uncertainty relations concerning the rest mass:
\[
\Delta m \Delta x_k \gtrsim \frac{\hbar |p_k|}{mc^2}, \quad \Delta m \Delta t \gtrsim \frac{\hbar E}{mc^2},
\] (60)

(where we have restored \( c \)). These relations indicate that a sharp localization

* Most of the relations to be derived below can be easily rewritten in the forms valid for the case \( m = 0 \) included, if we employ \( W_\mu \) instead of \( \mathbf{w}_\mu \) and \( R_\mu \) instead of \( r_\mu \) (cf. 4d).

** As is known, helicity becomes independent of the frame for the special case of neutrino.
of the particle in space or in time not only accompanies the well-known indeterminacy of its momentum or energy but also induces superposition of states of different rest masses. Especially the former relation of (60) means that the rest mass of the particle, when running nearly with the light velocity, becomes indeterminate to an extent comparable with its value \( \Delta m \approx \bar{m} \) if it is localized within a region of its Compton wavelength. (The other unusual consequence is that under external interaction the rest mass is no longer conserved in general as we shall discuss later, but this effect will not necessarily conflict with experimental evidences either).

On the other hand \( \Sigma \) commutes neither with \( x_\mu \) nor with \( S_\mu \) in general as we shall see presently.

4c. Next we consider the scalar and pseudoscalar

\[
\begin{align*}
\phi &= \frac{1}{4} S_\mu^2 = -\frac{1}{4} S_\mu^2, \\
\zeta &= \frac{1}{4} S_\mu \tilde{S}_\mu = S_{23} S_{10} + S_{21} S_{20} + S_{12} S_{30}.
\end{align*}
\]

(Note that \( \zeta \) is self-adjoint owing to the last equation of (57).) They are the group-invariants of the homogeneous Lorentz transformations for proper internal variables, hence

\[
[S_\mu, \phi] = 0, \quad [S_\mu, \zeta] = 0.
\]

Since they are proper internal variables themselves, they commute with all of \( (x_\mu, p_\mu, J_\mu, S_\mu) \).

They do not, however, commute with any proper internal vector quantity \( \xi_\mu \), since (56) yields

\[
\begin{align*}
[S_\mu, \phi] &= \frac{i\hbar}{2} \{S_\mu, \xi_\nu\}, \\
[S_\mu, \zeta] &= i\hbar S_\mu \xi_\nu = i\hbar \xi_\nu \tilde{S}_\mu.
\end{align*}
\]

As we shall see this fact is related to the fact that \( \phi \) and \( \zeta \) are not constants of motion in general.

For the \( S_\mu \) obeying (54) we have further two important formulas*:

\[
\begin{align*}
[S_\mu S_\nu - \tilde{S}_\mu \tilde{S}_\nu] &= 2 (\delta_\mu \phi + \hbar S_\nu), \\
[S_\mu \tilde{S}_\nu] &= \delta_\mu \zeta + \hbar \tilde{S}_\mu.
\end{align*}
\]

Using the former, \( \Sigma \) is expressed as

\[\text{* For instance, to derive (65) we use the general formula}
\]

\[A_\mu A_\nu - \tilde{A}_\mu \tilde{A}_\nu = \frac{1}{2} \delta_\mu A_\nu,\]

\[\text{for an arbitrary skew-tensor} \ A_\mu \text{, and the third equation of (57).}\]
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\[
\Sigma = \frac{p_\mu p_\nu}{2m^2} \{\tilde{S}_{\mu\nu}, \tilde{S}_{\alpha\beta}\} = 2\phi + \frac{p_\mu p_\nu}{2m^2} \{S_{\mu\nu}, S_{\alpha\beta}\}.
\]

Also, from \( S_{\mu\nu} \) we may construct further scalars:

\[
\begin{cases}
\phi^{(3)} = \frac{1}{4} S_{\mu\nu} S_{\alpha\beta} S_{\lambda\kappa}, \\
\phi^{(4)} = \frac{1}{4} S_{\mu\nu} S_{\alpha\beta} S_{\lambda\kappa},
\end{cases}
\]

By this notation, \( \phi = -\phi^{(3)} \). All those \( \phi^{(n)} \) are proper internal scalar quantities, hence \( [\phi^{(n)}, S_{\mu\nu}] = 0 \).

Since, however, we know that there exist only two group-invariants, \( \phi \) and \( \chi \), corresponding to the generators \( S_{\mu\nu} \), all \( \phi^{(n)} \) must be reduced to \( \phi \) and \( \chi \). In fact, using (65) and (66), we get the identities:

\[
\begin{cases}
\phi^{(3)} = i\hbar \phi, \\
\phi^{(4)} = \chi^2 + 2\phi^2 + \hbar^2 \phi,
\end{cases}
\]

4d. We now introduce the vector

\[
R_\mu = S_{\mu\nu} p_\nu,
\]

or, for case \( m \neq 0 \),

\[
r_\mu = \frac{1}{m^2} S_{\mu\nu} p_\nu, \quad (r_\mu p_\mu = 0)
\]

which represents the radius vector of the Zitterbewegung as we shall see in 5b. The commutation rules are derived to be

\[
\begin{cases}
[r_\mu, r_\nu] = -\frac{\hbar}{m^3} \varepsilon_{\rho\sigma\lambda\mu} w_\nu p_\lambda, \\
w_\mu r_\nu = \frac{\hbar}{m} \varepsilon_{\rho\sigma\lambda} r_\sigma p_\lambda,
\end{cases}
\]

which reduce in the \( \Pi \)-frame to the well-known commutation rules for \( S_{ij} \) and \( S_{kl} \).

In terms of \( w_\mu \) and \( r_\mu \) the skew-tensor \( S_{\mu\nu} \) is decomposed in the following form:

\[
S_{\mu\nu} = -\frac{i}{m} \varepsilon_{\rho\sigma\lambda} w_\nu p_\lambda - r_{(\mu} p_{\nu)}.
\]

Thus the set of quantities \( (p_\mu, S_{\mu\nu}) \) is equivalent to the set of vectors:

\[
(p_\mu, w_\mu, r_\mu),
\]

where both \( w_\mu \) and \( r_\mu \) are "space-like" vectors \( (w_\mu p_\mu = r_\mu p_\mu = 0) \).

In (71) the first term is the constant part due to the spin \( w_\mu \), and the
second term is the fluctuating part which has the form of an orbital angular momentum tensor with the radius vector \( r^\mu \).

Using (71) the conservation law for \( J_{\mu \nu} \) is expressed in the form

\[
J_{\mu \nu} = -\frac{i}{m} \varepsilon_{\nu\lambda\sigma} w^\lambda p^\sigma + y^\mu p^\nu = \text{const},
\]

with

\[
y^\mu = x^\mu - r^\mu, \quad \{ y^\mu, p^\nu \} = i\hbar \theta^\mu_{\nu},
\]

where both terms in the left side of (73) are separately conserved. The conserving "orbital" part \( y^\mu p^\nu \) is defined not with the particle position \( x^\mu \) but with the proper center of mass* \( y^\mu \) of the system, which is the position that excludes from \( x^\mu \) the Zitterbewegung \( r^\mu \). Taken in the II-frame, the space components of (73) are nothing but \( w_k^{(\nu)} = \text{const} \), while the time components are \( y_k^{(\mu)} = x_k^{(\mu)} - r_k^{(\mu)} = \text{const} \). This corresponds to the relation usually known as the conservation of center of mass (cf. Eq. (95)). It must be noted that since we are considering in the II-frame where \( p^\mu \)'s have sharp values (i.e. zero), \( y_k^{(\nu)} \)'s are completely indeterminate. More important is the fact that the components of \( r^\mu \) (and so of \( y^\mu \)) are not mutually commuting in contrast with those of \( x^\mu \). We have the simple relation:

\[
\{ y^\mu, y^\nu \} = -\{ r^\mu, r^\nu \} = \frac{1}{m^2} [w^\mu, w^\nu].
\]

Corresponding to the decomposition (71) the scalars \( \phi \) and \( \gamma \) are expressed as

\[
\begin{align*}
\phi &= \frac{1}{2} (\Sigma - m^2 \Lambda), \\
\gamma &= -m w^\mu r^\mu = -mr^\mu w^\mu,
\end{align*}
\]

where \( \Lambda \) denotes the magnitude of Zitterbewegung:

\[
\Lambda \equiv r^\mu r^\mu.
\]

It is worthwhile to note that the minus sign appearing in (75) and (76) is due to the Lorentz metric. The pseudoscalar \( \gamma \), (76), represents the non-orthogonality between spin and Zitterbewegung and we called it internal chirality*. The squared radius of the Zitterbewegung, (77), will also play an important role, and obeys

* The proper center of mass is such point in reference to which the total angular momentum tensor \( J_{\mu \nu} \) satisfies \( J_{\mu \nu} p^\nu = 0 \). The above \( y^\mu \) really satisfies this condition since

\[
J_{\mu \nu} = J_{\mu \nu} - y_{\nu} p^\mu = -\frac{i}{m} \varepsilon_{\mu\nu\lambda\sigma} w^\lambda p^\sigma.
\]
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Starting from the variables \( (S', p', r', \mu) \), i.e. \( (w', r', p', \mu) \), we thus have the commuting set of four independent invariant quantities:

\[
(P, \Sigma, \phi, \lambda),
\]

taking account of the identical relations \( p_w r_w = 0, p_r r_p = 0 \), (75) and (76). In (79) \( \phi \) may be replaced by \( \Lambda \). Of those four invariant quantities, \( P \) and \( \Sigma \) are conserved, but \( \phi \) (or \( \Lambda \)) and \( \lambda \) in general are not so.

Finally we consider the commutation relation between spin and position coordinate. First we get

\[
[w_w, x_\mu] = -\frac{i\hbar}{m} \tilde{S}_{\mu\nu} \left( \delta_{\mu\nu} + \frac{1}{m^2} p_{\nu} p_{\nu} \right) = \frac{1}{m} p_{\mu} \mu + \frac{i}{m} \tilde{S}_{\mu\nu} r_{\nu} p_{\nu},
\]

taking account of (59). This leads to

\[
[\Sigma, x_\mu] = -\frac{i\hbar}{m} \tilde{S}_{\mu\nu} \left( \delta_{\mu\nu} + \frac{1}{m^2} p_{\nu} p_{\nu} \right),
\]

which contains

\[
[\Sigma, w_w, x_\mu] = 0.
\]

If in particular the condition

\[
\{S_{\mu\nu}, S_{\sigma\rho}\} = \delta_{\mu\rho} \cdot (\text{scalar quantity})
\]

be satisfied, it must be

\[
\{S_{\mu\nu}, S_{\sigma\rho}\} = -\{\tilde{S}_{\mu\nu}, \tilde{S}_{\sigma\rho}\} = 2\delta_{\mu\rho} \phi,
\]

by noting the formula (65). In this case, (81) reduces to

\[
[\Sigma, x_\mu] = 0.
\]

This fact is evident because under the condition (82), Eq. (67) reduces to

\[
\Sigma = \phi,
\]

i.e. \( \Sigma \) becomes a proper internal quantity \( \phi \), independent of \( p_\nu \). This means, conversely speaking, that under this special condition \( \phi \) coincides with the magnitude of spin and hence is conserved. At the same time we have the relation

\[
m^2 \Lambda = -\Sigma = -\phi.
\]

The condition (82) represents a quite particular one, but we shall see that it is satisfied in the case of Dirac particle as it should (cf. Part II).

Eqs. (81) and (83) show that a sharp localization of the particle generally accompanies an indeterminacy of the magnitude of spin, \( \Sigma \), but for a special

* This relation is equivalent to \( 2S_{\mu\nu} = \phi S_{\mu\nu} \).
system satisfying (82) $\Sigma$ remains to be a definite value under a sharp localization.

§ 5. Velocity operator

Now we enter into the analysis of the properties directly connected with the time evolution of the system. We consider in the "Heisenberg picture", where every dynamical quantity develops with $\tau$.

5a. Taking the derivative of any quantity $F$ with respect to $\tau$ we get $\dot{F}$, but essentially new quantity thus obtained is the instantaneous 4-velocity corresponding to $x_{\mu}$:

$$\frac{dx_{\mu}}{d\tau} = \dot{x}_{\mu} = v_{\mu}$$ (25')

only, because the derivation of $p_{\mu}$ gives

$$\dot{p}_{\mu} = 0,$$ (38a)

while the derivative of $S_{\mu\nu}$ is related to the other quantities by

$$\dot{S}_{\mu\nu} = -v_{(\mu}p_{\nu)} = p_{(\mu}v_{\nu)},$$ (86)

which is the re-expression of the basic conservation law (38b), taking account of (25') and (38a). On the other hand, the derivation of (2) gives

$$[v_{\mu}, p_{\nu}] = 0, \quad [v_{\mu}, P] = 0,$$ (87)

and

$$[\dot{v}_{\mu}, p_{\nu}] = 0,$$ (88)

noting (38a), while the derivation of (1) gives

$$[v_{\mu}, x_{\nu}] = [v_{\nu}, x_{\mu}] = h_{\mu\nu},$$ (89)

and

$$-[\dot{v}_{\mu}, x_{\nu}] = [v_{\mu}, v_{\nu}] = ik_{\mu\nu}.$$ (90)

The relation (89) means that $h_{\mu\nu}$ is a symmetric tensor, while $k_{\mu\nu}$ in (90) is a skew-symmetric tensor. As stated in the Introduction, whether $h_{\mu\nu}$ vanishes or not is the criterion (C), and whether $k_{\mu\nu}$ vanishes or not is the criterion (B), for classifying point-like systems.

Adding the new variables $v_{\mu}$, we thus have the set of basic variables (7) for any point-like system. Among them the set of quantities (6) has universal commutation rules among themselves as we have seen in § 4, while those relating to $v_{\mu}$ are not definite in advance apart from (87). Also, there is arbitrariness in the definition of $\tau$, since any monotonously increasing function of $\tau$, $\tau' = f(\tau)$, may equally be employed as the ordering parameter. As stated before we must postulate certain inhomogeneous conditions on $v_{\mu}$ in order to define the kinematical properties of $v_{\mu}$ sufficiently as well as to suppress the
arbitrariness of $\tau$, and those conditions work to characterize point-like systems into various types. Here we consider the properties of $v_\mu$, taking the criterion (A) together with (B) into consideration, (while the criterion (C) is considered in $\delta c$ and thereafter).

(i) First we assume $\rho=\alpha_{\mu}^2$ is an absolute invariant, i.e. it commutes with $v_\mu$ and all other quantities. We have called such a system "of the normal class", which includes many familiar systems.* Then we can practically suppress the arbitrariness in the definition of $\tau$ by requiring that

$$\dot{\rho}=0, \text{ i.e. } \rho=\text{const.} \tag{91}$$

Since $\rho$ is dimensionless it thus takes numerical constant eigenvalues, but their absolute values have no physical significance and the only meaningful distinction is that between the cases $\rho\neq0$ and $\rho=0$. For the former case we may always normalize $v_\mu$ (i.e. $\tau$) such that

$$v_\mu^{2}=-\rho=-1. \tag{92}$$

In other words, for normal class $\rho$ is regarded to obey the idempotent condition

$$\rho^{2}=\rho, \quad (\rho^{\dagger}=\rho) \tag{93}$$

taking eigenvalues 1 and 0. We get from (91)

$$\{v_\mu, \dot{v}_\mu\}=0. \tag{94}$$

The case $\rho=0$ is quite different from the case $\rho=1$. It is to be noted that, since in our theory $v_\mu$ and $p_\mu$ are essentially different, the condition of zero rest mass, $p_\mu^{2}=0$, and the present condition, $v_\mu^{2}=0$, are also different in general. Thus we may have states in which $\rho=0$ but $m\neq0$. If we consider about the case of "classical model" ($[v_\mu, v_\mu]=0$), $\rho=1$ means a state where particle has the unitary velocity (cf. 2e), while $\rho=0$ means a state where particle runs always with the light velocity.*** (This refers to the instantaneous velocity including the Zitterbewegung; the velocity of the mean motion is still less than $c$ in general.) We anticipate that two distinct families of states, $\rho=1$ and $\rho=0$, for a system of normal class correspond to baryons and leptons, respectively. Then, speaking for the case of classical model, the conservations of baryon and lepton numbers correspond just to the impossibility of transitions between a state where the instantaneous velocity is less than $c$ and a state where it is $c$.

* Relativistic rotator\(^4\) as well as the Dirac particle are examples of normal class, but not the Kemmer particle (cf. Part II).

** For the case of "non-classical model" the normalization (92) is not necessarily the most suitable one, and, for example, for Dirac particle we take $\rho=4$ (cf. 2e).

*** In pure classical theory such kind of particle was considered by Weyssenhoff.\(^8\)
(ii) Next, we consider systems for which $\rho$ is not an absolute invariant. Then we cannot impose the condition (91) in advance. However, we shall see that there generally exists an absolute invariant $\rho'$, which is intimately related to $\rho = -v_\mu^2$. This $\rho'$ may then be regarded as the operator corresponding to baryon number (cf. 6f. (ii)).

5b. We are now in a position to show that the general aspect of the motion of any point-like system is governed by the conservation equations (38a) and (86). Taking account of (47'') and (71), Eq. (86) is equivalent to $\omega_\mu = \text{const.}$ (Eq. (47'')) together with

$$\dot{\rho}_\mu = \frac{\epsilon}{m^2} p_\mu, \quad \text{i.e.} \quad \dot{v}_\mu = v_\mu - \frac{\epsilon}{m^2} p_\mu,$$

where

$$\epsilon = -v_\mu p_\mu$$

is connected with $\dot{v}_\mu$ through the relation

$$v_\mu^2 = \frac{\epsilon^2}{m^2} - \rho,$$

which is a consequence of (95) itself. The $\epsilon$ means the rest mass times the magnitude of the velocity of Zitterbewegung subsisting in the $II$-frame:

$$\epsilon = mv_\mu^{(II)},$$

and plays an important physical role as will be seen later. Like $\Sigma$, it is an invariant quantity (so $[\epsilon, p_\mu] = 0, [\epsilon, J_\mu] = 0$) but does not commute with $x_\mu$ nor with $S_{\mu\nu}$.

Eq. (95) is expressed in the integrated form:

$$x_\mu = y_\mu + r_\mu, \quad y_\mu = \left( \int \epsilon d\tau \right) \frac{p_\mu - r_\mu^0}{m^2} \left( \frac{1}{m^2} J_{\mu\nu} p_\nu = \text{const.} \right)$$

which directly shows that particle makes Zitterbewegung $r_\mu$ around the mean rectilinear motion of the direction of $p_\mu$. The motion of any point-like system thus consists of three parts with mutual correlations: The first is the mean rectilinear translation represented by $p_\mu$, the second is a certain orbital Zitterbewegung represented by $r_\mu$ or $v_\mu$, and the third is pure internal motion (rotation, etc.) to be described by the other internal kinematical variables $\xi_\alpha$ (which have not yet appeared explicitly.)

It is important to note that $r_\mu$ always exists as dynamical variables except the trivial case of structureless Newtonian particle, since if all $r_\mu$'s identically vanish, all $\omega_\mu$'s must also vanish identically because of the commutation rule (69), and we are left with $S_{\mu\nu} = 0$. The appearance of Zitterbewegung is thus
a general and necessary consequence due to the coupling between relativity and quantum mechanics,*† and it will play an important role in our theory.

To see the motion more clearly we consider about the expectation values. First, for a "stationary state" satisfying (39), we had \( \langle \hat{F} \rangle = 0 \), (Eq. (40)), where \( F \) is any bounded operator that is not explicitly \( \tau \)-dependent. Applying this to Eq. (95), we get

\[
\langle \nu_{\mu} \rangle_{\lambda} = \langle \frac{\epsilon}{m^2} \hat{p}_{\mu} \rangle_{\lambda}.
\] (99)

This is rewritten, for an eigenstate of \( \hat{p}_{\mu} \), as

\[
\langle \nu_{\mu} \rangle_{\lambda} = \frac{\langle \epsilon \rangle_{\lambda}}{m^2} \hat{p}_{\mu},
\] (99')

and in particular in the II-frame

\[
\langle \nu_{\mu}^{(m)} \rangle_{\lambda} = 0.
\] (99'')

To consider for a general state, we construct an intrinsic-time average over an interval \( T \) of the expectation value of \( \hat{F} \):

\[
\frac{1}{T} \int_{0}^{T} \psi^* \hat{F} \psi \, d\tau = \frac{1}{T} [\psi^*(F(\tau) - F(0)) \psi].
\]

Then for any bounded operator \( F \) we have

\[
\langle \langle \hat{F} \rangle \rangle_{\tau} = 0,
\] (100)

where \( \langle \rangle_{\tau} \) denotes the intrinsic-time average over a sufficiently long interval. Thus for a general state, (99') and (99'') are replaced by

\[
\langle \langle \nu_{\mu} \rangle \rangle_{\tau} = \frac{\langle \epsilon \rangle_{\tau}}{m^2} \hat{p}_{\mu},
\]

\[
\langle \langle \nu_{\mu}^{(m)} \rangle \rangle_{\tau} = 0,
\]

respectively. This is a kind of virial theorem indicating that the velocity subsisting in the II-frame is essentially of quasi-periodical character.

5c. Now we take up the criterion (C) which concerns the commutator \( h_{\mu\nu} \) defined in (89). The one is systems of the first kind for which

\[
h_{\mu\nu} = [\nu_{\mu}, x_{\nu}] = 0,
\] (101)

* The Zitterbewegung for Dirac particle is merely a special example of this general situation.
† In pure classical theory \( w_{\mu} \) need not vanish even if \( r_{\mu} = 0 \). This is the case which we called "classical case" previously and is realized by a relativistic rotator as a constant rectilinear translation associated with pure spatial internal rotation in the rest frame of the translational motion but without any Zitterbewegung. Such classical motion does not have good quantum-mechanical correspondence.
while the other is systems of the second kind for which the commutator does not vanish identically: $h_{\mu\nu} \neq 0$. In the latter case, if it belongs to normal class, $h_{\mu\nu}$ needs to satisfy

$$\{h_{\mu\nu}, v_\nu\} = 0.$$ 

A simple example belonging to the second kind (and to the normal class) is given by relativistic Newtonian particle which is defined by imposing a strict connection between $p_\mu$ and $v_\nu$, i.e. the colinearity:

$$v_\nu = \nu p_\mu.$$ 

This corresponds to an invariant hamiltonian \textit{quadratic} in $p_\mu$:

$$H = \frac{\nu}{2} p_\mu^2 + \mathcal{K},$$

and the corresponding stationary state equation is of Klein-Gordon type.

In the following section we accord our attention to systems of the first kind. This corresponds to invariant hamiltonian \textit{linear} in $p_\mu$ and will represent more important systems with internal degrees of freedom.

§ 6. Systems with \textit{velocity} as internal variables

Hereafter we consider systems of the first kind exclusively, so that the \textit{velocity components commute with position coordinates}:

$$[v_\mu, x_\nu] = 0.$$ (101)

Then this condition leads to various general and remarkable consequences as we shall now see.

6a. The assumption (101) means that $v_\mu$'s are \textit{proper internal variables} (although they represent instantaneous orbital velocity) commuting with all external variables (note (87)).

Thus, according to (56), we have the new basic commutation rule:

$$[S_{\mu\nu}, v_\nu] = i\hbar \delta_{\mu\nu} v_\nu.$$ (102)*

We also note the following relations derivable from (102):

$$[\tilde{S}_{\mu\nu}, v_\nu] = i\hbar \varepsilon_{\mu\rho\nu} v_\rho, \quad [\tilde{S}_{\mu\nu}, v_\nu] = 0, \quad (\text{N.S.})$$ (103)

Since we now have the set of proper internal variables:

$$(S_{\mu\nu}, v_\nu),$$ (104)

we can construct, by their combination, two proper internal vectors: the pseudovector

* This can be verified directly by taking derivative of (53) and using (86), (2), and (101).
and the vector

$$K_\mu = \frac{1}{2} \{S_\mu, v_\nu\} = S_\mu v_\nu - \frac{3}{2} i \hbar v_\mu = v_\mu S_\mu - \frac{3}{2} i \hbar v_\mu.$$ (106)

Then, like $v_\mu$ itself, each of them must obey, according to (56), the same type of commutation rules as (102):

$$[S_\mu, H_\nu] = i \hbar \delta_{\mu\nu} H_\nu,$$
$$[S_\mu, K_\nu] = i \hbar \delta_{\mu\nu} K_\nu.$$ (107)

These $H_\mu$ and $K_\mu$ are self-adjoint vectors and satisfy

$$\{H_\mu, v_\nu\} = 0, \quad \{K_\mu, v_\nu\} = 0,$$ (108)
meaning that they are space-like in the classical limit.

Using the above relations we can verify that $S_\mu$ is decomposed in the form

$$\rho S_\mu = \frac{1}{2i} \varepsilon_{\mu\nu\sigma} \{H_\sigma, v_\nu\} - \frac{1}{2} \{K_{[\nu}, v_{\sigma]}\} - \frac{i \hbar}{2} [v_\mu, v_\nu].$$ (109)

Thus, unless $\rho = 0$, the set (104) is equivalent to the set of three proper internal vectors:

$$(v_\mu, H_\nu, K_\nu),$$ (110)

obeying the relations (108). This is the second triple of vectors which is important for any system under (101), as well as the first triple (72), and brings to light a new aspect different from that already brought by means of the first triple in § 4. The physical meaning of $H_\mu$ and $K_\mu$ will become clearer later.

The commutation rules between one of the vectors (110) and $\phi$ or $\chi$ are given by the general formula (63) and (64). First we obtain

$$\begin{cases} [\phi, v_\nu] = i \hbar K_\nu, \\ [\chi, v_\nu] = i \hbar H_\nu, \end{cases}$$ (111) (112)
indicating that $\phi$ is the generator of the transformation turning $v_\mu$ to the direction $K_\mu$ "normal" to $v_\mu$, keeping $(S_\mu, p_\mu)$ invariant, and the analogous meaning for $\chi$.

Here we insert the formulas

$$\frac{1}{2} \{S_\mu, H_\nu\} = \bar{S}_\mu K_\nu = K_\nu \bar{S}_\mu,$$
$$= - \frac{1}{2} \{\chi, v_\nu\} = - \chi v_\mu + \frac{i \hbar}{2} H_\mu, \quad \begin{cases} \\ \end{cases}$$ (113)

$$S_\mu H_\nu = -(\chi v_\mu + i \hbar H_\mu),$$
(which are obtained by using (66), (107), and (112)). Utilizing (113), Eqs. (63) and (64) give

$$[\phi, H_\mu] = [z, K_\mu] = -\frac{i\hbar}{2} \{z, v_\mu\}. \quad (114)$$

Further we get the relation

$$[H_\mu, z] + i\hbar \{\phi, v_\mu\} = [K_\mu, \phi] - \frac{3}{4} i\hbar^3 v_\mu = \frac{i\hbar}{4} \{S_{\mu\rho} S_{\mu\sigma} v_\lambda\} - \frac{3}{2} i\hbar^3 v_\mu. \quad (115)$$

The commutation rule between $w_\mu$ (or $r_\mu$) and any proper internal vector $\xi_\mu$ easily follow from (56). Thus we get

$$\begin{align*}
[w_\mu, v_\nu] &= -\frac{\hbar}{m} \varepsilon_{\mu\nu\rho\sigma} p_\rho v_\sigma = -\frac{i\hbar}{m} (\tilde{S}_{\mu\rho})', \\
[w_\mu, H_\nu] &= -\frac{\hbar}{m} \varepsilon_{\mu\nu\rho\sigma} p_\rho H_\sigma, \\
r_\mu, v_\nu] &= -\frac{i\hbar}{m^2} (\delta_\mu_\nu \varepsilon + v_\lambda p_\lambda), \\
r_\mu, H_\nu] &= \frac{i\hbar}{m^2} (\delta_\mu_\nu p_\rho H_\rho - H_\nu p_\rho),
\end{align*} \quad (116-119)$$

and analogous relations for $[w_\mu, K_\nu]$ and $[r_\mu, K_\nu]$. Especially, (116) and (118) supply

$$\begin{align*}
[w_\mu, \varepsilon] &= 0, \\
[r_\mu, \varepsilon] &= -i\hbar \left( v_\mu - \frac{\varepsilon}{m^2} p_\mu \right) = -i\hbar \dot{r}_\mu. \quad (120-121)
\end{align*}$$

6b. The next important fact for systems of the first kind is that $-\varepsilon$ works as the generator of the intrinsic-time displacement for any quantity that is function of $x_\mu, p_\mu$ and $S_\mu$ only. This is because, as was stated in § 1, the invariant hamiltonian of the system is linear in $p_\mu$, taking the general form

$$H = -\varepsilon + \mathcal{K}, \quad (9)$$

where $\mathcal{K}$ is a scalar function of proper internal variables alone and hence it commutes with $x_\mu, p_\mu$, and $S_\mu$, due to (55). The preceding equations (120) and (121) are simply examples of this general theorem (compare with (47'') and (95)). Also we really reproduce, e.g.

$$\frac{i}{\hbar} [S_\mu, \varepsilon] = p_\mu v_\mu = \dot{S}_\mu, \quad (122)$$

due to (102) and (86), and

$$\frac{i}{\hbar} [x_\mu, \varepsilon] = v_\mu = \dot{x}_\mu. \quad (123)$$
Noting (111) and (112), we also get
\[
\dot{\phi} = -\frac{m^2}{2} \dot{A} = \frac{i}{\hbar} [\phi, \epsilon] = p_\lambda K_\lambda = -\frac{m^2}{2} \{v_\mu, r_\mu\},
\]
(124)
\[
\dot{z} = \frac{i}{\hbar} [z, \epsilon] = p_\mu H_\mu = -mv_\mu w_\mu = -mv_\mu v_\mu.
\]
(125)

It is to be noted that the equation of motion for any function of the variables \(x_\mu, p_\mu\) and \(S_{\mu\nu}\) alone gives a relation universal for the first kind systems, such as (120)–(125), because of its commutability with \(\mathcal{K}\), and of the general relations (101) and (102). On the contrary, only \(v_\mu\) among the variables (7) does not necessarily commute with \(\mathcal{K}\), and the actual form of its equation of motion:
\[
\dot{v}_\mu = -\frac{i}{\hbar} [v_\mu, v_\nu] p_\nu + \frac{i}{\hbar} [\mathcal{K}, v_\mu]
\]
(126)
depends on further characteristics of the system (see Part II).

Finally we write the following commutation rules for future references:
\[
[r_\mu, r_\nu] = -\hbar \frac{\epsilon}{m^2} \left( \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right),
\]
(127)
so that
\[
[r_i^{(n)}, r_j^{(m)}] = -\hbar \frac{\epsilon_{ij}}{m} \delta_{ij},
\]
(127')
and
\[
[A, r_\mu] = -\hbar \frac{\epsilon}{m^2} \{\epsilon, r_\mu\}.
\]
(128)
We also have
\[
\begin{align*}
[r, v_\mu] &= i\hbar \left( -\{\epsilon, r_\mu\} + \frac{2}{m^2} \phi p_\mu + 2K_\mu \right), \\
[A, v_\mu] &= -\frac{i\hbar}{m^2} \{\epsilon, r_\mu\} + \dot{A} p_\mu,
\end{align*}
\]
(129)
(130)
obtained from (116) and (118). The former indicates that \(\Sigma\) does not generally commute with \(v_\mu\) either.

6c. Now the set of proper internal variables (104) or (110) allows us to construct, besides \(\phi\) and \(z\), the following six self-adjoint proper internal scalars:
\[
\begin{align*}
\rho &= -v_\mu^2, & H_\mu^2, & K_\mu^2, \\
H_\mu K_\mu &= K_\mu H_\mu, & iH_\mu v_\mu &= -iv_\mu H_\mu, & iK_\mu v_\mu &= -iv_\mu K_\mu.
\end{align*}
\]
(131)
Due to (55) each of (131) must commute with \(S_{\mu\nu}\).
where we have introduced the notation
\[ \theta = H^2. \]  
Thus each of (131) must also commute with \( w_\rho \) and \( r_\rho \):
\[
[w_\rho, p] = 0, \quad [r_\rho, p] = 0, \quad [w_\rho, \theta] = 0, \quad [r_\rho, \theta] = 0, \quad \cdots
\]
Of the quantities (131), however, only four are independent, e.g.
\[
(\rho, \theta, iH_\rho v_\rho, iK_\rho v_\rho),
\]
because we can prove that there hold the two important identical relations:
\[
\begin{align*}
H_\rho K_\rho &= K_\rho H_\rho = -\left(\rho Z + \frac{i\hbar}{2} H_\rho v_\rho\right), \\
H^2 - K^2 &= 2\rho \phi + i\hbar K_\rho v_\rho + \frac{3}{4} \hbar^2 \rho.
\end{align*}
\]
The four quantities of (135) do not necessarily commute with each other. However, if the commutators \([v_\rho, v_\sigma]\) are specified, we can generally form of (135) two independent quantities mutually commuting, and the other two of (135) are represented in terms of the former two (together with \( \phi \) and \( Z \)) or simply reduce to \( \theta \)-numbers (cf. Part II). We regard those two independent quantities to be \( \rho \) and \( \theta \). (For normal class \( \rho \) always commutes with others.)
Thus, starting from the quantities \((S_{\rho\rho}, v_\rho)\) we have the set of four independent invariant quantities which commute with one another and with \( p_\rho \) and \( S_{\rho\rho} \):
\[
(\phi, Z, \rho, \theta),
\]
On the other hand, starting from \((S_{\rho\rho}, p_\rho)\), we had the set of four independent invariant quantities (79). Then, combining (79) and (138), we get the set of six independent invariant quantities commutable with one another:
\[
(P, \Sigma, \rho, \theta, \phi, Z),
\]
which represents the set of invariant quantities to be formed of all basic variables:
\[
(p_\rho, S_{\rho\rho}, v_\rho),
\]
without using internal variables \( \xi_\alpha \) explicitly.
It remains to consider about the invariant quantities which mix the

* Besides (140) we had the variable \( x_\rho \) but this does not contribute in the construction of the invariant quantities because of the requirement of the translational invariance (cf. (ii) of §36).
(\(p_\mu, w_\mu, r_\mu\)) system and the \((\nu_\mu, H_\mu, K_\mu)\) system. The examples of such "mixed" scalar quantities have already occurred. They are \(\dot{\nu}, \dot{H}, \dot{K}\) (see Eqs. (124) and (125)). The remaining mixed scalars are the following three:

\[
\begin{align*}
\Omega &= H_\mu w_\mu = w_\mu H_\mu, \\
\Omega' &= \frac{m}{2} \{r_\mu, K_\mu\}, \\
\Omega'' &= K_\mu w_\mu = w_\mu K_\mu = \frac{m}{2} \{r_\mu, H_\mu\},
\end{align*}
\]

which are all self-adjoint, and especially \(\Omega\) will play an important physical role.

Those quantities, however, can be represented in terms of

\[
(\phi, \chi, \varepsilon),
\]

or

\[
(\phi, \chi, \dot{\phi}, \dot{\chi}),
\]

in the following form:

\[
\begin{align*}
\Omega &= -\frac{1}{m\hbar^2} \{\chi[\varepsilon]\} = i \frac{\hbar}{m\hbar} [\chi, \dot{\chi}], \\
\Omega' &= -\frac{1}{m\hbar^2} [\phi[\varepsilon]\} = i \frac{\hbar}{m\hbar} [\phi, \dot{\phi}], \\
\Omega'' &= -\frac{1}{2m} \{\chi, \varepsilon\} = -\frac{1}{m\hbar^2} [\chi[\phi]\} = -\frac{1}{m\hbar^2} [\phi[\chi]\}.
\end{align*}
\]

Thus \(\Omega\)'s are commutable with \(p_\mu\) and \(w_\mu\), since all of the quantities (142) are so.

We can furthermore derive the following relation:

\[
m(\Omega - \Omega') = \frac{1}{\hbar^2} ([\phi[\phi, \varepsilon]\} - [\chi[\chi, \varepsilon]\}] = \{\phi, \varepsilon\} + \frac{3}{4} \hbar^2 \varepsilon,
\]

by the aid of the formula (65). Eqs. (144) and (145) mean two non-linear relations between the three quantities of (142). Again, \(\Omega\) has the following remarkable property:

\[
[\Omega, \phi] = -\frac{1}{2m} [\varepsilon, \chi^2] = -\frac{i\hbar}{2m} (\chi^2)' = [\Omega'', \chi],
\]

which is obtained by (114) and (125).

From the foregoing discussions we see that all mixed invariant quantities can be generated out of the basic quantities (142), and therefore that, if we add

\[
(\varepsilon, iH_\mu \nu_\mu, iK_\mu \nu_\mu)
\]
to the set (139), all invariant quantities constructed of our variables (140) are represented by them.

We can adopt the set of commutable quantities, (139), as part of the quantities to specify eigenstates of the system, but this is not adequate since $\phi$ and $\mathcal{Z}$ are generally not conserved. On the contrary $\Theta$ is a constant of motion under a certain general assumption (cf. 6f – (i)). Thus the important problem is to replace $\phi$ and $\mathcal{Z}$ by two independent conserved quantities. To be general we have been developing the theory without entering into the more precise characteristics of the system (i.e. the commutator $[\mathcal{V}, \mathcal{V}]$ and the form of $\mathcal{N}$, or the equation of motion for $v_\alpha$), but if they are given we should be able to construct two independent quantities starting from the set (142), in such a way that they commute with $\Theta$ and with each other and are constants of motion. Evidently, one of them is nothing but the invariant hamiltonian $H$, while we write the other as $Z$. Thus we finally get the set of six conserved invariant quantities mutually commuting:

$$(P, \Sigma, \rho, \Theta, H, Z). \quad (148)$$

The physical significance of $\Theta$, $H$, and $Z$ will be seen later. It is to be remarked that, speaking most generally, $\rho$ and $\Theta$ also may not be conserved, and in such a case we may construct four independent conserved invariant quantities, $\rho'$, $\Theta'$, $H$ and $Z$, replacing $(\rho, \Theta, \phi, \mathcal{Z})$, so that we have

$$(P, \Sigma, \rho', \Theta', H, Z) \quad (148')$$

in place of (148). Here $\rho'$ and $\Theta'$ are intimately related to $\rho$ and $\Theta$, respectively (cf. Eq. (160)).

In any case the set of those six quantities exhausts possible independent conserved invariant quantities to be obtained by using our basic variables (140) alone. It is possible that some of those conserved quantities degenerate according to the special characteristics of the system. On the other hand, the pure internal variables $\xi_\alpha$ of the system are not necessarily implied by the variables $(S_{\alpha\nu}, v_\alpha)$ completely, and accordingly it is possible that the system has independent invariant conserved quantities over and above (148') which explicitly depend on $\xi_\alpha$.

6d. To see the relation between $H_\mu$ and $w_\mu$, we insert the dual of (71) into (105) and take (125) into account. We get

$$H_\mu = \frac{\varepsilon}{m} w_\mu - \frac{\dot{\xi}}{m^2} p_\mu + A_\mu, \quad (149)$$

where

$$A_\mu = -i\varepsilon_{\mu\nu\lambda} r_\nu v_\lambda p_\nu = i\varepsilon_{\mu\nu\lambda} p_\nu v_\lambda r_\nu = -i\varepsilon_{\mu\nu\lambda} r_\lambda p_\nu, \quad (150)$$

which satisfies

$$A_\mu p_\mu = 0, \quad \{A_\mu, v_\nu\} = 0, \quad \{A_\mu, r_\nu\} = 0. \quad (151)$$
The $A_\mu$ represents the kinematical angular momentum of the Zitterbewegung, since in the $II$-frame
\begin{equation}
A_\mu^{(m)} = m \varepsilon_{ijk} r_\nu^{(m)} v_\mu^{(m)}, \quad (A_\mu^{(0)} = 0)
\end{equation}
and (149) becomes
\begin{equation}
v_\nu^{(m)} w_\mu^{(m)} = H_\mu^{(m)} - A_\mu^{(m)},
\end{equation}
indicating that the spin $w_\mu^{(m)}$ is the difference of the angular momentum related to pure internal motion, $H_\mu^{(m)}$, and the kinematical angular momentum of the orbital precession, $A_\mu^{(m)}$ (apart from the factor $\nu_\nu^{(m)}$). The long-time average of (149) yields
\begin{equation}
\langle H_\mu \rangle = \frac{\langle \epsilon \rangle}{m} w_\mu + \langle A_\mu \rangle.
\end{equation}
We have the commutation rule:
\begin{equation}
[A_\mu, w_\nu] = -m r_\nu r_\mu.
\end{equation}

The $A_\mu$ does not provide new independent invariant quantities: First the projection of $A_\mu$ on $w_\mu$, defined by
\begin{equation}
A_w = A_\mu w_\mu = w_\nu A_\mu = -i \varepsilon_{\rho\sigma\lambda} \varepsilon_\nu p_\nu r_\rho w_\mu = -i \varepsilon_{\rho\sigma\lambda} w_\rho r_\lambda \dot{r}_\nu p_\nu,
\end{equation}
stands in an intimate connection with $\Omega$ defined in (141), since we get from (149):
\begin{equation}
\Omega = \frac{\epsilon}{m} \Sigma + A_w.
\end{equation}
Similarly we get
\begin{equation}
\begin{cases}
A_\mu H_\mu = \theta - \frac{\epsilon}{m} \Omega + \frac{1}{m^2} \dot{r}_\mu^2,
\end{cases}
\end{equation}
\begin{equation}
A_\mu^2 = \theta + \frac{\epsilon^2}{m^2} \Sigma + \frac{1}{m^2} \dot{r}_\mu^2 - \frac{1}{m} \{\epsilon, \Omega\}.
\end{equation}
These relations indicate that the invariant quantities involving $A_\mu$ are always reduced to the mixed invariant quantities, $\Omega$ and $\dot{r}_\mu$, already considered. The last relation (156) is especially important as we shall see in Part II.

Ge. Finally we briefly mention the Schrödinger equation. It has the general form
\begin{equation}
i \hbar \frac{d\phi}{d\tau} = (v_\mu p_\mu + \mathcal{K}) \phi,
\end{equation}
and in particular the stationary state equation becomes
\begin{equation}(v_\mu p_\mu + \mathcal{K}) \phi = \lambda \phi,
\end{equation}
or in short
This equation has similarity with the usual Dirac equation but is more general. A usual wave equation always satisfies, when iterated, the Klein-Gordon equation, \( p_\mu^2 \approx -m_0^2 \), meaning that it corresponds to an irreducible representation of the inhomogeneous Lorentz transformations, to describe particles of a unique rest mass \( m_0 \). On the other hand our wave equation (157) need not lead to the Klein-Gordon equation by iteration but it is compatible with the latter since \( H \) and \( P \) commute. A general solution of (10) is a superposition of waves corresponding to different rest mass states, and those possible rest mass values are to be determined by (157). If we classify the solutions of (157) by taking simultaneous eigenstates of all the quantities of (148), they are expected just to represent various possible elementary-particle states.

6f. In this section we have analysed general properties of point-like systems of the first kind. However, (101) does not yet determine the kinematical properties of \( v_\mu \) completely, and includes different types of systems, to be characterized according to the criteria (A) and (B) and possibly still to other ones (see below). As already mentioned it is most convenient to apply the criterion (B) first so as to divide systems of the first kind into two categories, namely into “classical models”:

\[
[v_\mu, v_\nu] = ik_{\mu\nu} = 0, \quad (158)
\]

and “non-classical models”: \( k_{\mu\nu} \neq 0 \).

(i) Now for classical models we can consider simultaneous eigenstates of velocity components and so \( v_\mu \)'s are not only internal variables but also the kinematical variables describing the internal configuration of the system. This is just part of the general conditions required for a system to be of realistic internal structure. Since in this case \([p, v_\mu] = 0\), it is appropriate to regard \( p \) as an absolute invariant restricted by \( p^2 = p \), namely to restrict the system to normal class (cf. 5a).

The remarkable fact for the present case is that the quantities \((S_{\mu\nu}, v_\mu)\) exactly satisfy among themselves the commutation rules for the generators of an inhomogeneous Lorentz group, taking account of (54), (102), and (158). Thus \( H_\mu \) plays for the set of quantities \((S_{\mu\nu}, v_\mu)\) the same role as \( W_\mu \) does for \((J_{\mu\nu}, p_\mu)\), hence takes integer and half-integer eigenvalues again; \( \rho \) and \( \theta \) represent the two group-invariants of this inhomogeneous Lorentz group generated by \((S_{\mu\nu}, v_\mu)\). However, we then have the general restriction that \( \Sigma \) and \( \Theta \) take either both integer or both half-integer eigenvalues at the same time.

This kind of systems will be treated more closely in Part II and Part III. (Simple examples are the Hönlpapapetrou particle and relativistic rotators.)

(ii) On the other hand non-classical models are also important, of which the simplest type is given by assuming Bhabha’s relation:
where $\kappa$ is an absolute constant of the dimension of action. Eq. (159) means the assumption that the internal angular momentum tensor $S_{\mu\nu}$ is produced out of $v_\mu$ only and has no contribution from the other internal degrees of freedoms of the system even if it has any.

In this case the totality of $S_{\mu\nu}$ and

$$S_{\phi\phi} = -S_{\phi\phi} = (\hbar\kappa)^{1/2} v_\phi$$

constitutes the generators of a homogeneous 5-dimensional Lorentz group, and

$$\begin{cases}
\phi' = \phi - \frac{2}{\hbar\kappa} \phi, \\
\theta' = \theta + \frac{1}{\hbar\kappa} \chi^2
\end{cases}$$

represent the two group-invariants of that group, giving general conserved invariant quantities.

If we take the limit $\kappa \to \infty$, a system of this type goes over to the case of classical model, but then $S_{\mu\nu}$ becomes decoupled from $v_\mu$ and this fact allows us to have broad varieties of possible "classical models".

Systems obeying (159) will also be analysed in Part II. They contain as its special examples the Dirac and Kemmer particles, and particle analogue corresponding to Heisenberg's non-linear field, etc. Indeed we shall see that if we further specialize the system by adding the simple relation

$$\{X, v_\mu\} = 0,$$

we get the system of Dirac type, while by assuming another relation

$$\{\rho, v_\mu\} = -\frac{\hbar}{\kappa} v_\mu$$

we get the system of Kemmer type.

Finally we mention two other criteria which are useful for characterizing systems of the first kind and are employed later. The previous equation (126) indicates that $\dot{v}_\mu$ depends not only on $k_\mu$ but also on $[X, v_\mu]$. Since $X$ is in general a function of the proper internal variables $(v_\mu, S_{\mu\nu}, \xi_\alpha)$ alone, if $[v_\mu, v_\mu]$ is specified, the form of $[X, v_\mu]$ is restricted to a certain extent, but not entirely, and this commutator gives our fourth criterion: (D) Whether or not $[X, v_\mu]$ vanishes identically. If it vanishes, $\dot{\xi}$ (and so $X$) becomes a constant of motion. Our final criterion is given by the formerly mentioned condition (cf. Eqs. (81) and (83)): (E) Whether or not $[\Sigma, v_\mu] = 0$.

The author would like to express his sincere thanks to Professor H. Yukawa, Professor S. Sakata and Dr. H. Fukutome for stimulating discussions about this work.
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